University Bonn — SS 2017 Ursula Hamenstädt

# Advanced Geometry II

Exercises 8 14. June 2017

**8.1.** (a) Show that the map $\gamma : t \to \gamma(t) = t + i \in \{z \mid \Im z > 0\} = \mathbb{H}^2$ is not a quasi-geodesic.

(b) Show that there is no reparametrization of the curve $\gamma$ above which makes it a quasi-geodesic.

**8.2.** Let $\Gamma_\theta$ be obtained from the fundamental group $\Gamma$ of a closed hyperbolic surface by bending ($\theta$ small). It leaves the pleated surface $H \subset \mathbb{H}^3$ invariant. Let $A \subset \mathbb{H}^3$ be closure of the union of all the images of geodesic arcs in $\mathbb{H}^3$ with both endpoints on $H$.

(a) Show that $A$ is invariant under the action of $\Gamma_\theta$ and has non-empty interior for $\theta \neq 0$.

(b) Show that $\Gamma \backslash A$ is compact.

(c) Show that $A$ is quasi-isometric to the hyperbolic plane $\mathbb{H}^2$.

**8.3.** Let $\Gamma$ be a finitely generated group, with finite symmetric generating set $\mathcal{S}$ which defines a word metric $d$. Show that the metric defined by a different finite generating set $\mathcal{S}'$ is quasi-isometric to $d$ and give an example which shows that this does not hold true if we allow infinite generating sets.

**8.4.** (a) Let $M$ be a smooth complete Riemannian manifold. Show that $\mathbb{H}^2 \times M$ equipped with the product metric is hyperbolic in the sense of Gromov if and only if the diameter of $M$ is finite.

(b) For each $m \geq 0$ construct a complete Riemannian manifold which is hyperbolic space in the sense of Gromov with respect to its induced distance and whose Gromov boundary consists of precisely $m$ points.

(c) Let $X$ be a hyperbolic length space and let $\gamma : \mathbb{R} \to X$ be a geodesic. Let furthermore $z \in X$ be arbitrary and let $y \in \gamma(\mathbb{R})$ be a point of smallest distance to $z$. Show that any geodesic connecting $z$ to some point on $\gamma$ passes through a uniformly bounded neighborhood of $y$.