T-Rep: A midsummer night's session on representation theory and tensor categories (July 3rd 2020)

Incompressible symmetric tensor categories

Pavel Etingof

MIT

etingof@math.mit.edu

July 4, 2020

arxiv: 2003.10499
joint with Dave Benson and Victor Ostrik
slides in collaboration with Victor Ostrik

Symmetric tensor categories

Symmetric tensor categories

Representation categories

Base field: $k = \bar{k}$.

Symmetric tensor categories

Representation categories

Base field: $k = \bar{k}$.

Given a (finite) group G, we can form

$\mathrm{Rep}(G) = \{$finite dimensional representations of $G\}$.

Symmetric tensor categories

Representation categories

Base field: $k = \bar{k}$.

Given a (finite) group G, we can form

$\mathrm{Rep}(G) = \{$finite dimensional representations of $G\}$.

What kind of mathematical object is $\mathrm{Rep}(G)$?

Symmetric tensor categories

Representation categories

Base field: $k = \bar{k}$.

Given a (finite) group G, we can form

$\mathrm{Rep}(G) = \{$finite dimensional representations of $G\}$.

What kind of mathematical object is $\mathrm{Rep}(G)$?

- $\mathrm{Rep}(G)$ is an abelian category;

Symmetric tensor categories

Representation categories

Base field: $k = \bar{k}$.

Given a (finite) group G, we can form

$\mathrm{Rep}(G) = \{$finite dimensional representations of $G\}$.

What kind of mathematical object is Rep(G)?

- $\mathrm{Rep}(G)$ is an abelian category;
- $\mathrm{Rep}(G)$ has a bifunctor $\otimes$;

Symmetric tensor categories

Representation categories

Base field: $k = \bar{k}$.

Given a (finite) group G, we can form

$\mathrm{Rep}(G) = \{$finite dimensional representations of $G\}$.

What kind of mathematical object is $\mathrm{Rep}(G)$?

- $\mathrm{Rep}(G)$ is an abelian category;
- $\mathrm{Rep}(G)$ has a bifunctor $\otimes$;
- $\otimes$ is associative, commutative

Symmetric tensor categories

Representation categories

Base field: $k = \bar{k}$.

Given a (finite) group G, we can form

$\mathrm{Rep}(G) = \{$finite dimensional representations of $G\}$.

What kind of mathematical object is $\mathrm{Rep}(G)$?

- $\mathrm{Rep}(G)$ is an abelian category;
- $\mathrm{Rep}(G)$ has a bifunctor $\otimes$;
- $\otimes$ is associative, commutative $\leftarrow$ structures;

Symmetric tensor categories

Representation categories

Base field: $k = \bar{k}$.

Given a (finite) group G, we can form

$\mathrm{Rep}(G) = \{$finite dimensional representations of $G\}$.

What kind of mathematical object is $\mathrm{Rep}(G)$?

- $\mathrm{Rep}(G)$ is an abelian category;
- $\mathrm{Rep}(G)$ has a bifunctor $\otimes$;
- $\otimes$ is associative, commutative $\leftarrow$ structures;
- we have the unit object $\mathbf{1}$ and duality $X \mapsto X^*$

Symmetric tensor categories

Representation categories

Base field: $k = \bar{k}$.

Given a (finite) group G, we can form

Rep(G) = {finite dimensional representations of G}.

What kind of mathematical object is Rep(G)?

- Rep(G) is an abelian category;
- Rep(G) has a bifunctor $\otimes$;
- $\otimes$ is associative, commutative $\leftarrow$ structures;
- we have the unit object $\mathbf{1}$ and duality $X \mapsto X^*$ $\leftarrow$ properties of $\otimes$ and associativity;

Symmetric tensor categories

Representation categories

Base field: $k = \bar{k}$.
Given a (finite) group G, we can form
$\mathrm{Rep}(G) = \{\text{finite dimensional representations of } G\}$.

What kind of mathematical object is $\mathrm{Rep}(G)$?

- $\mathrm{Rep}(G)$ is an abelian category;
- $\mathrm{Rep}(G)$ has a bifunctor $\otimes$;
- $\otimes$ is associative, commutative $\leftarrow$ structures;
- we have the unit object $\mathbf{1}$ and duality $X \mapsto X^*$ $\leftarrow$ properties of $\otimes$ and associativity;
- we have forgetful tensor functor $\mathrm{Rep}(G) \to \mathrm{Vec}$.

Symmetric tensor categories

Representation categories

Base field: $k = \bar{k}$.
Given a (finite) group G, we can form
$\mathrm{Rep}(G) = \{$finite dimensional representations of $G\}$.

What kind of mathematical object is Rep(G)?

- $\mathrm{Rep}(G)$ is an abelian category;
- $\mathrm{Rep}(G)$ has a bifunctor $\otimes$;
- $\otimes$ is associative, commutative $\leftarrow$ structures;
- we have the unit object $\mathbf{1}$ and duality $X \mapsto X^*$ $\leftarrow$ properties of $\otimes$ and associativity;
- we have forgetful tensor functor $\mathrm{Rep}(G) \to \mathrm{Vec}$.

Thus $\mathrm{Rep}(G)$ is a rigid symmetric tensor category equipped with a (symmetric) tensor functor to Vec.

Tannakian theory

Pre-Tannakian categories

A pre-Tannakian category is a $k-$linear rigid symmetric $\otimes$ category $\mathcal{C}$ satisfying

Tannakian theory

Pre-Tannakian categories

A pre-Tannakian category is a $k-$linear rigid symmetric $\otimes$ category $\mathcal{C}$ satisfying

- $\mathcal{C}$ is abelian;

Tannakian theory

Pre-Tannakian categories

A pre-Tannakian category is a $k-$linear rigid symmetric $\otimes$ category $\mathcal{C}$ satisfying

- $\mathcal{C}$ is abelian;
- $\dim \mathrm{Hom}(X, Y) < \infty$;

Tannakian theory

Pre-Tannakian categories

A pre-Tannakian category is a $k-$linear rigid symmetric $\otimes$ category $\mathcal{C}$ satisfying

- $\mathcal{C}$ is abelian;
- $\dim \mathrm{Hom}(X, Y) < \infty$;
- $\mathrm{length}(X) < \infty$;

Tannakian theory

Pre-Tannakian categories

A pre-Tannakian category is a $k-$linear rigid symmetric $\otimes$ category $\mathcal{C}$ satisfying

- $\mathcal{C}$ is abelian;
- $\dim \mathrm{Hom}(X, Y) < \infty$;
- $\mathrm{length}(X) < \infty$;
- $\mathbf{1}$ is simple.

Tannakian theory

Pre-Tannakian categories

A pre-Tannakian category is a $k-$linear rigid symmetric $\otimes$ category $\mathcal{C}$ satisfying

- $\mathcal{C}$ is abelian;
- $\dim \mathrm{Hom}(X, Y) < \infty$;
- $\mathrm{length}(X) < \infty$;
- $\mathbf{1}$ is simple.

Definition

Fiber functor: exact symmetric $\otimes$ functor $F : \mathcal{C} \to \mathrm{Vec}$.

Tannakian theory

Pre-Tannakian categories

A pre-Tannakian category is a $k-$linear rigid symmetric $\otimes$ category $\mathcal{C}$ satisfying

- $\mathcal{C}$ is abelian;
- $\dim \mathrm{Hom}(X, Y) < \infty$;
- $\mathrm{length}(X) < \infty$;
- $\mathbf{1}$ is simple.

Definition

Fiber functor: exact symmetric $\otimes$ functor $F : \mathcal{C} \to \mathrm{Vec}$.
$\mathcal{C}$ is Tannakian if it is pre-Tannakian and has a fiber functor.

Tannakian theory

Pre-Tannakian categories

A pre-Tannakian category is a $k-$linear rigid symmetric $\otimes$ category $\mathcal{C}$ satisfying

- $\mathcal{C}$ is abelian;
- $\dim \mathrm{Hom}(X, Y) < \infty$;
- $\mathrm{length}(X) < \infty$;
- $\mathbf{1}$ is simple.

Definition

Fiber functor: exact symmetric $\otimes$ functor $F : \mathcal{C} \to \mathrm{Vec}$.
$\mathcal{C}$ is Tannakian if it is pre-Tannakian and has a fiber functor.

Theorem (Grothendieck, Saavedra Rivano, Deligne-Milne)

Assume $\mathcal{C}$ is Tannakian. Then $\mathcal{C} = Rep(G)$ for some (unique) affine group scheme G. Namely, $G = \underline{\mathrm{Aut}}_{\otimes}(F)$.

Super-Tannakian categories

Example

The category of supervector spaces sVec is pre-Tannakian but not Tannakian ($\mathrm{char}(k) \neq 2$).

Super-Tannakian categories

Example

The category of supervector spaces sVec is pre-Tannakian but not Tannakian ($\mathrm{char}(k) \neq 2$).
Reminder: $\mathsf{sVec} = \mathsf{Rep}(\mathbb{Z}/2)$ but with modified commutativity constraint:
$u \otimes v \mapsto (-1)^{|u|\cdot|v|} v \otimes u$.

Super-Tannakian categories

Example

The category of supervector spaces sVec is pre-Tannakian but not Tannakian $(\mathrm{char}(k) \neq 2)$.
Reminder: sVec $=$ Rep$(\mathbb{Z}/2)$ but with modified commutativity constraint: $u \otimes v \mapsto (-1)^{|u|\cdot|v|} v \otimes u$.

Definition

super fiber functor: an exact symmetric $\otimes$ functor $F : \mathcal{C} \to$ sVec.

Super-Tannakian categories

Example

The category of supervector spaces sVec is pre-Tannakian but not Tannakian $(\mathrm{char}(k) \neq 2)$.
Reminder: sVec $=$ Rep$(\mathbb{Z}/2)$ but with modified commutativity constraint: $u \otimes v \mapsto (-1)^{|u|\cdot|v|} v \otimes u$.

Definition

super fiber functor: an exact symmetric $\otimes$ functor $F : \mathcal{C} \to$ sVec.
$\mathcal{C}$ is super-Tannakian if it is pre-Tannakian and has a super fiber functor.

Super-Tannakian categories

Example

The category of supervector spaces sVec is pre-Tannakian but not Tannakian ($\mathrm{char}(k) \neq 2$).
Reminder: $\mathrm{sVec} = \mathrm{Rep}(\mathbb{Z}/2)$ but with modified commutativity constraint:
$u \otimes v \mapsto (-1)^{|u|\cdot|v|} v \otimes u$.

Definition

super fiber functor: an exact symmetric $\otimes$ functor $F : \mathcal{C} \to \mathrm{sVec}$.
$\mathcal{C}$ is super-Tannakian if it is pre-Tannakian and has a super fiber functor.

Example

If G is an affine supergroup scheme (= supercommutative Hopf superalgebra)

Super-Tannakian categories

Example

The category of supervector spaces sVec is pre-Tannakian but not Tannakian ($\mathrm{char}(k) \neq 2$).
Reminder: sVec $=$ Rep$(\mathbb{Z}/2)$ but with modified commutativity constraint: $u \otimes v \mapsto (-1)^{|u|\cdot|v|} v \otimes u$.

Definition

super fiber functor: an exact symmetric $\otimes$ functor $F : \mathcal{C} \to$ sVec.
$\mathcal{C}$ is super-Tannakian if it is pre-Tannakian and has a super fiber functor.

Example

If G is an affine supergroup scheme ($=$ supercommutative Hopf superalgebra) then Rep(G) is super-Tannakian.

Super-Tannakian categories

Example

The category of supervector spaces sVec is pre-Tannakian but not Tannakian ($\mathrm{char}(k) \neq 2$).
Reminder: sVec $=$ Rep$(\mathbb{Z}/2)$ but with modified commutativity constraint:
$u \otimes v \mapsto (-1)^{|u|\cdot|v|} v \otimes u$.

