University Bonn — WS 2017/18	Ursula Hamenstädt
Advanced Geometry I	
Exercises 5	6. November 2017

5.1. Consider the complex projective line $\mathbb{C}P^1$.

(a) Show that the group of biholomorphic automorphisms of $\mathbb{C}P^1$ is transitive (it is even triply transitive).

(b) Use (a) to show that for any pair of points $p \neq q \in \mathbb{C}P^1$ there exists a meromorphic function f on $\mathbb{C}P^1$ with a simple pole at p, a simple zero at q and no other zeros or poles.

(c) Use (b) to show the following. Let $D_1 = p$, $D_2 = q$ for points $p, q \in \mathbb{C}P^1$. The line bundles determined by D_1, D_2 are equivalent (=connected by a holomorphic bundle isomorphism).

5.2. For $n \geq 2$ consider the complex projective space $\mathbb{C}P^n$. Let $H = \mathbb{C}P^{n-1} \subset \mathbb{C}P^n$ be a complex projective hyperplane.

(a) Show that there exists a holomorphic line bundle $\mathcal{O}(1) \to \mathbb{C}P^n$ with a holomorphic section σ vanishing to first order on H and which does not have any other zero.

(b) Show that the first Chern class of $\mathcal{O}(1)$ is not zero.

5.3. Let $\tau \to G(2, n)$ be the universal bundle over the Grassmannian of two-dimensional subspaces in $\mathbb{C}^n$.

(a) Show that the exterior produce $L = \tau \wedge \tau$ is a holomorphic line bundle over $G(2, n)$.

(b) Show that the first Chern class of L is non-trivial. (Hint: $G(2, 3) \subset G(2, n)$ is biholomorphic to $G(1, 3) = \mathbb{C}P^2$)

5.4. Let $\mathcal{O}(1) \to \mathbb{C}P^n$ be the holomorphic line bundle from Exercise 2. For $k \geq 2$ define $\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$ (k-fold tensor product). Show that any nontrivial homogeneous polynomial p on $\mathbb{C}^{n+1}$ of degree k (ie we have $p(az_0, \ldots, az_n) = a^k p(z_0, \ldots, z_n)$ for all $a \in \mathbb{C}$ and all $z_0, \ldots, z_n$) can be viewed as a section of $\mathcal{O}(k)$.