Chapter 4

The ending lamination conjecture

4.1 Convex cocompact manifolds

4.1.1 Limit sets

In this section we begin the investigation of complete hyperbolic 3-manifolds which are diffeomorphic to $S \times \mathbb{R}$ where as before, S is a closed surface of genus $g \geq 2$. Such a manifold M can be represented as a quotient $M = \mathbb{H}^3/\Gamma$ where Γ is a discrete subgroup of the group $PSL(2,\mathbb{C})$ of orientation preserving isometries of hyperbolic 3-space $\mathbb{H}^3$. We begin with collecting some general properties of discrete subgroups of $PSL(2,\mathbb{C})$.

Hyperbolic 3-space $\mathbb{H}^3$, viewed as a geodesic metric space, is hyperbolic in the sense of Gromov. Its Gromov boundary $\partial\mathbb{H}^3$ can naturally be identified with the space $T^1_X\mathbb{H}^3$ of all initial velocities of geodesic rays in $\mathbb{H}^3$ issuing from a fixed point x, which is just the standard sphere $\partial\mathbf{H}^3 = S^2$. The union $\mathbf{H}^3 \cup S^2$ is homeomorphic to a compact ball. The action of $PSL(2,\mathbb{C})$ extends to the standard action of $PSL(2,\mathbb{C})$ on $S^2 = \mathbb{C} \cup \{\infty\}$ by *linear fractional transformations*

$$z \to \frac{az+b}{cz+d} \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C}).$$

With this identification, the group multiplication in $PSL(2,\mathbb{C})$ transforms into concatenation of group elements.

Any element in $PSL(2,\mathbb{C})$ can be conjugated in $PSL(2,\mathbb{C})$ to an element in Jordan normal form. This means that every $A \in PSL(2,\mathbb{C})$ is conjugate to either $z \to z + a$ for some $a \in \mathbb{C} - \{0\}$ or to $z \to \lambda z$ for some $\lambda \in \mathbb{C} - \{0\}$.

An element g of $PSL(2,\mathbb{C})$ which is conjugate to $z \to z+a$ for some $a \neq 0$ is called *parabolic*. Its action on S^2 has precisely one fixed point $x \in S^2$. If $y \in S^2$ is arbitrary, then the sequence $g^i y$ converges as $i \to \pm\infty$ to x. The subgroup $< g >$ of $PSL(2,\mathbb{C})$ generated by a parabolic element g is infinite cyclic and acts freely on $\mathbb{H}^3$. Any closed curve in $\mathbb{H}^3/\langle g\rangle$ can be freely homotoped to a curve whose length is arbitrarily small.

An element g conjugate to $z \to \lambda z$ for some $\lambda \in \mathbb{C} - \{0\}$ with $|\lambda| \neq 1$ is called *loxodromic*. It acts on S^2 with *north-south dynamics* with respect to fixed points $x \neq y$: for every neighborhood U of x and V of y there is some $k > 0$ such that $g^k(S^2 - V) \subset U$ and $g^{-k}(S^2 - U) \subset V$. Moreover, the geodesic γ in $\mathbb{H}^3$ connecting x to y is an *axis* for g. This means that γ is invariant under g and that g acts on γ as a translation. The *translation length* of g is the distance between a point $z \in \gamma$ and its image under g. This translation length equals $2|\log(|\lambda|)|$.

An element conjugate to $z \to \lambda z$ with $|\lambda| = 1$ is called *elliptic*. An element $g \in PSL(2,\mathbb{C})$ is elliptic if and only if its action on $\mathbb{H}^3$ has a fixed point (see Proposition 1.16 of [**?**]) if and only if the closure in $PSL(2,\mathbb{C})$ of the group $\langle g\rangle$ generated by g is compact. In particular, an element of $PSL(2,\mathbb{C})$ of finite order is elliptic.

Definition 4.1.1. A *Kleinian group* is a discrete subgroup of $PSL(2,\mathbb{C})$.

Let $\Gamma < PSL(2,\mathbb{C})$ be any Kleinian group. Since the stabilizer in $PSL(2,\mathbb{C})$ of a point in $\mathbf{H}^3$ is compact, the intersection of Γ with the stabilizer of a point in $\mathbf{H}^3$ is *finite*. On the other hand, every finite subgroup of $PSL(2,\mathbb{C})$ consists of elliptic elements. This implies that the discrete group $\Gamma < PSL(2,\mathbb{C})$ is torsion free if and only if it does not contain elliptic elements. Under this assumption, the quotient space $\mathbb{H}^3/\Gamma$ is a smooth hyperbolic manifold whose fundamental group is isomorphic to Γ.

The group Γ is purely loxodromic if and only if it is torsion free and every conjugacy class in Γ can be represented by a (unique) closed geodesic in the manifold $M = \mathbb{H}^3/\Gamma$. This closed geodesic is the projection of a geodesic line in $\mathbb{H}^3$ which is invariant under some element $e \neq g \in \Gamma$. On the other hand, if Γ contains a parabolic element then the function which associates to $x \in \mathbb{H}^3/\Gamma$ the *injectivity radius* $\mathrm{inj}(x)$ *at* x, i.e. the supremum of all numbers $r > 0$ such that the metric ball of radius r about x is contractible, is not bounded from below by a positive constant. In other words, a torsion free group Γ such that the injectivity radius of $\mathbb{H}^3/\Gamma$ is bounded from below by a positive constant is purely loxodromic. Note that a lower bound for the injectivity radius is equivalent to a lower bound for the length of any non-contractible closed curve in M.