Definition

super fiber functor: an exact symmetric $\otimes$ functor $F : \mathcal{C} \to$ sVec.
$\mathcal{C}$ is super-Tannakian if it is pre-Tannakian and has a super fiber functor.

Example

If G is an affine supergroup scheme ($=$ supercommutative Hopf superalgebra) then Rep(G) is super-Tannakian.
More generally: $z \in G(k)$, $z^2 = 1$, $Ad(z) =$parity automorphism of G.

Super-Tannakian categories

Example

The category of supervector spaces sVec is pre-Tannakian but not Tannakian ($\mathrm{char}(k) \neq 2$).
Reminder: sVec $=$ Rep($\mathbb{Z}/2$) but with modified commutativity constraint:
$u \otimes v \mapsto (-1)^{|u|\cdot|v|} v \otimes u$.

Definition

super fiber functor: an exact symmetric $\otimes$ functor $F : \mathcal{C} \to$ sVec.
$\mathcal{C}$ is super-Tannakian if it is pre-Tannakian and has a super fiber functor.

Example

If G is an affine supergroup scheme ($=$ supercommutative Hopf superalgebra) then Rep(G) is super-Tannakian.
More generally: $z \in G(k)$, $z^2 = 1$, $Ad(z)$ =parity automorphism of G.
Rep(G, z) $= \{$objects X of Rep(G) such that the action of z is the parity automorphism of $X\}$

Super-Tannakian categories

Example

The category of supervector spaces sVec is pre-Tannakian but not Tannakian ($\mathrm{char}(k) \neq 2$).
Reminder: sVec $=$ Rep($\mathbb{Z}/2$) but with modified commutativity constraint: $u \otimes v \mapsto (-1)^{|u|\cdot|v|} v \otimes u$.

Definition

super fiber functor: an exact symmetric $\otimes$ functor $F : \mathcal{C} \to$ sVec.
$\mathcal{C}$ is super-Tannakian if it is pre-Tannakian and has a super fiber functor.

Example

If G is an affine supergroup scheme ($=$ supercommutative Hopf superalgebra) then Rep(G) is super-Tannakian.
More generally: $z \in G(k)$, $z^2 = 1$, $Ad(z)$ =parity automorphism of G.
Rep(G, z) $= \{$objects X of Rep(G) such that the action of z is the parity automorphism of $X\}$ then Rep(G, z) is super-Tannakian.

Super-Tannakian theory and generalization

Theorem (Deligne)

Assume $\mathcal{C}$ is super-Tannakian. Then $\mathcal{C} = Rep(G, z)$ for some affine supergroup scheme G and z as above. Namely, $G = \underline{\mathrm{Aut}}_{\otimes}(F)$ and $z \in G(k)$ is the parity element.

Super-Tannakian theory and generalization

Theorem (Deligne)

Assume $\mathcal{C}$ is super-Tannakian. Then $\mathcal{C} = Rep(G, z)$ for some affine supergroup scheme G and z as above. Namely, $G = \underline{\mathrm{Aut}}_{\otimes}(F)$ and $z \in G(k)$ is the parity element.

Generalization

Let $F : \mathcal{C} \to \mathcal{D}$ be an exact symmetric $\otimes$ functor.

Super-Tannakian theory and generalization

Theorem (Deligne)

Assume $\mathcal{C}$ is super-Tannakian. Then $\mathcal{C} = Rep(G, z)$ for some affine supergroup scheme G and z as above. Namely, $G = \underline{\mathrm{Aut}}_{\otimes}(F)$ and $z \in G(k)$ is the parity element.

Generalization

Let $F : \mathcal{C} \to \mathcal{D}$ be an exact symmetric $\otimes$ functor.
Then $\mathcal{C} = \mathrm{Rep}(G, \pi)$ where G is an affine group scheme in $\mathcal{D}$ and π is the fundamental group of $\mathcal{D}$.

Super-Tannakian theory and generalization

Theorem (Deligne)

Assume $\mathcal{C}$ is super-Tannakian. Then $\mathcal{C} = Rep(G, z)$ for some affine supergroup scheme G and z as above. Namely, $G = \underline{\mathrm{Aut}}_{\otimes}(F)$ and $z \in G(k)$ is the parity element.

Generalization

Let $F : \mathcal{C} \to \mathcal{D}$ be an exact symmetric $\otimes$ functor.
Then $\mathcal{C} = \mathrm{Rep}(G, \pi)$ where G is an affine group scheme in $\mathcal{D}$ and π is the fundamental group of $\mathcal{D}$.
Namely, $G = \underline{\mathrm{Aut}}_{\otimes}(F)$ and $\pi = \underline{\mathrm{Aut}}_{\otimes}(\mathrm{Id}_{\mathcal{D}})$. Then π acts canonically on every object of $\mathcal{D}$, we have a canonical homomorphism $\pi \to G$, and $\mathrm{Rep}(G, \pi)$ is the category of representations of G on objects of $\mathcal{D}$ whose restriction to π is the canonical action.

Super-Tannakian theory and generalization

Theorem (Deligne)

Assume $\mathcal{C}$ is super-Tannakian. Then $\mathcal{C} = Rep(G, z)$ for some affine supergroup scheme G and z as above. Namely, $G = \underline{\mathrm{Aut}}_{\otimes}(F)$ and $z \in G(k)$ is the parity element.

Generalization

Let $F : \mathcal{C} \to \mathcal{D}$ be an exact symmetric $\otimes$ functor.
Then $\mathcal{C} = \mathrm{Rep}(G, \pi)$ where G is an affine group scheme in $\mathcal{D}$ and π is the fundamental group of $\mathcal{D}$.
Namely, $G = \underline{\mathrm{Aut}}_{\otimes}(F)$ and $\pi = \underline{\mathrm{Aut}}_{\otimes}(\mathrm{Id}_{\mathcal{D}})$. Then π acts canonically on every object of $\mathcal{D}$, we have a canonical homomorphism $\pi \to G$, and $\mathrm{Rep}(G, \pi)$ is the category of representations of G on objects of $\mathcal{D}$ whose restriction to π is the canonical action.
In other words, $\mathcal{C}$ can be expressed in terms of "group theory in $\mathcal{D}$".

Super-Tannakian theory and generalization

Theorem (Deligne)

Assume $\mathcal{C}$ is super-Tannakian. Then $\mathcal{C} = Rep(G, z)$ for some affine supergroup scheme G and z as above. Namely, $G = \underline{\mathrm{Aut}}_{\otimes}(F)$ and $z \in G(k)$ is the parity element.

Generalization

Let $F : \mathcal{C} \to \mathcal{D}$ be an exact symmetric $\otimes$ functor.
Then $\mathcal{C} = \mathrm{Rep}(G, \pi)$ where G is an affine group scheme in $\mathcal{D}$ and π is the fundamental group of $\mathcal{D}$.
Namely, $G = \underline{\mathrm{Aut}}_{\otimes}(F)$ and $\pi = \underline{\mathrm{Aut}}_{\otimes}(\mathrm{Id}_{\mathcal{D}})$. Then π acts canonically on every object of $\mathcal{D}$, we have a canonical homomorphism $\pi \to G$, and $\mathrm{Rep}(G, \pi)$ is the category of representations of G on objects of $\mathcal{D}$ whose restriction to π is the canonical action.
In other words, $\mathcal{C}$ can be expressed in terms of "group theory in $\mathcal{D}$".

Question: What are the pre-Tannakian categories which can't be expressed in terms of "group theory" in smaller categories?

Question: What are the pre-Tannakian categories which can't be expressed in terms of "group theory" in smaller categories? Equivalently, which pre-Tannakian categories do not admit exact $\otimes$ functors to smaller categories?

Incompressible categories

Surjective functors

An exact $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is surjective if any object of $\mathcal{D}$ is a subquotient of $F(X)$.

Surjective functors

An exact $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is surjective if any object of $\mathcal{D}$ is a subquotient of $F(X)$.

Any such functor $F : \mathcal{C} \to \mathcal{D}$ factorizes as $\mathcal{C} \to \mathrm{Im}(F) \to \mathcal{D}$ where $\mathcal{C} \to \mathrm{Im}(F)$ is surjective and $\mathrm{Im}(F) \to \mathcal{D}$ is a fully faithful embedding.

Incompressible categories

Surjective functors

An exact $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is surjective if any object of $\mathcal{D}$ is a subquotient of $F(X)$.

Any such functor $F : \mathcal{C} \to \mathcal{D}$ factorizes as $\mathcal{C} \to \mathrm{Im}(F) \to \mathcal{D}$ where $\mathcal{C} \to \mathrm{Im}(F)$ is surjective and $\mathrm{Im}(F) \to \mathcal{D}$ is a fully faithful embedding.

Definition

A pre-Tannakian $\mathcal{C}$ is incompressible if any surjective $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence for any pre-Tannakian $\mathcal{D}$.

Incompressible categories

Surjective functors

An exact $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is surjective if any object of $\mathcal{D}$ is a subquotient of $F(X)$.
Any such functor $F : \mathcal{C} \to \mathcal{D}$ factorizes as $\mathcal{C} \to \mathrm{Im}(F) \to \mathcal{D}$ where $\mathcal{C} \to \mathrm{Im}(F)$ is surjective and $\mathrm{Im}(F) \to \mathcal{D}$ is a fully faithful embedding.

Definition

A pre-Tannakian $\mathcal{C}$ is incompressible if any surjective $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence for any pre-Tannakian $\mathcal{D}$.

Equivalently, $\mathcal{C}$ is incompressible if any exact $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is an embedding.

Incompressible categories

Surjective functors

An exact $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is surjective if any object of $\mathcal{D}$ is a subquotient of $F(X)$.
Any such functor $F : \mathcal{C} \to \mathcal{D}$ factorizes as $\mathcal{C} \to \mathrm{Im}(F) \to \mathcal{D}$ where $\mathcal{C} \to \mathrm{Im}(F)$ is surjective and $\mathrm{Im}(F) \to \mathcal{D}$ is a fully faithful embedding.

Definition

A pre-Tannakian $\mathcal{C}$ is incompressible if any surjective $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence for any pre-Tannakian $\mathcal{D}$.

Equivalently, $\mathcal{C}$ is incompressible if any exact $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is an embedding.

Example

Vec sVec

Incompressible categories

Surjective functors

An exact $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is surjective if any object of $\mathcal{D}$ is a subquotient of $F(X)$.
Any such functor $F : \mathcal{C} \to \mathcal{D}$ factorizes as $\mathcal{C} \to \mathrm{Im}(F) \to \mathcal{D}$ where $\mathcal{C} \to \mathrm{Im}(F)$ is surjective and $\mathrm{Im}(F) \to \mathcal{D}$ is a fully faithful embedding.

Definition

A pre-Tannakian $\mathcal{C}$ is incompressible if any surjective $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence for any pre-Tannakian $\mathcal{D}$.

Equivalently, $\mathcal{C}$ is incompressible if any exact $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is an embedding.

Example

Vec sVec

Any more examples?

Characteristic zero

Characteristic zero

Definition

We say that $\mathcal{C}$ is of sub-exponential growth if for any $X \in \mathcal{C}$ there is $a_X \in \mathbb{R}$ such that $\text{length}(X^{\otimes n}) \leq a_X^n$.

Characteristic zero

Definition

We say that $\mathcal{C}$ is of sub-exponential growth if for any $X \in \mathcal{C}$ there is $a_X \in \mathbb{R}$ such that $\mathrm{length}(X^{\otimes n}) \leq a_X^n$.

Theorem (Deligne)

Assume char $k = 0$ and let $\mathcal{C}$ be pre-Tannakian of sub-exponential growth. Then $\mathcal{C}$ is super-Tannakian. In particular, Vec and sVec are the only incompressible categories of sub-exponential growth.

