University Bonn — WS 2017/18 Ursula Hamenstädt

Advanced Geometry I

Exercises 6 14. November 2017

6.1. Let X be a smooth (local) section of $T\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C}$.

(a) Show that X is holomorphic if and only if it can be written in the form

$$X = \sum_i a_i \frac{\partial}{\partial z_i}$$

with holomorphic functions a_i.

(b) The Lie bracket $[X^1, X^2]$ of two vector fields X^1, X^2 is the vector field defined by $[X^1, X^2](f) = X^1(X^2(f)) - X^2(X^1(f))$ for every smooth function f. Compute this is local coordinates for vector fields $X^k = \sum_j a_j^k \frac{\partial}{\partial x_j} + \sum_j b_j^k \frac{\partial}{\partial y_j}$ (here $z_j = x_j + iy_j$, $k = 1, 2$) and use this to show that the Lie bracket is bilinear.

(c) Use (a) and (b) and $\mathbb{C}$-bilinear extension to compute (in local coordinates): If X is holomorphic then $[X, iY] = i[X, Y]$ for all vector fields Y (ie X automorphic).

6.2. Let $\omega = \frac{i}{2\pi} \sum_j dz_j \wedge d\bar{z}_j$ be the standard Kähler form on $\mathbb{C}^n$.

(a) Find for any point in $\mathbb{C}^n$ an explicit function ϕ such that $\omega = \frac{i}{2\pi}\partial\bar{\partial}\phi$.

(b) Let h be the standard Hermitian metric on $\mathbb{C}^n$. Show that any smooth real function f on $\mathbb{C}^n$ such that the metric $e^f h$ is Kähler is constant.

6.3. The canonical embedding $\mathbb{C}^n \to \mathbb{C}^{n+1}$ induces an embedding $\mathbb{C}P^{n-1} \to \mathbb{C}P^n$. Show that the restriction to $\mathbb{C}P^{n-1}$ of the Fubini Study metric on $\mathbb{C}P^n$ is the Fubini Study metric on $\mathbb{C}P^{n-1}$.

6.4. Let M, N be two Kähler manifolds. Show that $M \times N$ admits a Kähler metric.