Our goal is to find a geometric invariant for hyperbolic 3-manifolds $M = \mathbb{H}^3/\Gamma$ of positive injectivity radius which are diffeomorphic to $S \times \mathbb{R}$. Furthermore, we aim at a complete classification up to isometry, ie at an invariant which distinguishes non-isometric such manifolds. For this we remark first

Lemma 4.1.2. *Two complete hyperbolic 3-manifolds $M = \mathbb{H}^3/\Gamma_1, N = \mathbb{H}^3/\Gamma_2$ are isometric if and only if the groups Γ_1, Γ_2 are conjugate in $PSL(2, \mathbb{C})$.*

Proof. An isometry $M \to N$ lifts to an isometry of $\mathbb{H}^3$ which conjugates the action of the groups Γ_1, Γ_2 on $\mathbb{H}^3$. Vice versa, any element of $PSL(2, \mathbb{C})$ which conjugates the action of Γ_1 to the action of Γ_2 descends to an isometry of $\mathbb{H}^2/\Gamma_1$ onto $\mathbb{H}^3/\Gamma_2$. □

Thus for a classification of complete hyperbolic 3-manifolds of positive injectivity radius diffeomorphic to $S \times \mathbb{R}$ it suffices to classify conjugacy classes of purely loxodromic subgroups of $PSL(2, \mathbb{C})$ which are isomorphic to the fundamental group of S and with the additional property that the absolute value of the trace of each nontrivial element is uniformly bounded away from 2.

A particularly simple class of examples arises as follows. Choose any cocompact torsion free lattice $\Gamma < PSL(2, \mathbb{R})$ and view Γ as a subgroup of $PSL(2, \mathbb{C})$ via the embedding $PSL(2, \mathbb{R}) \to PSL(2, \mathbb{C})$. The resulting hyperbolic 3-manifold $M = \mathbf{H}^3/\Gamma$ contains a totally geodesic embedded surface $\mathbf{H}^2/\Gamma$. Any non-contractible closed curve in M is freely homotopic to a closed geodesic. This closed geodesic is the curve of minimal length in its free homotopy class, and it is contained in the surface $\mathbf{H}^2/\Gamma \subset M$.

A discrete subgroup of $PSL(2, \mathbb{C})$ which is conjugate to a subgroup of $PSL(2, \mathbb{R})$ is called *Fuchsian.* Thus a subgroup Γ of PSL$(2, \mathbb{C})$ isomorphic to the fundamental group of a closed surface is Fuchsian if and only if it stabilizes a totally geodesic hyperbolic plane $\mathbf{H}^2 \subset \mathbf{H}^3$, and this in turn is the case if and only if it stabilizes a round circle in S^2 (which is just the boundary of the invariant totally geodesic subspace $\mathbf{H}^2 \subset \mathbf{H}^3$). The plane $\mathbf{H}^2$ is the convex hull of this round circle in $\mathbf{H}^3$.

For discrete groups $\Gamma < PSL(2, \mathbb{C})$ which are not Fuchsian, no round circle in S^2 is invariant, but we can look for closed invariant subsets of S^2 of a more general form and try to relate the shape of such a set to the geometry of the quotient manifold. A particularly useful invariant closed subset of S^2 is described in the next definition.

Definition 4.1.3. The *limit set* Λ of a subgroup $\Gamma < PSL(2,\mathbb{C})$ is the set of accumulation points in the boundary $\partial\mathbf{H}^3 = S^2$ of a Γ-orbit in $\mathbf{H}^3$. The complement $\Omega = S^2 - \Lambda$ is called the *domain of discontinuity* for the action of Γ on S^2.

Lemma 4.1.4. *The limit set Λ of a Kleinian Γ is a closed Γ-invariant subset of S^2 which contains the fixed points of all loxodromic elements of Γ.*

Proof. If $x \in \mathbf{H}^3$, $\xi \in \Lambda$ and if $\{g_i\} \subset \Gamma$ are such that $g_i x \to \xi$ then for every $h \in \Gamma$ the sequence $\{hg_i x\}$ converges to $h\xi$. Thus the limit set Λ of Γ is a Γ-invariant subset of S^2 which moreover is closed by construction. The domain of discontinuity of Γ is open and Γ-invariant.

Since a loxodromic element of $PSL(2,\mathbb{C})$ preserves a geodesic γ in $\mathbf{H}^3$ and acts on γ as a group of translations, the endpoints of γ are contained in the limit set of any subgroup of $PSL(2,\mathbb{C})$ containing g. Moreover, these endpoints are just the fixed points for the action of g on S^2. This shows the lemma. □

The following easy observation explains the significance of the limit set. For its formulation, we define

Definition 4.1.5. A Kleinian group $\Gamma < PSL(2,\mathbb{C})$ is called *elementary* if its limit set consists of at most two points.

Elementary Kleinian groups can easily be classified (see [**?**]. We do not use this classification in the sequel, so we omit to discuss it here. All we need is the following remark.

Lemma 4.1.6. *A a torsion free purely loxodromic elementary Kleinian group is infinite cyclic.*

Proof. Let Γ be a torsion free purely loxodromic Kleinain group, and let $e \neq g \in \Gamma$ be a loxodromic element with fixed points ξ, η. Then ξ, η are contained in the limit set of Γ. Since Γ is elementary by assumption, the limit set of Γ is the set $\{\xi, \eta\}$, and this set is Γ-invariant.

Thus each $u \in \Gamma$ preserves the geodesic γ in $\mathbf{H}^3$ connecting ξ and η and acts on γ as an isometry. This shows that there is a homomorphism ρ of Γ into the isometry group of the geodesic line γ. Since Γ is torsion free and hence does not contain elliptic elements, the homomorphism ρ is injective and its image consists of translations. Since $\Gamma < PSL(2,\mathbb{C})$ is discrete, the group $\rho(\Gamma)$ is discrete. Therefore Γ is isomorphic to a discrete group of translations of the real line, i.e. Γ is infinite cyclic. □

The next easy observation explains the significance of the limit set.

Lemma 4.1.7. *Let $\Gamma < PSL(2,\mathbb{C})$ be any Kleinian group and let Λ be the limit set of Γ.*

1. *Γ acts properly discontinuously on the domain of discontinuity.*
2. *If Γ contains at least one loxodromic element then the limit set Λ of Γ is the closure of the fixed points of all loxodromic elements.*
3. *If Γ is non-elementary and contains at least one loxodromic element then Λ is the smallest non-empty closed Γ-invariant subset of S^2.*

Proof. Let Γ be in the lemma. We show first that Γ acts properly discontinuously on its domain of discontinuity.

Let Ω be the domain of discontinuity of Γ. Then Ω is an open Γ-invariant subset of the compact sphere $\partial\mathbf{H}^3 = S^2$. Since Γ acts as a group of homeomorphisms on S^2, it suffices to show that no point $\xi \in \Omega$ is an accumulation point of a Γ-orbit in Ω.