Characteristic zero

Definition

We say that $\mathcal{C}$ is of sub-exponential growth if for any $X \in \mathcal{C}$ there is $a_X \in \mathbb{R}$ such that $\text{length}(X^{\otimes n}) \leq a_X^n$.

Theorem (Deligne)

Assume char $k = 0$ and let $\mathcal{C}$ be pre-Tannakian of sub-exponential growth. Then $\mathcal{C}$ is super-Tannakian. In particular, Vec and sVec are the only incompressible categories of sub-exponential growth.

Example

Deligne categories $\text{Rep}(GL_t), \text{Rep}(O_t), \text{Rep}(S_t)$ $(t \in k)$ are of super-exponential growth.

Characteristic zero

Definition

We say that $\mathcal{C}$ is of sub-exponential growth if for any $X \in \mathcal{C}$ there is $a_X \in \mathbb{R}$ such that $\text{length}(X^{\otimes n}) \leq a_X^n$.

Theorem (Deligne)

Assume char $k = 0$ and let $\mathcal{C}$ be pre-Tannakian of sub-exponential growth. Then $\mathcal{C}$ is super-Tannakian. In particular, Vec and sVec are the only incompressible categories of sub-exponential growth.

Example

Deligne categories $\text{Rep}(GL_t), \text{Rep}(O_t), \text{Rep}(S_t)$ $(t \in k)$ are of super-exponential growth.

They typically admit surjective functors like $\text{Rep}(S_t) \to \text{Rep}(S_{t-1})$, so are **not** incompressible.

Characteristic zero

Definition

We say that $\mathcal{C}$ is of sub-exponential growth if for any $X \in \mathcal{C}$ there is $a_X \in \mathbb{R}$ such that $\text{length}(X^{\otimes n}) \leq a_X^n$.

Theorem (Deligne)

Assume char $k = 0$ and let $\mathcal{C}$ be pre-Tannakian of sub-exponential growth. Then $\mathcal{C}$ is super-Tannakian. In particular, Vec and sVec are the only incompressible categories of sub-exponential growth.

Example

Deligne categories $\text{Rep}(GL_t), \text{Rep}(O_t), \text{Rep}(S_t)$ $(t \in k)$ are of super-exponential growth.

They typically admit surjective functors like $\text{Rep}(S_t) \to \text{Rep}(S_{t-1})$, so are **not** incompressible.

Conjecture: No more incompressible categories in characteristic zero.

Semisimplification

Semisimplification

Let $\mathcal{T}$ be a rigid symmetric monoidal category (perhaps non-abelian)

Semisimplification

Let $\mathcal{T}$ be a rigid symmetric monoidal category (perhaps non-abelian)

Negligible morphisms

We say that $f : X \to Y$ is negligible if for any $g : Y \to X$ we have $\mathrm{Tr}(fg) = 0$.

Semisimplification

Let $\mathcal{T}$ be a rigid symmetric monoidal category (perhaps non-abelian)

Negligible morphisms

We say that $f : X \to Y$ is negligible if for any $g : Y \to X$ we have $\mathrm{Tr}(fg) = 0$.

Negligible morphisms form a $\otimes$ ideal $\mathcal{N}$.

Semisimplification

Let $\mathcal{T}$ be a rigid symmetric monoidal category (perhaps non-abelian)

Negligible morphisms

We say that $f : X \to Y$ is negligible if for any $g : Y \to X$ we have $\mathrm{Tr}(fg) = 0$.

Negligible morphisms form a $\otimes$ ideal $\mathcal{N}$.

Define $\overline{\mathcal{T}}$: the same objects as in $\mathcal{T}$ but $\mathrm{Hom}_{\overline{\mathcal{T}}}(X, Y) = \mathrm{Hom}_{\mathcal{T}}(X, Y)/\mathcal{N}$.

Semisimplification

Let $\mathcal{T}$ be a rigid symmetric monoidal category (perhaps non-abelian)

Negligible morphisms

We say that $f : X \to Y$ is negligible if for any $g : Y \to X$ we have $\mathrm{Tr}(fg) = 0$.

Negligible morphisms form a $\otimes$ ideal $\mathcal{N}$.

Define $\overline{\mathcal{T}}$: the same objects as in $\mathcal{T}$ but $\mathrm{Hom}_{\overline{\mathcal{T}}}(X, Y) = \mathrm{Hom}_{\mathcal{T}}(X, Y)/\mathcal{N}$.

$\overline{\mathcal{T}}$ is again a rigid symmetric monoidal category.

Semisimplification

Let $\mathcal{T}$ be a rigid symmetric monoidal category (perhaps non-abelian)

Negligible morphisms

We say that $f : X \to Y$ is negligible if for any $g : Y \to X$ we have $\mathrm{Tr}(fg) = 0$.

Negligible morphisms form a $\otimes$ ideal $\mathcal{N}$.

Define $\overline{\mathcal{T}}$: the same objects as in $\mathcal{T}$ but $\mathrm{Hom}_{\overline{\mathcal{T}}}(X, Y) = \mathrm{Hom}_{\mathcal{T}}(X, Y)/\mathcal{N}$.

$\overline{\mathcal{T}}$ is again a rigid symmetric monoidal category.

Theorem (U. Jannsen)

Assume $\dim Hom(X, Y) < \infty$ *and any nilpotent endomorphism in* $\mathcal{T}$ *has trace zero. Then* $\overline{\mathcal{T}}$ *is semisimple (and so abelian). Moreover*

Irreducibles of $\overline{\mathcal{T}} \leftrightarrow$ *Indecomposables of* $\mathcal{T}$ *of nonzero dimension.*

Semisimplification

Let $\mathcal{T}$ be a rigid symmetric monoidal category (perhaps non-abelian)

Negligible morphisms

We say that $f : X \to Y$ is negligible if for any $g : Y \to X$ we have $\mathrm{Tr}(fg) = 0$.

Negligible morphisms form a $\otimes$ ideal $\mathcal{N}$.

Define $\overline{\mathcal{T}}$: the same objects as in $\mathcal{T}$ but $\mathrm{Hom}_{\overline{\mathcal{T}}}(X, Y) = \mathrm{Hom}_{\mathcal{T}}(X, Y)/\mathcal{N}$.

$\overline{\mathcal{T}}$ is again a rigid symmetric monoidal category.

Theorem (U. Jannsen)

Assume $\dim Hom(X, Y) < \infty$ *and any nilpotent endomorphism in* $\mathcal{T}$ *has trace zero. Then* $\overline{\mathcal{T}}$ *is semisimple (and so abelian). Moreover*

Irreducibles of $\overline{\mathcal{T}} \leftrightarrow$ *Indecomposables of* $\mathcal{T}$ *of nonzero dimension.*

Remark: assume $F : \mathcal{T} \to \mathcal{C}$ is a $\otimes$ functor to abelian $\mathcal{C}$. Then any nilpotent endomorphism in $\mathcal{T}$ has trace zero.

Semisimple Verlinde categories

Assume char $k = p > 0$.

Semisimple Verlinde categories

Assume char $k = p > 0$.

S. Gelfand-Kazhdan and Georgiev-Mathieu

Let G be a simple group, e.g. $G = SL_n$.

Semisimple Verlinde categories

Assume char $k = p > 0$.

S. Gelfand-Kazhdan and Georgiev-Mathieu

Let G be a simple group, e.g. $G = SL_n$. Let $\mathcal{T} = \{\text{tilting } G\text{−modules}\}$.

Semisimple Verlinde categories

Assume char $k = p > 0$.

S. Gelfand-Kazhdan and Georgiev-Mathieu

Let G be a simple group, e.g. $G = SL_n$. Let $\mathcal{T} = \{\text{tilting } G\text{−modules}\}$. Then $\mathrm{Ver}(G) := \overline{\mathcal{T}}$ is a semisimple pre-Tannakian category; it has finitely many irreducibles provided $p \geq$ Coxeter number $h(G)$ of G (e.g. $h(SL_n) = n$).

Semisimple Verlinde categories

Assume char $k = p > 0$.

S. Gelfand-Kazhdan and Georgiev-Mathieu

Let G be a simple group, e.g. $G = SL_n$. Let $\mathcal{T} = \{\text{tilting } G-\text{modules}\}$. Then $\mathrm{Ver}(G) := \overline{\mathcal{T}}$ is a semisimple pre-Tannakian category; it has finitely many irreducibles provided $p \geq$ Coxeter number $h(G)$ of G (e.g. $h(SL_n) = n$).

Example

$\mathrm{Ver}_p := \mathrm{Ver}(SL_2)$.

Semisimple Verlinde categories

Assume char $k = p > 0$.

S. Gelfand-Kazhdan and Georgiev-Mathieu

Let G be a simple group, e.g. $G = SL_n$. Let $\mathcal{T} = \{\text{tilting } G\text{−modules}\}$. Then $\mathrm{Ver}(G) := \overline{\mathcal{T}}$ is a semisimple pre-Tannakian category; it has finitely many irreducibles provided $p \geq$ Coxeter number $h(G)$ of G (e.g. $h(SL_n) = n$).

Example

$\mathrm{Ver}_p := \mathrm{Ver}(SL_2)$.
Simple objects $L_1 = \mathbf{1}, L_2, \ldots, L_{p-1}$.

Semisimple Verlinde categories

Assume char $k = p > 0$.

S. Gelfand-Kazhdan and Georgiev-Mathieu

Let G be a simple group, e.g. $G = SL_n$. Let $\mathcal{T} = \{\text{tilting } G-\text{modules}\}$. Then $\mathrm{Ver}(G) := \overline{\mathcal{T}}$ is a semisimple pre-Tannakian category; it has finitely many irreducibles provided $p \geq$ Coxeter number $h(G)$ of G (e.g. $h(SL_n) = n$).

Example

$\mathrm{Ver}_p := \mathrm{Ver}(SL_2)$.
Simple objects $L_1 = \mathbf{1}, L_2, \ldots, L_{p-1}$.
$L_2 \otimes L_i = L_{i-1} \oplus L_{i+1}$ with convention $L_0 = L_p = 0$.

Semisimple Verlinde categories

Assume char $k = p > 0$.

S. Gelfand-Kazhdan and Georgiev-Mathieu

Let G be a simple group, e.g. $G = SL_n$. Let $\mathcal{T} = \{\text{tilting } G\text{−modules}\}$. Then $\mathrm{Ver}(G) := \overline{\mathcal{T}}$ is a semisimple pre-Tannakian category; it has finitely many irreducibles provided $p \geq$ Coxeter number $h(G)$ of G (e.g. $h(SL_n) = n$).

Example

$\mathrm{Ver}_p := \mathrm{Ver}(SL_2)$.
Simple objects $L_1 = \mathbf{1}, L_2, \ldots, L_{p-1}$.
$L_2 \otimes L_i = L_{i-1} \oplus L_{i+1}$ with convention $L_0 = L_p = 0$.
This implies: $L_{p-1} \otimes L_{p-1} = \mathbf{1}$ and $\langle \mathbf{1}, L_{p-1} \rangle = \mathrm{sVec}$ for $p > 2$.

Semisimple Verlinde categories

Assume char $k = p > 0$.

S. Gelfand-Kazhdan and Georgiev-Mathieu

Let G be a simple group, e.g. $G = SL_n$. Let $\mathcal{T} = \{\text{tilting } G\text{−modules}\}$. Then $\mathrm{Ver}(G) := \overline{\mathcal{T}}$ is a semisimple pre-Tannakian category; it has finitely many irreducibles provided $p \geq$ Coxeter number $h(G)$ of G (e.g. $h(SL_n) = n$).

Example

$\mathrm{Ver}_p := \mathrm{Ver}(SL_2)$.
Simple objects $L_1 = \mathbf{1}, L_2, \ldots, L_{p-1}$.
$L_2 \otimes L_i = L_{i-1} \oplus L_{i+1}$ with convention $L_0 = L_p = 0$.
This implies: $L_{p-1} \otimes L_{p-1} = \mathbf{1}$ and $\langle \mathbf{1}, L_{p-1} \rangle = \mathrm{sVec}$ for $p > 2$.
For $p = 5$: $L_3 \otimes L_3 = \mathbf{1} \oplus L_3$ spans the Fibonacci category Fib.

