University Bonn — WS 2017/18	Ursula Hamenstädt
Advanced Geometry I	
Exercises 10	12. December 2017

10.1. Let V be an $2n$-dimensional real vector space with a complex structure J and a J-invariant inner product $\langle , \rangle$. Recall that these data define a Hermitean metric h on V.

(a) Show that for all $p, q \in \{0, \dots, n\}$ the Hermitean metric h induces a Hermitean metric on $\Lambda^{p,q}V^*$, the tensor space of antisymmetric p-fold complex linear q-fold complex antilinear $\mathbb{C}$-valued functionals on V. (Note that the complex structure is inherited from the complex structure of $V \otimes_{\mathbb{R}} \mathbb{C}$). Show that the decomposition $\Lambda^m V^* = \oplus_{p+q=m} \Lambda^{p,q} V^*$ is orthogonal for this metric.

(b) Show that the Hodge star operator $* : \Lambda^{p,q}V^* \to \Lambda^{n-p,n-q}V^*$ is self-adjoint up to a factor $(-1)^u$ for some u. Compute this u.

10.2. Let M be a compact complex manifold of dimension n.

(a) Show that $\Delta_{\bar\partial} * = * \Delta_{\bar\partial}$.

(b) Show that $H^{p,q}_{\bar\partial}(M)$ is isomorphic to $H^{n-p,n-q}_{\bar\partial}(M)$.

10.3. Let $M = \mathbb{C}^n/\Lambda$ be a compact complex torus of dimension n. Here $\Lambda \sim \mathbb{Z}^{2n}$ is a lattice in $\mathbb{C}^n$. Compute the dimension of $H^{1,0}_{\bar\partial}(M), H^{0,1}_{\bar\partial}(M)$ and write down an explicit basis of these complex vector spaces.

10.4. Let P be the regular octagon in $\mathbb{C}$, centered at the origin. Let S be the Riemann surface obtained from P by identifying opposite sides.

(a) Show that the one-form dz on $\mathbb{C}$ descends to a holomorphic one-form η on S.

(b) Compute the number of zeros of η and their multiplicities.

(c) Use (b) to show that there exists a two-sheeted branched cover $S \to \mathbb{C}P^1$. Write this cover down explicity and determine the number of its branch points.