Let Λ be the limit set of Γ. Assume to the contrary that there are points $\nu, \xi \in \Omega$ and there is a sequence $\{g_i\} \subset \Gamma$ of pairwise distinct elements such that $g_i\nu \to \xi$. Let γ be the hyperbolic geodesic joining ν to some point $\zeta \in \Lambda$. Then for each i, the geodesic $g_i\gamma$ connects $g_i\nu$ to $g_i\zeta \in \Lambda$. Since Λ is closed, up to passing to a subsequence we may assume that the points $g_i\zeta \in \Lambda$ converge to a point $\eta \in \Lambda$. Now $\xi \notin \Lambda$ and therefore the geodesics $g_i\gamma$ converge to the geodesic ρ connecting ξ to η. This means the following. For each fixed $x \in \mathbb{H}^3$, a shortest geodesic connecting x to $g_i\gamma$ converges to a shortest geodesic connecting x to ρ, and the directions of $g_i\gamma$ at the endpoints of these geodesic arcs converge to the direction of ρ. In particular, the geodesics $g_i\gamma$ pass through a fixed compact subset B of $\mathbf{H}^3$.

Let $y \in B \cap \gamma$. Then $g_i y$ is a point on the geodesic $g_i\gamma$. Since Γ acts properly on $\mathbb{H}^3$ and the elements $g_i \in \Gamma$ are pairwise distinct, the distance between $g_i y$ and B tends to infinity as $i \to \infty$. Thus up to passing to a subsequence, we may assume that $g_i y$ converges as $i \to \infty$ in the compact space $\mathbb{H}^3 \cup S^2$ to an endpoint of the geodesic ρ connecting ξ to η. Since $\xi \notin \Lambda$ we have $g_i y \to \eta$.

On the other hand, if $y_i \in g_i\gamma \cap B$ then the distance between $g_i y_i$ and $g_i y \in \gamma_i$ is bounded from above by the diameter of B. In particular, we have $g_i y_i \to \eta$ in $\mathbb{H}^3 \cup S^2$. Then for sufficiently large i, the oriented geodesic segment connecting $y_i \in g_i\gamma$ to $g_i y \in g_i\gamma$ is contained in the subray of γ_i connecting y_i to $g_i\zeta$. By invariance, the oriented geodesic segment connecting y to $g_i^{-1}y_i \in \gamma$ is contained in the subray of γ connecting y to ν. Since $d(g_i^{-1}y_i, y) \to \infty$, this just means that $g_i^{-1}y_i \to \nu$ $(i \to \infty)$. Now the diameter of B is finite and hence for *every* $z \in B$ we have $g_i^{-1}z \to \nu$. Then $\nu \in \Lambda$ which is a contradicts to the assumption that $\nu \notin \Lambda$. The first part of the lemma is proven.

By Lemma 4.1.6, for the proof of the second and the third part of the lemma, we may assume that Γ is non-elementary and contains a loxodromic element g. Then g acts on S^2 with north-south-dynamics and hence the fixed points ξ, ζ for the action of g on S^2 are contained in *any* non-trivial closed g-invariant subset of S^2 which contains at least three points. For every $h \in \Gamma$ the points $h\xi, h\zeta$ are fixed points for the loxodromic element hgh^{-1}. Since the set of fixed points of loxodromic elements of Γ is invariant under Γ, its closure is a closed Γ-invariant subset A of the limit set Λ of Γ. Moreover, it is the smallest non-empty closed Γ-invariant subset of S^2.

We have to show that every point $\xi \in \Lambda$ is contained in A. For this assume without loss of generality that ξ is not fixed by any loxodromic element of Γ. Let $x \in \mathbb{H}^3$ and let $\{g_i\} \subset \Gamma$ be a sequence with $g_i x \to \xi$. Let $h \in \Gamma$ be a loxodromic element. Then the axis γ of h passes through a compact neighborhood B of x. For each i, the axis $g_i\gamma$ of the loxodromic element $g_i h g_i^{-1} \in \Gamma$ passes through the set $g_i B$. But $g_i B \to \xi$ $(i \to \infty)$ and hence there are two cases possible. In the first case, up to passing to a subsequence, there is a compact subset K of $\mathbf{H}^3$ which meets each of the geodesics $g_i\gamma$. But then after possibly passing to another subsequence, the geodesics $g_i\gamma$ converge to a geodesic with one endpoint ξ and hence one of the fixed points of $g_i h g_i^{-1}$ converges to ξ. If $g_i\gamma$ eventually leaves every compact set of $\mathbb{H}^3$ then $g_i\gamma \to \xi$ in the Hausdorff topology of compact subsets of $\mathbb{H}^3 \cup \partial\mathbb{H}^3$ and once again, we conclude that $\xi \in A$. This shows that Λ is contained in the closure of the fixed points of loxodromic elements and completes the proof of the lemma. □

The domain of discontinuity Ω of Γ is a Γ-invariant open subset of S^2, and by Lemma 4.1.7, the group Γ acts on Ω properly discontinuously as a group of biholomorphic automorphisms. If Γ is torsion free, then Ω/Γ is a Riemann surface. By the celebrated *Ahlfors finiteness theorem*, if Γ is finitely generated then this surface is of *finite type*. However, we will not need this beautiful and deep fact in the sequel, and we refer to Chapter 4 of [**?**] for a discussion and for references.

4.1.2 Quasifuchsian groups

In this subsection we relate the limit set of a torsion free Kleinian group Γ to the geometry of the quotient manifold $\mathbb{H}^3/\Gamma$ in a particularly simple special case.

If $A \subset S^2$ is any closed set, then the *convex hull* Hull(A) of A is defined to be the intersection of all closed half-spaces in $\mathbb{H}^3$ whose closures in $\mathbb{H}^3 \cup S^2$

contain A. Clearly Hull(A) is closed and convex, furthermore the *limit set* of Hull(A), i.e the set of all accumulation points of sequences $(x_i) \subset$ Hull(A) which leave every compact set contains A.

Lemma 4.1.8. *The limit set of* Hull(A) *equals* A.