Semisimple Verlinde categories

Assume char $k = p > 0$.

S. Gelfand-Kazhdan and Georgiev-Mathieu

Let G be a simple group, e.g. $G = SL_n$. Let $\mathcal{T} = \{\text{tilting } G\text{−modules}\}$. Then $\mathrm{Ver}(G) := \overline{\mathcal{T}}$ is a semisimple pre-Tannakian category; it has finitely many irreducibles provided $p \geq$ Coxeter number $h(G)$ of G (e.g. $h(SL_n) = n$).

Example

$\mathrm{Ver}_p := \mathrm{Ver}(SL_2)$.
Simple objects $L_1 = \mathbf{1}, L_2, \ldots, L_{p-1}$.
$L_2 \otimes L_i = L_{i-1} \oplus L_{i+1}$ with convention $L_0 = L_p = 0$.
This implies: $L_{p-1} \otimes L_{p-1} = \mathbf{1}$ and $\langle \mathbf{1}, L_{p-1} \rangle = \mathrm{sVec}$ for $p > 2$.
For $p = 5$: $L_3 \otimes L_3 = \mathbf{1} \oplus L_3$ spans the Fibonacci category Fib.
$\mathrm{Ver}_2 = \mathrm{Vec}$; $\mathrm{Ver}_3 = \mathrm{sVec}$; $\mathrm{Ver}_5 = \mathrm{Fib} \boxtimes \mathrm{sVec}$.

Semisimple Verlinde categories

Assume char $k = p > 0$.

S. Gelfand-Kazhdan and Georgiev-Mathieu

Let G be a simple group, e.g. $G = SL_n$. Let $\mathcal{T} = \{\text{tilting } G\text{−modules}\}$. Then $\mathrm{Ver}(G) := \overline{\mathcal{T}}$ is a semisimple pre-Tannakian category; it has finitely many irreducibles provided $p \geq$ Coxeter number $h(G)$ of G (e.g. $h(SL_n) = n$).

Example

$\mathrm{Ver}_p := \mathrm{Ver}(SL_2)$.
Simple objects $L_1 = \mathbf{1}, L_2, \ldots, L_{p-1}$.
$L_2 \otimes L_i = L_{i-1} \oplus L_{i+1}$ with convention $L_0 = L_p = 0$.
This implies: $L_{p-1} \otimes L_{p-1} = \mathbf{1}$ and $\langle \mathbf{1}, L_{p-1} \rangle = \mathrm{sVec}$ for $p > 2$.
For $p = 5$: $L_3 \otimes L_3 = \mathbf{1} \oplus L_3$ spans the Fibonacci category Fib.
$\mathrm{Ver}_2 = \mathrm{Vec}$; $\mathrm{Ver}_3 = \mathrm{sVec}$; $\mathrm{Ver}_5 = \mathrm{Fib} \boxtimes \mathrm{sVec}$. More generally, for $p > 2$ $\mathrm{Ver}_p = \mathrm{Ver}_p^+ \boxtimes \mathrm{sVec}$, where $\mathrm{Ver}_p^+ \subset \mathrm{Ver}_p$ is spanned by L_i with odd i.

Semisimple Verlinde categories

Alternatively Ver_p = semisimplification of $\mathrm{Rep}(\mathbb{Z}/p)$.

Semisimple Verlinde categories

Alternatively $\mathrm{Ver}_p =$ semisimplification of $\mathrm{Rep}(\mathbb{Z}/p)$.

Fact: the category Ver_p is incompressible, and so are its tensor subcategories $\mathrm{Vec}, \mathrm{sVec}, \mathrm{Ver}_p^+$.

Semisimple Verlinde categories

Alternatively $\mathrm{Ver}_p =$ semisimplification of $\mathrm{Rep}(\mathbb{Z}/p)$.

Fact: the category Ver_p is incompressible, and so are its tensor subcategories $\mathrm{Vec}, \mathrm{sVec}, \mathrm{Ver}_p^+$.

Theorem (Ostrik, 2015)

For any pre-Tannakian $\mathcal{C}$ which is semisimple with finitely many irreducibles (i.e., a fusion category) there exists an exact $\otimes$ functor $\mathcal{C} \to \mathrm{Ver}_p$. In particular, theonly incompressible pre-Tannakian fusion categories are Ver_p and its tensor subcategories.

Semisimple Verlinde categories

Alternatively Ver_p = semisimplification of $\mathrm{Rep}(\mathbb{Z}/p)$.

Fact: the category Ver_p is incompressible, and so are its tensor subcategories $\mathrm{Vec}, \mathrm{sVec}, \mathrm{Ver}_p^+$.

Theorem (Ostrik, 2015)

For any pre-Tannakian $\mathcal{C}$ which is semisimple with finitely many irreducibles (i.e., a fusion category) there exists an exact $\otimes$ functor $\mathcal{C} \to Ver_p$. In particular, theonly incompressible pre-Tannakian fusion categories are Ver_p and its tensor subcategories.

Conjecture: These are the only semisimple incompressible pre-Tannakian categories (at least of sub-exponential growth).

Semisimple Verlinde categories

Alternatively Ver_p = semisimplification of $\mathrm{Rep}(\mathbb{Z}/p)$.
Fact: the category Ver_p is incompressible, and so are its tensor subcategories $\mathrm{Vec}, \mathrm{sVec}, \mathrm{Ver}_p^+$.

Theorem (Ostrik, 2015)

For any pre-Tannakian $\mathcal{C}$ which is semisimple with finitely many irreducibles (i.e., a fusion category) there exists an exact $\otimes$ functor $\mathcal{C} \to \mathrm{Ver}_p$. In particular, theonly incompressible pre-Tannakian fusion categories are Ver_p and its tensor subcategories.

Conjecture: These are the only semisimple incompressible pre-Tannakian categories (at least of sub-exponential growth).
What about non-semisimple examples?

Semisimple Verlinde categories

Alternatively $\mathrm{Ver}_p =$ semisimplification of $\mathrm{Rep}(\mathbb{Z}/p)$.
Fact: the category Ver_p is incompressible, and so are its tensor subcategories $\mathrm{Vec}, \mathrm{sVec}, \mathrm{Ver}_p^+$.

Theorem (Ostrik, 2015)

For any pre-Tannakian $\mathcal{C}$ which is semisimple with finitely many irreducibles (i.e., a fusion category) there exists an exact $\otimes$ functor $\mathcal{C} \to \mathrm{Ver}_p$. In particular, theonly incompressible pre-Tannakian fusion categories are Ver_p and its tensor subcategories.

Conjecture: These are the only semisimple incompressible pre-Tannakian categories (at least of sub-exponential growth).
What about non-semisimple examples?

Example

For $p = 2$ S. Venkatesh constructed an example $\mathcal{V}$ by modifying the commutativity constraint in Rep("$\mathbb{Z}/2$")

Semisimple Verlinde categories

Alternatively $\mathrm{Ver}_p =$ semisimplification of $\mathrm{Rep}(\mathbb{Z}/p)$.
Fact: the category Ver_p is incompressible, and so are its tensor subcategories $\mathrm{Vec}, \mathrm{sVec}, \mathrm{Ver}_p^+$.

Theorem (Ostrik, 2015)

For any pre-Tannakian $\mathcal{C}$ which is semisimple with finitely many irreducibles (i.e., a fusion category) there exists an exact $\otimes$ functor $\mathcal{C} \to \mathrm{Ver}_p$. In particular, theonly incompressible pre-Tannakian fusion categories are Ver_p and its tensor subcategories.

Conjecture: These are the only semisimple incompressible pre-Tannakian categories (at least of sub-exponential growth).
What about non-semisimple examples?

Example

For $p = 2$ S. Venkatesh constructed an example $\mathcal{V}$ by modifying the commutativity constraint in $\mathrm{Rep}(\text{“}\mathbb{Z}/2\text{”})$ $\text{“}\mathbb{Z}/2\text{”} = (\mathbb{G}_a)_1$

p=2

Benson-E., 2018

For $p = 2$ there is a strictly increasing sequence of incompressible pre-Tannakian categories
$\mathcal{C}_0 = \mathrm{Vec} \subset \mathcal{C}_1 = \mathcal{V} \subset \mathcal{C}_2 = \mathcal{T}(SL_2)/\mathcal{I}_2 \subset \mathcal{C}_3 \subset \ldots$

p=2

Benson-E., 2018

For $p = 2$ there is a strictly increasing sequence of incompressible pre-Tannakian categories
$\mathcal{C}_0 = \mathrm{Vec} \subset \mathcal{C}_1 = \mathcal{V} \subset \mathcal{C}_2 = \mathcal{T}(SL_2)/\mathcal{I}_2 \subset \mathcal{C}_3 \subset \dots$
Technology: Hopf algebras (in categories) and graded extensions.

p=2

Benson-E., 2018

For $p = 2$ there is a strictly increasing sequence of incompressible pre-Tannakian categories
$\mathcal{C}_0 = \mathrm{Vec} \subset \mathcal{C}_1 = \mathcal{V} \subset \mathcal{C}_2 = \mathcal{T}(SL_2)/\mathcal{I}_2 \subset \mathcal{C}_3 \subset \dots$
Technology: Hopf algebras (in categories) and graded extensions.
The category $\mathcal{C}_2$ was previously constructed by Ostrik.

General $p > 0$

General $p > 0$

More on tilting modules for SL_2

Let V be a tautological 2-dimensional representation of SL_2

General $p > 0$

More on tilting modules for SL_2

Let V be a tautological 2-dimensional representation of SL_2

$\mathcal{T}(SL_2) :=$ additive $\otimes$ category generated by V.

General $p > 0$

More on tilting modules for SL_2

Let V be a tautological 2-dimensional representation of SL_2

$\mathcal{T}(SL_2) :=$ additive $\otimes$ category generated by V.

$\{$indecomposables of $\mathcal{T}(SL_2)\} = \{$indecomposable summands of $V^{\otimes n}\}$

General $p > 0$

More on tilting modules for SL_2

Let V be a tautological 2-dimensional representation of SL_2
$\mathcal{T}(SL_2) :=$ additive $\otimes$ category generated by V.
$\{$indecomposables of $\mathcal{T}(SL_2)\} = \{$indecomposable summands of $V^{\otimes n}\}$
Fact: indecomposables are classified by highest weight. Thus we have $T_0 = \mathbf{1}$, $T_1 = V$, T_2, $T_3, \ldots$.

General $p > 0$

More on tilting modules for SL_2

Let V be a tautological 2-dimensional representation of SL_2
$\mathcal{T}(SL_2) :=$ additive $\otimes$ category generated by V.
$\{$indecomposables of $\mathcal{T}(SL_2)\} = \{$indecomposable summands of $V^{\otimes n}\}$
Fact: indecomposables are classified by highest weight. Thus we have
$T_0 = \mathbf{1}, T_1 = V, T_2, T_3, \ldots.$
Steinberg modules: $T_0, T_{p-1}, T_{p^2-1}, \ldots, \mathrm{St}_r = T_{p^r-1}, \ldots$

General $p > 0$

More on tilting modules for SL_2

Let V be a tautological 2-dimensional representation of SL_2

$\mathcal{T}(SL_2) :=$ additive $\otimes$ category generated by V.

$\{$indecomposables of $\mathcal{T}(SL_2)\} = \{$indecomposable summands of $V^{\otimes n}\}$

Fact: indecomposables are classified by highest weight. Thus we have $T_0 = \mathbf{1}$, $T_1 = V$, T_2, T_3, ...

