University Bonn — SS 2017 Ursula Hamenstädt

**Advanced Geometry II**

Exercises 11 11. July 2017

**11.1.** Let $M$ be a hyperbolic 3-manifold with positive injectivity radius which is homeomorphic to $S \times \mathbb{R}$. Assume that the convex core $\mathrm{Core}(M)$ of $M$ is not compact.

(a) Show that $M$ has precisely two ends.

(b) Show that there is one end $E$ of $M$ and a globally minimizing geodesic ray $\gamma : [0, \infty) \to \mathrm{Core}(M)$ which is exits the end $E$.

(c) Show that if $\mathrm{Core}(M) = M$ then there exists a globally minimizing geodesic line $\gamma : \mathbb{R} \to M$.

(d) Let $g$ be the intrinsic path metric on $S \times \{0\}$. Show that there exists a sequence $\gamma_i$ of closed geodesics in $M$ so that $\ell(\gamma_i)/\ell_S(\gamma_i) \to 0$ where $\ell_S(\gamma_i)$ is the shortest length of a closed curve on $S$ which is freely homotopic to $\gamma_i$.

**11.2.** Let $S$ be a closed hyperbolic surface with fundamental group $\Gamma < PSL(2, \mathbb{R})$.

(a) Show that the limit set for the action of $\Gamma$ on the hyperbolic plane equals the entire circle $S^1 = \partial \mathbb{H}^2$.

(b) A fixed point $\xi \in S^1$ of an element $e \neq g \in \Gamma$ is *attracting* if $g^k \zeta \to \xi$ uniformly for all $\zeta$ outside compact neighborhoods of a point $\eta \in S^1$. Show that every element $g \in \Gamma$ admits an attracting fixed point.

(c) Show that for every $e \neq g \in \Gamma$, the attracting fixed point of $g$ is distinct from the repelling fixed point (i.e. the attracting fixed point of $g^{-1}$).

(d) (Harder) Assume that $h \in \Gamma$ is another element and let $\zeta \neq \xi$ be the attracting fixed point of $h^{-1}$. Assume furthermore that the attracting fixed point of $h$ is distinct from the attracting fixed point of $g^{-1}$. Show that for any neighborhood $U$ of $\xi$, $V$ of $\zeta$ there exists a number $k > 0$ such that the element $g^k \circ h^k$ has attracting fixed point in $U$ and repelling fixed point in $V$.

(d) Show that for every non-empty open subset $U$ of $S$, there exists a closed geodesic in $S$ passing through $U$.

**11.3.** (Harder) Let $M$ be a hyperbolic 3-manifold with injectivity radius bounded from below. Show that through every non-empty open subset $U$of $\mathrm{Core}(M)$ passes a closed geodesic.