Proof. If $x \in S^2 - A$ the since A is closed by assumption, there exists a round disk about x whose closure is entirely contained in $S^2 - A$. The boundary of this disk is the boundary of a totally geodesic hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^3$ which bounds a halfspace containing A in its closure but not x. Then x is not contained in the limit set of Hull(A). □

Note that if Λ is the limit set of a Kleinian group Λ then Hull(Λ) is invariant under Γ. We next give some geometric information on Hull(Λ) in this case. To this end define a *supporting hyperplane* H for Hull(Λ) at a point $x \in \partial$Hull(Λ) to be a totally geodesic embedded hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^3$ which contains x and bounds a half-space containing Λ in its closure.

The *boundary* ∂Hull(Λ) of Hull(Λ) is the set of all points $x \in$ Hull(Λ) such that every neighborhood of x contains a point in $\mathbb{H}^3 -$ Hull(Λ).

The next proposition implies among others that through every point $x \in \partial$Hull(Λ) passes a supporting hyperplane for Hull(Λ).

Proposition 4.1.9. *Let* $\Gamma < PSL(2,\mathbb{C})$ *be an arbitrary Kleinian group, with limit set* $\Lambda(\Gamma)$*, and let* $x \in \partial$Hull($\Lambda(\Gamma)$)*. Then either*

1. *there is a supporting hyperplane* H *for* Hull($\Lambda(\Gamma)$) *at* x*, and there is a geodesic line* γ *through* x *such that* $\gamma = H \cap \partial$Hull($\Lambda(\Gamma)$)*, or*
2. *there is a unique supporting hyperplane* H *for* Hull($\Lambda(\Gamma)$) *through* x*, and there is an ideal triangle* $\Delta \subset H \cap \partial$Hull($\Lambda(\Gamma)$) *containing* x *in its interior.*

Proof. Let $x \in \partial$Hull($\Lambda(\Gamma)$) and define

$$D_1 = \{w \in T_x^1\mathbb{H}^3 \mid \gamma_w(\infty) \in \Lambda(\Gamma)\}$$

where γ_w is the geodesic with initial velocity w. We claim that the closure A of the convex hull of D_1 in $T_x\mathbb{H}^3$ contains 0.

Note first that this closure is contained in a halfspace of $T_x\mathbb{H}^3$ which is equivalent to stating that 0 is not contained in the interior of the convex hull of D_1. Namely, otherwise there are four points in D_1 which span a simplex containing 0 in its interior, and the convex hull of the geodesic rays with these vectors as initial velocities contains x in its interior contradicting the assumption that $x \in \partial \mathrm{Hull}(\Lambda)$.

To show the claim assume otherwise. Then there exists a unit vector $w \in T^1_x \mathbb{H}^3$ with $\langle w, v\rangle \geq \delta > 0$ for all $v \in A$. By continuity, for small $\epsilon > 0$ we have

$$\langle \gamma'_w(\epsilon), u\rangle \geq \delta/2$$

for all $u \in D_2 = \{z \in T^1_{\gamma_w(\epsilon)}\mathbb{H}^3 \mid \gamma_z(\infty) \in \Lambda(\Gamma)\}$. Then if $H \subset \mathbb{H}^3$ is a closed half-space whose boundary contains $\gamma_w(\epsilon)$ and whose tangent space at that point equals $(\gamma'_w(\epsilon))^\perp$, then $H \subset \Lambda(\Gamma)$ and hence $\mathrm{Hull}(\Lambda(\Gamma)) \subset H$. But $x \in \mathrm{Hull}(\Lambda(\Gamma))$ and $x \notin H$ which is a contradiction. This shows the claim.

There are now two cases. In the first case, there exists $v \in D_1$ such that $-v \in D_1$. Then the geodesic γ with initial velocity $\gamma'(0) = v$ is contained in $\partial\mathrm{Hull}(\Lambda(\Gamma))$. This case corresponds to the first case stated in the proposition.

Since D_1 is contained in a closed half-space of $T_x\mathbb{H}^3$, in the second case 0 is contained in the closure of the convex hull of the boundary of such a half space. Then there are vectors $v_1, v_2, v_3 \in D_1$ contained in a two-dimensional subspace of $T^1_x\mathbb{H}^3$ which span a triangle in this plane containing 0 in its interior. The convex hull of the geodesic rays with these velocities is a totally geodesic triangle in $\mathbb{H}^3$ which is contained in $H \cap \partial\mathrm{Hull}(\Lambda)$. Thus the properties stated in the second case of the proposition are fullfilled. □

Lemma 4.1.9 shows that under the assumption that $\mathrm{Hull}(\Lambda) \subset \mathbb{H}^3$ has non-empty interior, the boundary of $\mathrm{Hull}(\Lambda)$ is a union of *flat* pieces, ie open subsets of totally geodesic embedded hyperbolic planes, and geodesic lines. Furthermore, if γ is such a geodesic line, then γ is contained in any supporting hyperplane through any of its points.

If $x \in \mathbb{H}^3$ is any point, then the function $y \to d(x, y)$ is strictly convex and hence by convexity, it assumes a unique minimum on $\mathrm{Hull}(\Lambda)$. Assigning to x this minimum defines the shortest distance projection $\Pi : \mathbb{H}^3 \to \mathrm{Hull}(\Lambda)$ which is equivariant with respect to the action of Γ. We note

Corollary 4.1.10. *If $x \notin \mathrm{Hull}(\Lambda)$ then the geodesic arc connecting x to $\Pi(x)$ is orthogonal to some supporting hyperplane at $\Pi(x)$.*

Proof. Let $D \subset T_{\Pi(x)}\mathbb{H}^3$ be the tangent plane of the distance sphere of radius $d(x, \mathrm{Hull}(\Lambda))$ about x. The discussion in the proof of Proposition 4.1.9 shows that D is tangent to a supporting hyperplane if there does not exist a vector v so that the geodesic γ_v converges to a point in $\partial\mathrm{Hull}(\Lambda)$ and such that the angle between v and the tangent of the geodesic arc connecting $\Pi(x)$ to x is smaller than $\pi/2$. However, this characterises a shortest distance projection. □

Definition 4.1.11. The *convex core* Core(M) of a hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ is the quotient under Γ of the convex hull Hull(Λ) of the limit set Λ of Γ.

Note that by the discussion in the previous section, Core(M) contains every closed geodesic in M. The convex core of M is a strictly convex closed subset of M. If the interior of this set is not empty then it is a (topological) submanifold of M with boundary.