Steinberg modules: $T_0, T_{p-1}, T_{p^2-1}, \ldots, \mathrm{St}_r = T_{p^r-1}, \ldots$

Each St_r generates a (thick) tensor ideal $\mathcal{P}_r = \langle T_{p^r-1}, T_{p^r}, T_{p^r+1}, \ldots \rangle$

General $p > 0$

More on tilting modules for SL_2

Let V be a tautological 2-dimensional representation of SL_2
$\mathcal{T}(SL_2) :=$ additive $\otimes$ category generated by V.
$\{$indecomposables of $\mathcal{T}(SL_2)\} = \{$indecomposable summands of $V^{\otimes n}\}$
Fact: indecomposables are classified by highest weight. Thus we have $T_0 = \mathbf{1}$, $T_1 = V$, T_2, $T_3, \dots$.
Steinberg modules: T_0, T_{p-1}, $T_{p^2-1}, \dots, \mathrm{St}_r = T_{p^r-1}, \dots$
Each St_r generates a (thick) tensor ideal $\mathcal{P}_r = \langle T_{p^r-1}, T_{p^r}, T_{p^r+1}, \dots \rangle$
Each thick tensor ideal above gives a tensor ideal $\mathcal{I}_r$: $\mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \dots$

More on tilting modules for SL_2

Let V be a tautological 2-dimensional representation of SL_2

$\mathcal{T}(SL_2) :=$ additive $\otimes$ category generated by V.

$\{$indecomposables of $\mathcal{T}(SL_2)\} = \{$indecomposable summands of $V^{\otimes n}\}$

Fact: indecomposables are classified by highest weight. Thus we have $T_0 = \mathbf{1}$, $T_1 = V$, T_2, $T_3, \ldots$.

Steinberg modules: T_0, T_{p-1}, $T_{p^2-1}, \ldots, \mathrm{St}_r = T_{p^r-1}, \ldots$

Each St_r generates a (thick) tensor ideal $\mathcal{P}_r = \langle T_{p^r-1}, T_{p^r}, T_{p^r+1}, \ldots \rangle$

Each thick tensor ideal above gives a tensor ideal $\mathcal{I}_r$: $\mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \ldots$

Fact: Any nontrivial tensor ideal in $\mathcal{T}(SL_2)$ is one of $\mathcal{I}_r$.

General $p > 0$

More on tilting modules for SL_2

Let V be a tautological 2-dimensional representation of SL_2

$\mathcal{T}(SL_2) :=$ additive $\otimes$ category generated by V.

$\{$indecomposables of $\mathcal{T}(SL_2)\} = \{$indecomposable summands of $V^{\otimes n}\}$

Fact: indecomposables are classified by highest weight. Thus we have $T_0 = \mathbf{1}$, $T_1 = V$, T_2, $T_3, \ldots$.

Steinberg modules: $T_0, T_{p-1}, T_{p^2-1}, \ldots, \mathrm{St}_r = T_{p^r-1}, \ldots$

Each St_r generates a (thick) tensor ideal $\mathcal{P}_r = \langle T_{p^r-1}, T_{p^r}, T_{p^r+1}, \ldots \rangle$

Each thick tensor ideal above gives a tensor ideal $\mathcal{I}_r$: $\mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \ldots$

Fact: Any nontrivial tensor ideal in $\mathcal{T}(SL_2)$ is one of $\mathcal{I}_r$.

Quotients

Define $\mathcal{T}_{p,r} := \mathcal{T}(SL_2)/\mathcal{I}_r$, e.g. $\mathcal{T}_{p,1} = \mathrm{Ver}_p$.

General $p > 0$

More on tilting modules for SL_2

Let V be a tautological 2-dimensional representation of SL_2
$\mathcal{T}(SL_2) :=$ additive $\otimes$ category generated by V.
$\{$indecomposables of $\mathcal{T}(SL_2)\} = \{$indecomposable summands of $V^{\otimes n}\}$
Fact: indecomposables are classified by highest weight. Thus we have $T_0 = \mathbf{1}$, $T_1 = V$, T_2, T_3, ...
Steinberg modules: $T_0, T_{p-1}, T_{p^2-1}, \ldots, \mathrm{St}_r = T_{p^r-1}, \ldots$
Each St_r generates a (thick) tensor ideal $\mathcal{P}_r = \langle T_{p^r-1}, T_{p^r}, T_{p^r+1}, \ldots \rangle$
Each thick tensor ideal above gives a tensor ideal $\mathcal{I}_r$: $\mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \ldots$
Fact: Any nontrivial tensor ideal in $\mathcal{T}(SL_2)$ is one of $\mathcal{I}_r$.

Quotients

Define $\mathcal{T}_{p,r} := \mathcal{T}(SL_2)/\mathcal{I}_r$, e.g. $\mathcal{T}_{p,1} = \mathrm{Ver}_p$.
$\mathcal{T}_{p,r}$ is non-semisimple and non-abelian for $r > 1$ (except $r = 2$ and $p = 2$).

General $p > 0$

More on tilting modules for SL_2

Let V be a tautological 2-dimensional representation of SL_2
$\mathcal{T}(SL_2) :=$ additive $\otimes$ category generated by V.
$\{$indecomposables of $\mathcal{T}(SL_2)\} = \{$indecomposable summands of $V^{\otimes n}\}$
Fact: indecomposables are classified by highest weight. Thus we have
$T_0 = \mathbf{1}, T_1 = V, T_2, T_3, \ldots$.
Steinberg modules: $T_0, T_{p-1}, T_{p^2-1}, \ldots, \mathrm{St}_r = T_{p^r-1}, \ldots$
Each St_r generates a (thick) tensor ideal $\mathcal{P}_r = \langle T_{p^r-1}, T_{p^r}, T_{p^r+1}, \ldots \rangle$
Each thick tensor ideal above gives a tensor ideal $\mathcal{I}_r$: $\mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \ldots$
Fact: Any nontrivial tensor ideal in $\mathcal{T}(SL_2)$ is one of $\mathcal{I}_r$.

Quotients

Define $\mathcal{T}_{p,r} := \mathcal{T}(SL_2)/\mathcal{I}_r$, e.g. $\mathcal{T}_{p,1} = \mathrm{Ver}_p$.
$\mathcal{T}_{p,r}$ is non-semisimple and non-abelian for $r > 1$ (except $r = 2$ and $p = 2$).
$\mathcal{T}_{p,r}$ contains a tensor ideal $\mathcal{P}_{r-1} = \mathcal{I}_{r-1}/\mathcal{I}_r$.

Main Theorem

Observation: (Benson-E., 2018): the category $\mathcal{C}_{2r}$ contains $\mathcal{T}_{2,r}$;

Main Theorem

Observation: (Benson-E., 2018): the category $\mathcal{C}_{2r}$ contains $\mathcal{T}_{2,r}$;
$\{$ the ideal $\mathcal{P}_{r-1}\} = \{$ subcategory of projective objects in $\mathcal{C}_{2r}\}$.

Main Theorem

Observation: (Benson-E., 2018): the category $\mathcal{C}_{2r}$ contains $\mathcal{T}_{2,r}$;
$\{$ the ideal $\mathcal{P}_{r-1}\} = \{$ subcategory of projective objects in $\mathcal{C}_{2r}\}$.

Theorem (Benson-E.-Ostrik, 2020)

There exists a unique pre-Tannakian category Ver_{p^n} containing $\mathcal{T}_{p,n}$ and such that $\mathcal{P}_{n-1}$ coincides with the ideal of projective objects. The category Ver_{p^n} is incompressible.

The first statement was also independently proved by K. Coulembier.

Main Theorem

Observation: (Benson-E., 2018): the category $\mathcal{C}_{2r}$ contains $\mathcal{T}_{2,r}$;
$\{$ the ideal $\mathcal{P}_{r-1}\} = \{$ subcategory of projective objects in $\mathcal{C}_{2r}\}$.

Theorem (Benson-E.-Ostrik, 2020)

There exists a unique pre-Tannakian category Ver_{p^n} containing $\mathcal{T}_{p,n}$ and such that $\mathcal{P}_{n-1}$ coincides with the ideal of projective objects. The category Ver_{p^n} is incompressible.

The first statement was also independently proved by K. Coulembier.

Split morphisms

A morphism $f : X \to Y$ (in an additive category) is split if it is a projection to a direct summand followed by an inclusion of a direct summand.

Main Theorem

Observation: (Benson-E., 2018): the category $\mathcal{C}_{2r}$ contains $\mathcal{T}_{2,r}$;
$\{$ the ideal $\mathcal{P}_{r-1}\} = \{$ subcategory of projective objects in $\mathcal{C}_{2r}\}$.

Theorem (Benson-E.-Ostrik, 2020)

There exists a unique pre-Tannakian category Ver_{p^n} containing $\mathcal{T}_{p,n}$ and such that $\mathcal{P}_{n-1}$ coincides with the ideal of projective objects. The category Ver_{p^n} is incompressible.

The first statement was also independently proved by K. Coulembier.

Split morphisms

A morphism $f : X \to Y$ (in an additive category) is split if it is a projection to a direct summand followed by an inclusion of a direct summand.

Example. If X and Y are indecomposable, f is split $\Leftrightarrow$ f is an isomorphism or $f = 0$.

Main Theorem

Observation: (Benson-E., 2018): the category $\mathcal{C}_{2r}$ contains $\mathcal{T}_{2,r}$;
$\{$ the ideal $\mathcal{P}_{r-1}\} = \{$ subcategory of projective objects in $\mathcal{C}_{2r}\}$.

Theorem (Benson-E.-Ostrik, 2020)

There exists a unique pre-Tannakian category Ver_{p^n} containing $\mathcal{T}_{p,n}$ and such that $\mathcal{P}_{n-1}$ coincides with the ideal of projective objects. The category Ver_{p^n} is incompressible.

The first statement was also independently proved by K. Coulembier.

Split morphisms

A morphism $f : X \to Y$ (in an additive category) is split if it is a projection to a direct summand followed by an inclusion of a direct summand.

Example. If X and Y are indecomposable, f is split $\Leftrightarrow$ f is an isomorphism or $f = 0$.
Exercise. Let P be a projective object and $f : X \to Y$ be any morphism. Then $\mathrm{id}_P \otimes f : P \otimes X \to P \otimes Y$ is split.

Splitting ideals

Splitting ideals

Let $\mathcal{T}$ be a rigid (symmetric) monoidal category (perhaps non-abelian), $\dim \mathrm{Hom}(X, Y) < \infty$.

Splitting ideals

Let $\mathcal{T}$ be a rigid (symmetric) monoidal category (perhaps non-abelian), $\dim \mathrm{Hom}(X, Y) < \infty$. Let $\mathcal{P} \subset \mathcal{T}$ be a thick tensor ideal. Assume that $\mathcal{P}$ is faithful, i.e., any morphism f s.t. $\mathrm{id}_P \otimes f = 0$ for any $P \in \mathcal{P}$ is zero.

Splitting ideals

Let $\mathcal{T}$ be a rigid (symmetric) monoidal category (perhaps non-abelian), $\dim \mathrm{Hom}(X, Y) < \infty$. Let $\mathcal{P} \subset \mathcal{T}$ be a thick tensor ideal. Assume that $\mathcal{P}$ is faithful, i.e., any morphism f s.t. $\mathrm{id}_P \otimes f = 0$ for any $P \in \mathcal{P}$ is zero. We say that $\mathcal{P}$ is a splitting ideal if for any morphism $f : X \to Y$ in $\mathcal{T}$ and $P \in \mathcal{P}$ the morphism $\mathrm{id}_P \otimes f$ is split.

