University Bonn — SS 2017 Ursula Hamenstädt

Advanced Geometry II

Exercises 2 26.April 2017

2.1. The group $PSL(2,\mathbb{R})$ has a natural topology as a quotient of $SL(2,\mathbb{R})$. Call a subgroup $\Gamma < PSL(2,\mathbb{R})$ *discrete* if the induced topology on Γ is discrete. Show that for an abstract group Γ, the following are equivalent.

(i) Γ is the fundamental group of a complete oriented hyperbolic surface.

(ii) There exists an embedding $\Gamma \to PSL(2,\mathbb{R})$ whose image is a discrete subgroup not containing any elliptic elements.

2.2. The *injectivity radius* $i(p)$ of a hyperbolic surface S at a point p is the supremum of all numbers $r > 0$ such that the open ball of radius r about p is contractible.

(i) Show: If there exists some p so that $i(p) = \infty$ then S is isometric to $\mathbb{H}^2$.

(ii) Show: If the group Γ as in the first exercise contains a parabolic element then there exists a sequence of points $p_i \subset S$ such that $i(p_i) \to 0$.

2.3. Let again S be a complete hyperbolic surface and let $\gamma \subset S$ be a periodic geodesic. Show that if γ' is any curve freely homotopic to γ whose trace is different from the trace of γ, then the length of γ' is strictly bigger than the length of γ.

2.4. (i) Determine the set of all triples $(\alpha, \beta, \gamma) \in [0, \pi]$ so that there exists a hyperbolic triangle with angles α, β, γ.

(ii) Determine the set of all triples $(a, b, c) \in [0, \infty)$ so that there exists a hyperbolic triangle with side lengths a, b, c.