Assume as before that $\Gamma < PSL(2,\mathbb{C})$ is a discrete subgroup which is isomorphic to the fundamental group $\pi_1(S)$ of the closed surface S. The next proposition gives a geometric description of a particular case of this situation (compare with Section 4.4 of [**?**]).

Proposition 4.1.12. *Let $M = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold whose fundamental group Γ is isomorphic to the surface group $\pi_1(S)$. Then the following are equivalent.*

1. Core(M) *is compact.*
2. *The limit set Λ of Γ is an embedded circle in S^2, and Γ preserves each of the components of its complement.*

Proof. We begin with showing that the second statement in the proposition is a consequence of the first.

Let $\Gamma < PSL(2,\mathbb{C})$ be a discrete subgroup isomorphic to $\pi_1(S)$, let $M = \mathbf{H}^3/\Gamma$ and assume that Core(M) is compact. Since Γ is torsion free, it does not contain elliptic elements. Thus M is a smooth manifold.

Let Hull(Λ) be the convex hull of the limit set Λ of Γ. Then Hull(Λ) is a closed convex subset of $\mathbf{H}^3$. In particular, the restriction of the hyperbolic metric on $\mathbf{H}^3$ to Hull(Λ) is a complete geodesic metric. A geodesic in Hull(Λ) is a geodesic in $\mathbf{H}^3$ and hence Hull(Λ) is hyperbolic in the sense of Gromov.

If Core(M) is compact then the group Γ acts isometrically, properly and cocompactly on Hull(Λ). By Proposition 3.1.4, for a given group isomorphism $\rho : \pi_1(S) \to \Gamma$ and for every $x \in$ Hull(Λ) the inclusion $g \in \pi_1(S) \to \rho(g)x \in$ Hull(Λ) is an equivariant quasi-isometry between the surface group $\pi_1(S)$ and Hull(Λ). Thus Hull(Λ) is ρ-equivariantly quasi-isometric to the hyperbolic plane $\mathbf{H}^2$. By Proposition 3.1.8, there is a ρ-equivariant homeomorphism f of the Gromov boundary S^1 of $\pi_1(S)$ onto the Gromov boundary of Hull(Λ). Since Hull(Λ) is proper, the Gromov boundary of Hull(Λ) can abstractly be identified with the space of equivalence classes of quasi-geodesic rays in Hull(Λ) where two quasi-geodesic rays are equivalent if and only if the Hausdorff distance of their images is finite.

A uniform quasi-geodesic in Hull(Λ) is a uniform quasi-geodesic in $\mathbf{H}^3$. Therefore such a quasi-geodesic converges to a point ξ in the boundary S^2 of $\mathbf{H}^3$ which is necessarily contained in the limit set Λ of Γ. Thus the Gromov boundary $f(S^1)$ of Hull(Λ) is naturally a closed subset of Λ. By equivariance, $f(S^1)$ is Γ-invariant and hence by Lemma 4.1.7 we have $f(S^1) \supset \Lambda$. This shows that the limit set Λ of Γ is homeomorphic to a circle. More precisely, it is the image under the map f of the boundary S^1 of the hyperbolic plane.

The orientation of $S^1 = \partial\mathbf{H}^2$ induces an orientation of Λ. Since the action of every element of $\pi_1(S)$ on S^1 is orientation preserving, by equivariance the induced orientation of Λ is preserved by the action of Γ. However, Γ acts on S^2 as a group of biholomorphic maps thus preserving the orientation of S^2. As a consequence, an element of Γ preserves each of the two components of $S - \Lambda$. The implication 1) $\to$ 2) in the proposition is established.

To show that the second property implies the first, note that by the Jordan curve theorem, an embedded topological circle in S^2 divides S^2 into two open discs D_1, D_2. If the group Γ satisfies the hypothesis in the second statement, then the domain of discontinuity Ω for Γ is a disjoint union of two discs D_1, D_2 which are invariant under Γ. The group Γ acts on them as a group of biholomorphic transformations. By uniformization, the discs D_1, D_2 are biholomorphic to the standard unit disc in $\mathbb{C}$, and this identification is unique up to an element of $PSL(2,\mathbb{R})$. Thus by restriction to D_i, the group Γ defines a subgroup of $PSL(2,\mathbb{R})$, unique up to conjugation. By Lemma 4.1.4, for every $g \in \Gamma$, the fixed points for the action of g on S^2 are contained in the limit set of Γ and hence the action of Γ on $\Omega = D_1 \cup D_2$ is free. By Lemma 4.1.7, Γ acts properly discontinuously on Ω and hence D_i and therefore the quotient of D_i under Γ is a complete marked oriented hyperbolic surface. Since Γ is isomorphic to $\pi_1(S)$, the fundamental groups of these quotient surfaces are isomorphic to $\pi_1(S)$. Now an oriented complete hyperbolic surface S is non-compact if and only if its fundamental group is free. This implies that the quotient surfaces are compact and diffeomorphic to S.

Let C be the closed one-neighborhood of Hull(Λ). Then C is a closed convex Γ-invariant submanifold of $\mathbf{H}^3$ with smooth boundary ∂C (see [**?**]). By convexity, there is a Γ-equivariant shortest distance projection $P : \Omega \to C$. The image of a point $x \in \Omega$ is the unique point $Px \in C$ with the property that the x is the endpoint of the geodesic ray issuing from Px whose initial velocity is the outer normal of C at Px. By equivariance, the quotient $\partial C/\Gamma$ is a compact disjoint union of two surfaces $\partial C_1, \partial C_2$ which are homeomorphic to S. A fixed homomorphism $\rho : \pi_1(S) \to \Gamma$ determines two homeomorphisms $f_1, f_2 : S \to \partial C_i$. The composition of each of these homeomorphisms f_i with the inclusion $\partial C_i \to M$ is a homotopy equivalence. In particular, the maps f_i are homotoic, which means that there is a continuous map $F : S \times [1,2] \to M$ such that $F|S \times \{i\} = f_i$ $(i = 1,2)$.