Splitting ideals

Let $\mathcal{T}$ be a rigid (symmetric) monoidal category (perhaps non-abelian), $\dim \mathrm{Hom}(X,Y) < \infty$. Let $\mathcal{P} \subset \mathcal{T}$ be a thick tensor ideal. Assume that $\mathcal{P}$ is faithful, i.e., any morphism f s.t. $\mathrm{id}_P \otimes f = 0$ for any $P \in \mathcal{P}$ is zero. We say that $\mathcal{P}$ is a splitting ideal if for any morphism $f : X \to Y$ in $\mathcal{T}$ and $P \in \mathcal{P}$ the morphism $\mathrm{id}_P \otimes f$ is split.

General construction

Given a splitting ideal $\mathcal{P} \subset \mathcal{T}$ as above, we construct an abelian rigid tensor category $\mathcal{C} \supset \mathcal{T}$ such that $\mathcal{P}$ is the subcategory of projective objects.

Splitting ideals

Let $\mathcal{T}$ be a rigid (symmetric) monoidal category (perhaps non-abelian), $\dim \mathrm{Hom}(X, Y) < \infty$. Let $\mathcal{P} \subset \mathcal{T}$ be a thick tensor ideal. Assume that $\mathcal{P}$ is faithful, i.e., any morphism f s.t. $\mathrm{id}_P \otimes f = 0$ for any $P \in \mathcal{P}$ is zero. We say that $\mathcal{P}$ is a splitting ideal if for any morphism $f : X \to Y$ in $\mathcal{T}$ and $P \in \mathcal{P}$ the morphism $\mathrm{id}_P \otimes f$ is split.

General construction

Given a splitting ideal $\mathcal{P} \subset \mathcal{T}$ as above, we construct an abelian rigid tensor category $\mathcal{C} \supset \mathcal{T}$ such that $\mathcal{P}$ is the subcategory of projective objects. **Hint** on the construction of $\mathcal{C}$: consider complexes of objects of $\mathcal{P}$ finite in the positive direction.

Splitting ideals

Let $\mathcal{T}$ be a rigid (symmetric) monoidal category (perhaps non-abelian), $\dim \mathrm{Hom}(X, Y) < \infty$. Let $\mathcal{P} \subset \mathcal{T}$ be a thick tensor ideal. Assume that $\mathcal{P}$ is faithful, i.e., any morphism f s.t. $\mathrm{id}_P \otimes f = 0$ for any $P \in \mathcal{P}$ is zero. We say that $\mathcal{P}$ is a splitting ideal if for any morphism $f : X \to Y$ in $\mathcal{T}$ and $P \in \mathcal{P}$ the morphism $\mathrm{id}_P \otimes f$ is split.

General construction

Given a splitting ideal $\mathcal{P} \subset \mathcal{T}$ as above, we construct an abelian rigid tensor category $\mathcal{C} \supset \mathcal{T}$ such that $\mathcal{P}$ is the subcategory of projective objects.
Hint on the construction of $\mathcal{C}$: consider complexes of objects of $\mathcal{P}$ finite in the positive direction.
Some challenges: What is the unit object of $\mathcal{C}$?

Splitting ideals

Let $\mathcal{T}$ be a rigid (symmetric) monoidal category (perhaps non-abelian), $\dim \mathrm{Hom}(X, Y) < \infty$. Let $\mathcal{P} \subset \mathcal{T}$ be a thick tensor ideal. Assume that $\mathcal{P}$ is faithful, i.e., any morphism f s.t. $\mathrm{id}_P \otimes f = 0$ for any $P \in \mathcal{P}$ is zero. We say that $\mathcal{P}$ is a splitting ideal if for any morphism $f : X \to Y$ in $\mathcal{T}$ and $P \in \mathcal{P}$ the morphism $\mathrm{id}_P \otimes f$ is split.

General construction

Given a splitting ideal $\mathcal{P} \subset \mathcal{T}$ as above, we construct an abelian rigid tensor category $\mathcal{C} \supset \mathcal{T}$ such that $\mathcal{P}$ is the subcategory of projective objects.

Hint on the construction of $\mathcal{C}$: consider complexes of objects of $\mathcal{P}$ finite in the positive direction.

Some challenges: What is the unit object of $\mathcal{C}$? Why is $\mathcal{C}$ is rigid?

Splitting ideals

Let $\mathcal{T}$ be a rigid (symmetric) monoidal category (perhaps non-abelian), $\dim \mathrm{Hom}(X, Y) < \infty$. Let $\mathcal{P} \subset \mathcal{T}$ be a thick tensor ideal. Assume that $\mathcal{P}$ is faithful, i.e., any morphism f s.t. $\mathrm{id}_P \otimes f = 0$ for any $P \in \mathcal{P}$ is zero.

We say that $\mathcal{P}$ is a splitting ideal if for any morphism $f : X \to Y$ in $\mathcal{T}$ and $P \in \mathcal{P}$ the morphism $\mathrm{id}_P \otimes f$ is split.

General construction

Given a splitting ideal $\mathcal{P} \subset \mathcal{T}$ as above, we construct an abelian rigid tensor category $\mathcal{C} \supset \mathcal{T}$ such that $\mathcal{P}$ is the subcategory of projective objects.
Hint on the construction of $\mathcal{C}$: consider complexes of objects of $\mathcal{P}$ finite in the positive direction.
Some challenges: What is the unit object of $\mathcal{C}$? Why is $\mathcal{C}$ is rigid?

Key Lemma

The ideal $\mathcal{P}_{n-1} \subset \mathcal{T}_{p,n}$ is splitting.

Splitting ideals

Let $\mathcal{T}$ be a rigid (symmetric) monoidal category (perhaps non-abelian), $\dim \mathrm{Hom}(X, Y) < \infty$. Let $\mathcal{P} \subset \mathcal{T}$ be a thick tensor ideal. Assume that $\mathcal{P}$ is faithful, i.e., any morphism f s.t. $\mathrm{id}_P \otimes f = 0$ for any $P \in \mathcal{P}$ is zero. We say that $\mathcal{P}$ is a splitting ideal if for any morphism $f : X \to Y$ in $\mathcal{T}$ and $P \in \mathcal{P}$ the morphism $\mathrm{id}_P \otimes f$ is split.

General construction

Given a splitting ideal $\mathcal{P} \subset \mathcal{T}$ as above, we construct an abelian rigid tensor category $\mathcal{C} \supset \mathcal{T}$ such that $\mathcal{P}$ is the subcategory of projective objects. **Hint** on the construction of $\mathcal{C}$: consider complexes of objects of $\mathcal{P}$ finite in the positive direction.
Some challenges: What is the unit object of $\mathcal{C}$? Why is $\mathcal{C}$ is rigid?

Key Lemma

The ideal $\mathcal{P}_{n-1} \subset \mathcal{T}_{p,n}$ is splitting.

Wanted: more examples of splitting tensor ideals!

Properties of Ver_{p^n}

Projectives and Cartan matrix

Projective objects: T_i with $p^{n-1} - 1 \le i < p^n - 1$; $\# = p^{n-1}(p-1)$.

Properties of Ver_{p^n}

Projectives and Cartan matrix

Projective objects: T_i with $p^{n-1} - 1 \le i < p^n - 1$; $\# = p^{n-1}(p-1)$.
We set $P_i = T_{i-1}$ where $i = [i_1 i_2 \dots i_n]_p$ has precisely n digits ($i_1 \neq 0$!)

Properties of Ver_{p^n}

Projectives and Cartan matrix

Projective objects: T_i with $p^{n-1} - 1 \le i < p^n - 1$; $\# = p^{n-1}(p-1)$.
We set $P_i = T_{i-1}$ where $i = [i_1 i_2 \dots i_n]_p$ has precisely n digits ($i_1 \neq 0$!)
The Cartan matrix $C_{ij} := \dim \mathrm{Hom}(P_i, P_j)$.

Properties of Ver_{p^n}

Projectives and Cartan matrix

Projective objects: T_i with $p^{n-1}-1 \le i < p^n-1$; $\# = p^{n-1}(p-1)$.
We set $P_i = T_{i-1}$ where $i = [i_1 i_2 \dots i_n]_p$ has precisely n digits ($i_1 \neq 0$!)
The Cartan matrix $C_{ij} := \dim \mathrm{Hom}(P_i, P_j)$.
Negative digits game: make some digits negative.
$[23045]_p \rightsquigarrow 2(-3)0(-4)5 = 2p^4 - 3p^3 - 4p + 5$.
Descendants of i: all positive numbers you get in this way.
E.g. $[23045]_p$ has $2^3 = 8$ descendants.

Properties of Ver_{p^n}

Projectives and Cartan matrix

Projective objects: T_i with $p^{n-1}-1 \le i < p^n - 1$; $\# = p^{n-1}(p-1)$.
We set $P_i = T_{i-1}$ where $i = [i_1 i_2 \dots i_n]_p$ has precisely n digits ($i_1 \neq 0!$)
The Cartan matrix $C_{ij} := \dim \mathrm{Hom}(P_i, P_j)$.
Negative digits game: make some digits negative.
$[23045]_p \rightsquigarrow 2(-3)0(-4)5 = 2p^4 - 3p^3 - 4p + 5$.
Descendants of i: all positive numbers you get in this way.
E.g. $[23045]_p$ has $2^3 = 8$ descendants.
Tubbenhauer-Wedrich: $C_{ij} = |\{\text{descendants of } i\} \cap \{\text{descendants of } j\}|$.

Properties of Ver_{p^n}

Projectives and Cartan matrix

Projective objects: T_i with $p^{n-1} - 1 \leq i < p^n - 1$; $\# = p^{n-1}(p-1)$.
We set $P_i = T_{i-1}$ where $i = [i_1 i_2 \dots i_n]_p$ has precisely n digits ($i_1 \neq 0$!)
The Cartan matrix $C_{ij} := \dim \mathrm{Hom}(P_i, P_j)$.
Negative digits game: make some digits negative.
$[23045]_p \rightsquigarrow 2(-3)0(-4)5 = 2p^4 - 3p^3 - 4p + 5$.
Descendants of i: all positive numbers you get in this way.
E.g. $[23045]_p$ has $2^3 = 8$ descendants.
Tubbenhauer-Wedrich: $C_{ij} = |\{\text{descendants of } i\} \cap \{\text{descendants of } j\}|$.
Exercise: $C_{ij} = 0$ or power of 2.

Projectives and Cartan matrix

Projective objects: T_i with $p^{n-1} - 1 \le i < p^n - 1$; $\# = p^{n-1}(p-1)$.
We set $P_i = T_{i-1}$ where $i = [i_1 i_2 \dots i_n]_p$ has precisely n digits ($i_1 \neq 0$!)
The Cartan matrix $C_{ij} := \dim \mathrm{Hom}(P_i, P_j)$.
Negative digits game: make some digits negative.
$[23045]_p \rightsquigarrow 2(-3)0(-4)5 = 2p^4 - 3p^3 - 4p + 5$.
Descendants of i: all positive numbers you get in this way.
E.g. $[23045]_p$ has $2^3 = 8$ descendants.
Tubbenhauer-Wedrich: $C_{ij} = |\{\text{descendants of } i\} \cap \{\text{descendants of } j\}|$.
Exercise: $C_{ij} = 0$ or power of 2.
Exercise: $\det C_{ij} =$ power of p.

Properties of Ver_{p^n}

Projectives and Cartan matrix

Projective objects: T_i with $p^{n-1} - 1 \le i < p^n - 1$; $\# = p^{n-1}(p-1)$.
We set $P_i = T_{i-1}$ where $i = [i_1 i_2 \dots i_n]_p$ has precisely n digits ($i_1 \neq 0$!)
The Cartan matrix $C_{ij} := \dim \mathrm{Hom}(P_i, P_j)$.
Negative digits game: make some digits negative.
$[23045]_p \rightsquigarrow 2(-3)0(-4)5 = 2p^4 - 3p^3 - 4p + 5$.
Descendants of i: all positive numbers you get in this way.
E.g. $[23045]_p$ has $2^3 = 8$ descendants.
Tubbenhauer-Wedrich: $C_{ij} = |\{\text{descendants of } i\} \cap \{\text{descendants of } j\}|$.
Exercise: $C_{ij} = 0$ or power of 2.
Exercise: $\det C_{ij} =$ power of p.