For each $x \in S$ let γ_x be the unique geodesic arc in M parametrized on $[0,1]$ proportional to arc length which connects $f_1(x)$ to $f_2(X)$ and is homotopic to the arc $t \to F(x,t)$. Then γ_x depends continuously on x and is contained in C. Using these arcs we obtain a new homotopy equivalence $F' : S \times [0,1] \to C$ which maps the boundary of $S \times [0,1|$ homeomorphically onto the boundary of C. The double F' of this map is a homotopy equivalence of the compact manifold $S \times S^1$ onto the double of C doubled along the boundary. In particular, since $S \times S^1$ is a closed manifold, the map F' is of degree one and hence surjective. But this just means that C and hence Core(M) is compact. This completes the proof of the lemma. □

Definition 4.1.13. A subgroup $\Gamma < PSL(2,\mathbb{C})$ which is isomorphic to the fundamental group of a closed oriented surface and such that Core$(\mathbf{H}^3/\Gamma)$ is compact is called *quasi-fuchsian*, and the quotient manifold $\mathbf{H}^3/\Gamma$ is called *convex cocompact*.

Remark 4.1.14. From the proof of Proposition 4.1.12 we also obtain the following. Let $\Gamma < PSL(2,\mathbb{C})$ be a quasi-fuchsian group with limit set Λ. Let $D \subset S^2$ be a component of the domain of discontinuity for Γ. Then there is a discrete cocompact subgroup $\Gamma_0 < PSL(2,\mathbb{R})$, an isomorphism $\rho_0 : \Gamma_0 \to \Gamma$ and a ρ_0-equivariant biholomorphic map $f : \mathbf{H}^2 \to D$ which extends to a ρ_0-equivariant homeomorphism $\partial\mathbf{H}^2 \to \Lambda$.

The following deep result of Sullivan ([**?**], see also Section 5.2 of [**?**]) is one of the main ingredients for the proof of the ending lamination conjecture. Its proof goes beyond the score of this book and will be omitted. Sullivan's theorem is more general than the version stated here.

Theorem 4.1.15. *Let $M = \mathbf{H}^3/\Gamma, N = \mathbf{H}^3/\Lambda$ be two hyperbolic 3-manifolds which are homotopy equivalent to $S \times \mathbb{R}$. Assume that there is a homotopy equivalence $F : M \to N$ which is a quasi-isometry. Let $\tilde{F}$ be an equivariant lift of F to a quasi-isometry of $\mathbf{H}^3$. If the extension of F to the boundary sphere S^2 induces a conformal homeomorphism of the domain of discontinuity for Γ onto the domain of discontinuity for Λ then M, N are isometric.*

We use Theorem 4.1.15 to show.

Theorem 4.1.16. *A quasi-fuchsian group Γ is determined up to conjugacy by the Γ-quotient of its domain of discontinuity.*

Proof. Let $\Gamma < PSL(2,\mathbb{C})$ be a quasi-fuchsian group and let $\Gamma' < PSL(2,\mathbb{C})$ be another discrete group isomorpic to $\pi_1(S)$ which defines a quotient manifold $N = \mathbf{H}^3/\Gamma$ with Core(N) compact. Assume that the quotient of the domain of discontinuity under the action of Γ' is biholomorphic equivalent to the quotient of the domain of discontinuity for Γ. Then the domain of discontinuity for

Γ' consists of two discs D_1', D_2'. We choose these discs in such a way that D_i'/Γ' is marked biholomorphic to D_i/Γ $(i = 1, 2)$. This means that there is an isomorphism $\rho : \Gamma \to \Gamma'$ and there is a ρ-equivariant biholomorphic map $\Phi_i : D_i \to D_i'$. We use these maps to define a quasi-isometry between $M = \mathbf{H}^3/\Gamma$ onto N which satisfies the assumptions in the statement of Theorem 4.1.15.

Namely, let C, C' be the closed one-neighborhood of the convex hulls of the limit sets of Γ, Γ'. Then C, C' are smooth manifolds with boundary $\partial C, \partial C'$ which are diffeomorphic to $\mathbf{H}^2 \times [-1, 1]$. Moreover, the subsets C, C' of $\mathbf{H}^3$ are convex. The groups Γ, Γ' act on C, C' cocompactly as a group of isometries. There is a Γ, Γ'-equivariant shortest distance projection $P_i : D_i \to C, P_i' : D_i' \to C'$ which is defined as follows. For each $x \in D_i$, the image of x is the unique point on ∂C with the property that the geodesic ray which issues from the point and whose unit tangent is perpendicular to ∂C at that point has its endpoint at x. This shortest distance projection is a homeomorphism.

Define now a ρ-equivariant map $F : \mathbf{H}^3 \to \mathbf{H}^3$ as follows. Compose the inverse of P_i with the given ρ-equivariant conformal map $D_i \to D_i'$ and the projection P_i'. This defines a ρ-equivariant homeomorphism F_0 of the boundary ∂C of C onto the boundary $\partial C'$ of C' which can be extended to $\mathbf{H}^3$ as follows. A geodesic ray which issues from $x \in \partial C$ and is orthogonal to ∂C is mapped isometrically onto the ray issuing from $F_0(x)$ and which is orthogonal to $\partial C'$. The resulting map can be extended to an equivariant homeomorphism $M \to N$ which maps C onto C'. Standard hyperbolic geometry implies that F is a quasi-isometry which extends to a homeomorphims of S^2 whose restriction to S^2 is conformal. The theorem follows. □

The following result is due to Bers. Together with Proposition 4.1.12 it gives a complete classification of conjugacy classes of convex cocompact subgroups of $PSL(2, \mathbb{C})$ which are isomorphic to surface groups.

Theorem 4.1.17. *For every pair $(g, h) \in \mathcal{T}(S) \times \mathcal{T}(S)$ there is a quasi-fuchsian group Γ such that the quotient of the domain of discontinuity for the action of Γ on S^2 consists of the disjoint union of the Riemann surfaces $(S, g), (S, h)$.*

By Lemma 4.1.16, a group as in Theorem 4.1.17 is unique up to conjugation. A proof of the theorem can be found in the book [**?**]. The theorem will not be used in the sequel.

4.2 Degenerate ends

In this section we begin the investigation of manifolds which are homotopy equivalent to $S \times \mathbb{R}$ but which are not convex cocompact.

We begin with the following elementary result from differential geometry.

Lemma 4.2.1. *If* Core(M) *is non-compact then for every* $x \in$ Core(M) *there is a globally minimizing geodesic ray* $\gamma : [0, \infty) \to$ Core(M) *whose lifts to* $\mathbf{H}^3$ *are geodesic rays in* Hull(Λ) *with endpoint in* Λ.

Proof. Assume that Core(M) is non-compact and let $x \in$ Core(M). Choose a sequence of points $y_i \in$ Core(M) whose distances d_i to x tend to infinity as $i \to \infty$. For each i let $\gamma_i : [0, d_i] \to M$ be a minimal geodesic connecting $\gamma_i(0) = x$ to $\gamma_i(d_i) = y_i$ parametrized by arc length. Since Core(M) is convex, these geodesics are entirely contained in Core(M). By passing to a subsequence we may assume that the parametrized geodesic segments γ_i converge as $i \to \infty$ uniformly on compact subsets of $[0, \infty)$ to a geodesic ray γ. Since Core(M) is closed, we have $\gamma[0, \infty) \subset$ Core(M), moreover γ is globally minimizing by continuity of the distance function.

By convexity, a lift $\tilde{\gamma}$ of γ to $\mathbf{H}^3$ is contained in Hull(Λ) and hence the endpoint of $\tilde{\gamma}$ is contained in Λ. □

Our next goal is to obtain a better understanding of the geometry at infinity of a hyperbolic 3-manifold which is not convex cocompact. To give precise meaning of this idea we need the following

Definition 4.2.2. Lete M be a hyperbolic 3-manifold and let $K_1 \subset K_2 \subset \dots$ be an ascending sequence of compact subsets of M whose interiors cover M. An *end* of M is given by a sequence $U_1 \supset U_2 \supset \dots$ where U_i is a component of $M - K_i$. This does not depend on the choice of the sequence K_i. A *neighborhood* of an end is an open set V so that $V \supset U_n$ for some n,

Example 4.2.3. The real line $\mathbb{R}$ has two ends. A regular infinite three-valent tree has uncoutably many ends.

Consider for the moment an arbitrary non-elementary torsion free purely loxodromic Kleinian group $\Gamma < PSL(2, \mathbb{C})$ with quotient manifold $M = \mathbf{H}^3/\Gamma$, limit set $\Lambda \subset S^2$ and convex core Core$(M) \subset M$. Assume that there is at least one end E of M such that the intersection $E \cap$ Core(M) is unbounded. This means that for $U \cap$ Core$(M) \neq \emptyset$ for every neighborhood of E.

By Lemma 4.2.1, for every $x \in$ Core(M) there is a globally minimizing geodesic ray $\gamma : [0, \infty) \to$ Core(M) with $\gamma(0) = x$ whose lifts to Hull(Λ) are geodesic rays which end at points in the limit set Λ of Γ. By Lemma 4.1.7, this limit set is the closure of the fixed points of loxodromic elements in Γ. This easily implies that every neighborhood of the end E is intersected by a closed geodesics in M (the proof of Theorem 4.2.4 below gives a detailed argument for this fact).

It turns out that any such neighborhood *contains* a closed geodesic. Namely, we say that a family $\{c_i\}$ of closed curves in M *exits* an end E of M if $c_i \subset E$ for every i and if moreover for every compact subset K of M the intersection $c_i \cap K = \emptyset$ for all but finitely many i. The next result is due to Bonahon and is true in much bigger generality than stated below [**?**].

Theorem 4.2.4. *Let Γ be a torsion free purely loxodromic Kleinian group and let $M = \mathbf{H}^3/\Gamma$. Let E be an end of M such that $E \cap \mathrm{Core}(M)$ is not compact. Then there is a sequence (c_i) of closed geodesics in M which exit the end.*

Proof. Let E be an end of M whose intersection with Core(M) is unbounded and let U_0 be an arbitrary neighborhood of E. Assume without loss of generality that the boundary $\partial U_0 \subset M$ of U_0 is compact and intersects Core(M). Our goal is to show that there is a neighborhood N of U_0 whose radius is bounded independent of U_0, and there is a closed geodesic β which is entirely contained in N.

We observed in Lemma 4.2.1 that there is at least one distance minimizing geodesic ray $x = \gamma : [0, \infty) \to \mathrm{Core}(M) \cap E$ issuing from a point $\gamma(0) \in \partial U_0 \cap \mathrm{Core}(M)$. A lift $\tilde{\gamma}$ of such a geodesic ray to the universal covering $\mathbf{H}^3$ of M is contained in the convex hull Hull(Λ) of the limit set Λ of Γ.

By Lemma 4.1.7 fixed points of loxodromic elements of Γ are dense in Λ. Furthermore, a geodesic ray in $\mathbf{H}^3$ depends continuously on its initial point and its endpoint in S^2 in the following sense. If $\zeta_i : [0, \infty) \to \mathbb{H}^3$ is a sequence of such geodesics with $\zeta_i(0) = y$ for a fixed point y whose endpoints converge to the endpoint of a geodesic ray ζ with $\zeta(0) = y$ then the rays ζ_i converge uniformly on compact sets to the ray ζ.

Now the geodesic ray $\tilde{\gamma}$ can be approximated by rays in Hull(Λ) which issue from $\tilde{\gamma}(0)$ and whose endpoints are fixed points of some loxodromic elements of Γ. The projection η to M of such a geodesic *spirals* about the projection ν of the axis of this loxodromic element which is a closed geodesic in M. Here spiraling means that the distance between $\eta(t)$ and the closed geodesic ν tends to zero as $t \to \infty$.

If one of the closed geodesics α_i is entirely contained in U_0 then we are done. Otherwise assume without loss of generality that each of the geodesics α_i intersects ∂U_0. Since the geodesic rays γ_i approximate the minimizing ray γ, their initial segments enter arbitrarily deeply into the end E as $i \to \infty$. As γ_i spirals about α_i and α_i is not contained in U_0, initial subsegments of the geodesics γ_i define an infinite family β_i of geodesic arcs in U_0 with endpoints $\gamma(0), x_i$ on the boundary ∂U_0 of U_0 which meet every neighborhood of E.