Embeddings

We have $\mathrm{Ver}_p \subset \mathrm{Ver}_{p^2} \subset \mathrm{Ver}_{p^3} \subset \dots$

Properties of Ver_{p^n}

Projectives and Cartan matrix

Projective objects: T_i with $p^{n-1} - 1 \leq i < p^n - 1$; $\# = p^{n-1}(p-1)$.
We set $P_i = T_{i-1}$ where $i = [i_1 i_2 \dots i_n]_p$ has precisely n digits ($i_1 \neq 0$!)
The Cartan matrix $C_{ij} := \dim \mathrm{Hom}(P_i, P_j)$.
Negative digits game: make some digits negative.
$[23045]_p \rightsquigarrow 2(-3)0(-4)5 = 2p^4 - 3p^3 - 4p + 5$.
Descendants of i: all positive numbers you get in this way.
E.g. $[23045]_p$ has $2^3 = 8$ descendants.
Tubbenhauer-Wedrich: $C_{ij} = |\{\text{descendants of } i\} \cap \{\text{descendants of } j\}|$.
Exercise: $C_{ij} = 0$ or power of 2.
Exercise: $\det C_{ij} =$ power of p.

Embeddings

We have $\mathrm{Ver}_p \subset \mathrm{Ver}_{p^2} \subset \mathrm{Ver}_{p^3} \subset \dots$
For $p > 2$, $\mathrm{Ver}_{p^n} = \mathrm{Ver}^+_{p^n} \boxtimes \mathrm{sVec}$.

More properties of Ver_{p^n}

Grothendieck ring ($p > 2$)

More properties of Ver_{p^n}

Grothendieck ring ($p > 2$)

$K(\mathrm{Ver}_{p^n}^+) = \mathbb{Z}[\xi + \xi^{-1}]$. where ξ is a primitive p^n-th root of 1

More properties of Ver_{p^n}

Grothendieck ring ($p > 2$)

$K(\mathrm{Ver}_{p^n}^+) = \mathbb{Z}[\xi + \xi^{-1}]$. where ξ is a primitive p^n-th root of 1
$K(\mathrm{Ver}_{p^n}) = \mathbb{Z}[\xi + \xi^{-1}][\mathbb{Z}/2]$.

More properties of Ver_{p^n}

Grothendieck ring ($p > 2$)

$K(\mathrm{Ver}^+_{p^n}) = \mathbb{Z}[\xi + \xi^{-1}]$. where ξ is a primitive p^n−th root of 1
$K(\mathrm{Ver}_{p^n}) = \mathbb{Z}[\xi + \xi^{-1}][\mathbb{Z}/2]$.

Simples

Some simples: $T_0 = \mathbf{1}, T_1, \dots, T_{p-1}$ are simples $T_i^{[n]}$ in Ver_{p^n} ($n > 1$)

More properties of Ver_{p^n}

Grothendieck ring ($p > 2$)

$K(\mathrm{Ver}_{p^n}^+) = \mathbb{Z}[\xi + \xi^{-1}]$. where ξ is a primitive p^n−th root of 1
$K(\mathrm{Ver}_{p^n}) = \mathbb{Z}[\xi + \xi^{-1}][\mathbb{Z}/2]$.

Simples

Some simples: $T_0 = \mathbf{1}, T_1, \ldots, T_{p-1}$ are simples $T_i^{[n]}$ in Ver_{p^n} $(n > 1)$
Tensor Product Theorem: for $i = [i_1 \ldots i_n]_p$ with $i_1 \neq p - 1$.
$L_i = T_{i_1}^{[1]} \otimes T_{i_2}^{[2]} \ldots T_{i_n}^{[n]}$ is simple (and this is a complete list).

More properties of Ver_{p^n}

Grothendieck ring ($p > 2$)

$K(\mathrm{Ver}_{p^n}^+) = \mathbb{Z}[\xi + \xi^{-1}]$. where ξ is a primitive p^n-th root of 1
$K(\mathrm{Ver}_{p^n}) = \mathbb{Z}[\xi + \xi^{-1}][\mathbb{Z}/2]$.

Simples

Some simples: $T_0 = \mathbf{1}, T_1, \dots, T_{p-1}$ are simples $T_i^{[n]}$ in Ver_{p^n} ($n > 1$)
Tensor Product Theorem: for $i = [i_1 \dots i_n]_p$ with $i_1 \neq p-1$.
$L_i = T_{i_1}^{[1]} \otimes T_{i_2}^{[2]} \dots T_{i_n}^{[n]}$ is simple (and this is a complete list).
$P(L_i) = P_s$ where $s = [(i_1+1)i_2^* \dots i_n^*]$ where $i^* = p-1-i$.

More properties of Ver_{p^n}

Grothendieck ring ($p > 2$)

$K(\mathrm{Ver}_{p^n}^+) = \mathbb{Z}[\xi + \xi^{-1}]$. where ξ is a primitive p^n-th root of 1
$K(\mathrm{Ver}_{p^n}) = \mathbb{Z}[\xi + \xi^{-1}][\mathbb{Z}/2]$.

Simples

Some simples: $T_0 = \mathbf{1}, T_1, \ldots, T_{p-1}$ are simples $T_i^{[n]}$ in Ver_{p^n} ($n > 1$)
Tensor Product Theorem: for $i = [i_1 \ldots i_n]_p$ with $i_1 \neq p-1$.
$L_i = T_{i_1}^{[1]} \otimes T_{i_2}^{[2]} \ldots T_{i_n}^{[n]}$ is simple (and this is a complete list).
$P(L_i) = P_s$ where $s = [(i_1+1)i_2^* \ldots i_n^*]$ where $i^* = p-1-i$.
Extensions ($p > 2$): $\mathrm{Ext}^1(L_i, L_j) = 0$ or k.

More properties of Ver_{p^n}

Grothendieck ring ($p > 2$)

$K(\mathrm{Ver}_{p^n}^+) = \mathbb{Z}[\xi + \xi^{-1}]$. where ξ is a primitive p^n-th root of 1
$K(\mathrm{Ver}_{p^n}) = \mathbb{Z}[\xi + \xi^{-1}][\mathbb{Z}/2]$.

Simples

Some simples: $T_0 = \mathbf{1}, T_1, \dots, T_{p-1}$ are simples $T_i^{[n]}$ in Ver_{p^n} $(n > 1)$
Tensor Product Theorem: for $i = [i_1 \dots i_n]_p$ with $i_1 \neq p-1$.
$L_i = T_{i_1}^{[1]} \otimes T_{i_2}^{[2]} \dots T_{i_n}^{[n]}$ is simple (and this is a complete list).
$P(L_i) = P_s$ where $s = [(i_1+1)i_2^* \dots i_n^*]$ where $i^* = p-1-i$.
Extensions ($p > 2$): $\mathrm{Ext}^1(L_i, L_j) = 0$ or k.
$\mathrm{Ext}^1(L_i, L_j) \neq 0 \Leftrightarrow i$ and j differ only in two consecutive digits, of which the first ones differ by 1 and the second ones add up to $p-2$.

More properties of Ver_{p^n}

Grothendieck ring ($p > 2$)

$K(\mathrm{Ver}_{p^n}^+) = \mathbb{Z}[\xi + \xi^{-1}]$. where ξ is a primitive p^n-th root of 1
$K(\mathrm{Ver}_{p^n}) = \mathbb{Z}[\xi + \xi^{-1}][\mathbb{Z}/2]$.

Simples

Some simples: $T_0 = \mathbf{1}, T_1, \ldots, T_{p-1}$ are simples $T_i^{[n]}$ in Ver_{p^n} $(n > 1)$
Tensor Product Theorem: for $i = [i_1 \ldots i_n]_p$ with $i_1 \neq p-1$.
$L_i = T_{i_1}^{[1]} \otimes T_{i_2}^{[2]} \ldots T_{i_n}^{[n]}$ is simple (and this is a complete list).
$P(L_i) = P_s$ where $s = [(i_1+1)i_2^* \ldots i_n^*]$ where $i^* = p-1-i$.
Extensions ($p > 2$): $\mathrm{Ext}^1(L_i, L_j) = 0$ or k.
$\mathrm{Ext}^1(L_i, L_j) \neq 0 \Leftrightarrow i$ and j differ only in two consecutive digits, of which the first ones differ by 1 and the second ones add up to $p-2$.
Blocks: $n(p-1)$ of them of sizes $1, p-1, p^2-p, \ldots, p^{n-1}-p^{n-2}$.

More properties of Ver_{p^n}

Grothendieck ring ($p > 2$)

$K(\mathrm{Ver}_{p^n}^+) = \mathbb{Z}[\xi + \xi^{-1}]$. where ξ is a primitive p^n-th root of 1
$K(\mathrm{Ver}_{p^n}) = \mathbb{Z}[\xi + \xi^{-1}][\mathbb{Z}/2]$.

Simples

Some simples: $T_0 = \mathbf{1}, T_1, \ldots, T_{p-1}$ are simples $T_i^{[n]}$ in Ver_{p^n} $(n > 1)$
Tensor Product Theorem: for $i = [i_1 \ldots i_n]_p$ with $i_1 \neq p - 1$.
$L_i = T_{i_1}^{[1]} \otimes T_{i_2}^{[2]} \ldots T_{i_n}^{[n]}$ is simple (and this is a complete list).
$P(L_i) = P_s$ where $s = [(i_1 + 1)i_2^* \ldots i_n^*]$ where $i^* = p - 1 - i$.
Extensions ($p > 2$): $\mathrm{Ext}^1(L_i, L_j) = 0$ or k.
$\mathrm{Ext}^1(L_i, L_j) \neq 0 \Leftrightarrow i$ and j differ only in two consecutive digits, of which the first ones differ by 1 and the second ones add up to $p - 2$.
Blocks: $n(p-1)$ of them of sizes $1, p-1, p^2-p, \ldots, p^{n-1}-p^{n-2}$.
Ver_{p^n} is a mod p reduction of a semisimple category in char. zero.

More properties of Ver_{p^n}

Grothendieck ring ($p > 2$)

$K(\mathrm{Ver}_{p^n}^+) = \mathbb{Z}[\xi + \xi^{-1}]$. where ξ is a primitive p^n-th root of 1
$K(\mathrm{Ver}_{p^n}) = \mathbb{Z}[\xi + \xi^{-1}][\mathbb{Z}/2]$.

Simples

Some simples: $T_0 = \mathbf{1}, T_1, \dots, T_{p-1}$ are simples $T_i^{[n]}$ in Ver_{p^n} $(n > 1)$
Tensor Product Theorem: for $i = [i_1 \dots i_n]_p$ with $i_1 \neq p-1$.
$L_i = T_{i_1}^{[1]} \otimes T_{i_2}^{[2]} \dots T_{i_n}^{[n]}$ is simple (and this is a complete list).
$P(L_i) = P_s$ where $s = [(i_1+1)i_2^* \dots i_n^*]$ where $i^* = p-1-i$.
Extensions ($p > 2$): $\mathrm{Ext}^1(L_i, L_j) = 0$ or k.
$\mathrm{Ext}^1(L_i, L_j) \neq 0 \Leftrightarrow i$ and j differ only in two consecutive digits, of which the first ones differ by 1 and the second ones add up to $p-2$.
Blocks: $n(p-1)$ of them of sizes $1, p-1, p^2-p, \dots, p^{n-1}-p^{n-2}$.
Ver_{p^n} is a mod p reduction of a semisimple category in char. zero.
Corollary: $C = DD^T$ where D is the decomposition matrix.

Universal property of Ver_{p^n}

When $\wedge^2 X = \mathbf{1}$?

Universal property of Ver_{p^n}

When $\wedge^2 X = \mathbf{1}$?