Since ∂U_0 is compact, we may assume that the sequence $(x_i) \subset \partial U_0$ converges. By passing to a subsequence, we may moreover assume that for each i, there is a point on β_i whose distance to β_j for $j < i$ is at least 100. Then the simple closed curve $\gamma_{i,j}$ which we obtain as the concatenation of β_j, a minimal geodesic connecting x_j to x_i and of β_i^{-1} is not homotopic to zero. Namely, otherwise lifts of β_i, β_j to the universal covering $\mathbb{H}^3$ with the same initial point are geodesic arcs with endpoints of small distance whose Hausdorff distance is at least 100. By hyperbolicity, this is impossible.

Since the fundamental group $\pi_1(M)$ of M does not contain any parabolic element, there is a simple closed geodesic $\gamma_{i,j}^*$ which is freely homotopic to $\gamma_{i,j}$. We claim that this simple closed geodesic is contained in a tubular neighborhood of $\gamma_{i,j}$ of uniformly bounded radius.

To see that this is indeed the case, let $\psi : S^1 \times [0,1] \to M$ be a homotopy connecting $S^1 \times \{0\} = \gamma_{i,j}$ to $S^1 \times \{1\} = \gamma_{i,j}^*$ and let $a_1, a_2, a_3 \in S^1 \times \{0\}$ be the three breakpoints of $\gamma_{i,j}$ (namely, the points $\gamma(0), x_i, x_j$). Choose three points $b_1 \neq b_2 \neq b_3$ on $S^1 \times \{1\}$. There is then a triangulation of the annulus $S^1 \times [0,1]$ into six triangles whose vertices are the points a_i, b_j.

The image under ψ of each side of such a triangle can be straightened in M to a homotopic geodesic arc with the same endpoints. Note that the sides contained in the curves $\gamma_{i,j}, \gamma_{i,j}^*$ are unchanged. Since the union of the three sides of a triangle in $S^1 \times [0,1]$ is mapped by ψ to a contractible curve in M, a lift of their straightenings to $\mathbf{H}^3$ is the boundary of a totally geodesic triangle in $\mathbf{H}^3$. This triangle then projects back to a totally geodesic immersed triangle in M. In other words, the straightenings of the three sides of each such triangle bound a totally geodesic immersed triangle in M. We may assume that none of these triangles is degenerate. Then the hyperbolic metric on M induces via pull-back a metric on $S^1 \times [0,1]$ which is hyperbolic (i.e. of constant curvature -1), with piecewise geodesic boundary. Namely, even though two totally geodesic immersed triangles in M which share a common side may meet along this side with a non-trivial angle, the intrinsic Riemannian metric on these triangles extends smoothly across the side.

It is now enough to show that through every point $x \in \gamma_{i,j}^* = S^1 \times \{1\}$ passes a geodesic arc which connects x to a point in $\gamma_{i,j}$ and whose length is bounded from above by a universal constant. For this note first that since $\gamma_{i,j}^*$ is smooth, the sum of the angles at any singular point $y \in \gamma_{i,j}^*$ of the two totally geodesic immersed hyperbolic triangles which come together at y is not smaller than π. In particular, $\gamma_{i,j}^*$ is a convex subset of the piecewise hyperbolic annulus $A = S^1 \times [0,1|$: Every geodesic arc in A with both endpoints on $\gamma_{i,j}^*$ is entirely contained in $\gamma_{i,j}^*$.

For any $x \in \gamma_{i,j}^* - \{b_1, b_2, b_3\}$ let α_x be the geodesic arc in A issuing from x which is orthogonal to $\gamma_{i,j}^*$. If $x \in \{b_1, b_2, b_3\}$ then we let α_x be any geodesic segment with the property that the angle between α_x and the two subarcs of $\gamma_{i,j}^*$ issuing from x is at least $\pi/2$. Let $\epsilon > 0$ be a lower bound for the injectivity radius of M and for $z \in \alpha_x$ let β_z be a geodesic arc in A of length $\epsilon/4$ which is orthogonal to α_x and which points in the direction determined by the orientation of A and the orientation of α_x; we call this side *positive*.

By the choice of ϵ, the images in M of any such arc β_z is contained in a contractible subset of M. Thus if they may end on $\gamma_{i,j}$ then the distance between z and $\gamma_{i,j}$ is at most $\epsilon/4$. Let $\ell > 0$ be the maximal length of a connected subarc of α_x beginning at x which consists of starting points z for an embedded geodesic arc β_z of length $\epsilon/4$ as above whose interior is disjoint from $\gamma_{i,j}$. Fermi coordinates now yield that the area of the union of these arcs is not smaller than

$$\ell\epsilon/4.$$

On the other hand, the area of this union is not larger than the area of A. Since A consists of 6 hyperbolic geodesic triangles, its area is not bigger than 6π. This shows that $\ell \leq 24\pi/\epsilon = p$.

Then for every $z \in \gamma^*_{i,j}$, there exists a point $z' \in \gamma_{i,j}$ whose intrinsic distance in A to z is at most $p + \epsilon/4$. But this just means that the geodesic $\gamma^*_{i,j}$ is contained in the $\ell + \epsilon/4$-neighborhood of $\gamma_{i,j}$. ☐

Embarking from Theorem 4.2.4, the main idea for an understanding of the geometry of an end E in M which intersects Core(M) in a non-compact set is to obtain a detailed understading of those closed geodesics which exit E and their location in E. We begin this investigation with an estimate of intersection numbers Bonahon [**?**]. For this and later use we need the following elementary observation.

Lemma 4.2.5. *Let M be any hyperbolic 3-manifold and let c be a closed curve in M which is freely homotopic to a closed geodesic c^*. If the distance between c^* and c is a least $D > 0$, then*

$$\ell(c) \geq \cosh(D)\ell(c^*).$$

Proof. Consider the covering $\hat{M}$ of M whose fundamental group is generated by c^* and is isomorphic to $\mathbb{Z}$. The manifold $\hat{M}$ is diffeomorphic to a ball bundle over S^1. By elementary hyperbolic geometry, a closed curve on M homotopic to c^* which does not intersect the r-tubular neighborhood of c^* has length at least $\cosh(r)\ell(c^*)$ where $\ell(c^*)$ is the length of c^*. Since the distance in $\hat{M}$ between c^* and any lift of c is not smaller than the distance between c^* and c in M, the lemma follows. ☐