Assume $\mathcal{C}$ is generated by X such that $\wedge^2 X = \mathbf{1}$, i.e. any object of $\mathcal{C}$ is a subquotient of direct sums of $X^{\otimes n} \otimes (X^*)^{\otimes m}$.

Universal property of Ver_{p^n}

When $\wedge^2 X = \mathbf{1}$?

Assume $\mathcal{C}$ is generated by X such that $\wedge^2 X = \mathbf{1}$, i.e. any object of $\mathcal{C}$ is a subquotient of direct sums of $X^{\otimes n} \otimes (X^*)^{\otimes m}$.

Then $(\mathcal{C}, X)$ is one of the following:

Universal property of Ver_{p^n}

When $\wedge^2 X = \mathbf{1}$?

Assume $\mathcal{C}$ is generated by X such that $\wedge^2 X = \mathbf{1}$, i.e. any object of $\mathcal{C}$ is a subquotient of direct sums of $X^{\otimes n} \otimes (X^*)^{\otimes m}$.

Then $(\mathcal{C}, X)$ is one of the following:

• $\mathcal{C} = \mathrm{Rep}(\Gamma)$ where $\Gamma \subset SL_2 = SL(X)$;

Universal property of Ver_{p^n}

When $\wedge^2 X = \mathbf{1}$?

Assume $\mathcal{C}$ is generated by X such that $\wedge^2 X = \mathbf{1}$, i.e. any object of $\mathcal{C}$ is a subquotient of direct sums of $X^{\otimes n} \otimes (X^*)^{\otimes m}$.

Then $(\mathcal{C}, X)$ is one of the following:

- $\mathcal{C} = \mathrm{Rep}(\Gamma)$ where $\Gamma \subset SL_2 = SL(X)$;
- $\mathcal{C} = \mathrm{Ver}_{p^n}$ (for $p^n > 2$), $X = T_1$;

Universal property of Ver_{p^n}

When $\wedge^2 X = \mathbf{1}$?

Assume $\mathcal{C}$ is generated by X such that $\wedge^2 X = \mathbf{1}$, i.e. any object of $\mathcal{C}$ is a subquotient of direct sums of $X^{\otimes n} \otimes (X^*)^{\otimes m}$.

Then $(\mathcal{C}, X)$ is one of the following:

- $\mathcal{C} = \mathrm{Rep}(\Gamma)$ where $\Gamma \subset SL_2 = SL(X)$;
- $\mathcal{C} = \mathrm{Ver}_{p^n}$ (for $p^n > 2$), $X = T_1$;
- $\mathcal{C} = \mathrm{sVec}$, X – odd line ($p \neq 2, 3$).

Universal property of Ver_{p^n}

When $\wedge^2 X = \mathbf{1}$?

Assume $\mathcal{C}$ is generated by X such that $\wedge^2 X = \mathbf{1}$, i.e. any object of $\mathcal{C}$ is a subquotient of direct sums of $X^{\otimes n} \otimes (X^*)^{\otimes m}$.

Then $(\mathcal{C}, X)$ is one of the following:

- $\mathcal{C} = \mathrm{Rep}(\Gamma)$ where $\Gamma \subset SL_2 = SL(X)$;
- $\mathcal{C} = \mathrm{Ver}_{p^n}$ (for $p^n > 2$), $X = T_1$;
- $\mathcal{C} = \mathrm{sVec}$, X – odd line ($p \neq 2, 3$).

Representation type

Ver_{p^n} is of finite representation type if $n \leq 2$.

Universal property of Ver_{p^n}

When $\wedge^2 X = \mathbf{1}$?

Assume $\mathcal{C}$ is generated by X such that $\wedge^2 X = \mathbf{1}$, i.e. any object of $\mathcal{C}$ is a subquotient of direct sums of $X^{\otimes n} \otimes (X^*)^{\otimes m}$.

Then $(\mathcal{C}, X)$ is one of the following:

- $\mathcal{C} = \mathrm{Rep}(\Gamma)$ where $\Gamma \subset SL_2 = SL(X)$;
- $\mathcal{C} = \mathrm{Ver}_{p^n}$ (for $p^n > 2$), $X = T_1$;
- $\mathcal{C} = \mathrm{sVec}$, X – odd line ($p \neq 2, 3$).

Representation type

Ver_{p^n} is of finite representation type if $n \leq 2$.
Thus the semisimplification $\overline{\mathrm{Ver}_{p^2}}$ is manageable.

Universal property of Ver_{p^n}

When $\wedge^2 X = \mathbf{1}$?

Assume $\mathcal{C}$ is generated by X such that $\wedge^2 X = \mathbf{1}$, i.e. any object of $\mathcal{C}$ is a subquotient of direct sums of $X^{\otimes n} \otimes (X^*)^{\otimes m}$.

Then $(\mathcal{C}, X)$ is one of the following:

- $\mathcal{C} = \mathrm{Rep}(\Gamma)$ where $\Gamma \subset SL_2 = SL(X)$;
- $\mathcal{C} = \mathrm{Ver}_{p^n}$ (for $p^n > 2$), $X = T_1$;
- $\mathcal{C} = \mathrm{sVec}$, X – odd line ($p \neq 2, 3$).

Representation type

Ver_{p^n} is of finite representation type if $n \leq 2$.

Thus the semisimplification $\overline{\mathrm{Ver}_{p^2}}$ is manageable.

$\overline{\mathrm{Ver}_{p^2}} = (\mathrm{Ver}_p \boxtimes \mathrm{Ver}_p \boxtimes \mathrm{Rep}(\mathbb{Z}_{2p-2}, z))_{\mathbb{Z}_2}$. $\leftarrow$ de-equivariantization

Universal property of Ver_{p^n}

When $\wedge^2 X = \mathbf{1}$?

Assume $\mathcal{C}$ is generated by X such that $\wedge^2 X = \mathbf{1}$, i.e. any object of $\mathcal{C}$ is a subquotient of direct sums of $X^{\otimes n} \otimes (X^*)^{\otimes m}$.

Then $(\mathcal{C}, X)$ is one of the following:

- $\mathcal{C} = \mathrm{Rep}(\Gamma)$ where $\Gamma \subset SL_2 = SL(X)$;
- $\mathcal{C} = \mathrm{Ver}_{p^n}$ (for $p^n > 2$), $X = T_1$;
- $\mathcal{C} = \mathrm{sVec}$, X – odd line ($p \neq 2, 3$).

Representation type

Ver_{p^n} is of finite representation type if $n \leq 2$.

Thus the semisimplification $\overline{\mathrm{Ver}_{p^2}}$ is manageable.

$\overline{\mathrm{Ver}_{p^2}} = (\mathrm{Ver}_p \boxtimes \mathrm{Ver}_p \boxtimes \mathrm{Rep}(\mathbb{Z}_{2p-2}, z))_{\mathbb{Z}_2}$. $\leftarrow$ de-equivariantization

Here $\mathrm{Ver}_p \boxtimes \mathrm{Ver}_p$ comes from the simple objects of Ver_{p^2}.

Universal property of Ver_{p^n}

When $\wedge^2 X = \mathbf{1}$?

Assume $\mathcal{C}$ is generated by X such that $\wedge^2 X = \mathbf{1}$, i.e. any object of $\mathcal{C}$ is a subquotient of direct sums of $X^{\otimes n} \otimes (X^*)^{\otimes m}$.

Then $(\mathcal{C}, X)$ is one of the following:

- $\mathcal{C} = \mathrm{Rep}(\Gamma)$ where $\Gamma \subset SL_2 = SL(X)$;
- $\mathcal{C} = \mathrm{Ver}_{p^n}$ (for $p^n > 2$), $X = T_1$;
- $\mathcal{C} = \mathrm{sVec}$, X – odd line ($p \neq 2, 3$).

Representation type

Ver_{p^n} is of finite representation type if $n \leq 2$.

Thus the semisimplification $\overline{\mathrm{Ver}_{p^2}}$ is manageable.

$\overline{\mathrm{Ver}_{p^2}} = (\mathrm{Ver}_p \boxtimes \mathrm{Ver}_p \boxtimes \mathrm{Rep}(\mathbb{Z}_{2p-2}, z))_{\mathbb{Z}_2}$. $\leftarrow$ de-equivariantization

Here $\mathrm{Ver}_p \boxtimes \mathrm{Ver}_p$ comes from the simple objects of Ver_{p^2}.

$\mathrm{Rep}(\mathbb{Z}_{2p-2}, z)$ comes from Heller shift of $\mathbf{1}$.

Universal property of Ver_{p^n}

When $\wedge^2 X = \mathbf{1}$?

Assume $\mathcal{C}$ is generated by X such that $\wedge^2 X = \mathbf{1}$, i.e. any object of $\mathcal{C}$ is a subquotient of direct sums of $X^{\otimes n} \otimes (X^*)^{\otimes m}$.

Then $(\mathcal{C}, X)$ is one of the following:

- $\mathcal{C} = \mathrm{Rep}(\Gamma)$ where $\Gamma \subset SL_2 = SL(X)$;
- $\mathcal{C} = \mathrm{Ver}_{p^n}$ (for $p^n > 2$), $X = T_1$;
- $\mathcal{C} = \mathrm{sVec}$, X – odd line ($p \neq 2, 3$).

Representation type

Ver_{p^n} is of finite representation type if $n \leq 2$.

Thus the semisimplification $\overline{\mathrm{Ver}_{p^2}}$ is manageable.

$\overline{\mathrm{Ver}_{p^2}} = (\mathrm{Ver}_p \boxtimes \mathrm{Ver}_p \boxtimes \mathrm{Rep}(\mathbb{Z}_{2p-2}, z))_{\mathbb{Z}_2}$. $\leftarrow$ de-equivariantization

Here $\mathrm{Ver}_p \boxtimes \mathrm{Ver}_p$ comes from the simple objects of Ver_{p^2}.

$\mathrm{Rep}(\mathbb{Z}_{2p-2}, z)$ comes from Heller shift of $\mathbf{1}$.

$\overline{\mathrm{Ver}_{p^n}}$ contains $\mathrm{Ver}_p^{\boxtimes n}$ generated by the simples.

Some open questions

Some open questions

Module categories

Question 1. What are exact module categories over Ver_{p^n}?

Some open questions

Module categories

Question 1. What are exact module categories over Ver_{p^n}?

Extensions

Question 2. What is $\mathrm{Ext}^{\bullet}_{\mathrm{Ver}_{p^n}}(\mathbf{1},\mathbf{1})$? (Work in progress with D. Benson containing a conjecture and partial proof).

Some open questions

Module categories

Question 1. What are exact module categories over Ver_{p^n}?

Extensions

Question 2. What is $\mathrm{Ext}^\bullet_{\mathrm{Ver}_{p^n}}(\mathbf{1},\mathbf{1})$? (Work in progress with D. Benson containing a conjecture and partial proof).

More examples?

Question 3. Are there any other incompressible pre-Tannakian categories than tensor subcategories of $\mathrm{Ver}_{p^\infty} = \cup_n \mathrm{Ver}_{p^n}$? At least of sub-exponential growth?

Some open questions

Module categories

Question 1. What are exact module categories over Ver_{p^n}?

Extensions

Question 2. What is $\mathrm{Ext}^\bullet_{\mathrm{Ver}_{p^n}}(\mathbf{1},\mathbf{1})$? (Work in progress with D. Benson containing a conjecture and partial proof).

More examples?

Question 3. Are there any other incompressible pre-Tannakian categories than tensor subcategories of $\mathrm{Ver}_{p^\infty} = \cup_n \mathrm{Ver}_{p^n}$? At least of sub-exponential growth?

Universality

Let $\mathcal{C}$ be a pre-Tannakian category of sub-exponential growth.
Question 4. Is there an exact tensor functor $\mathcal{C} \to \mathrm{Ver}_{p^\infty}$?

Thanks for listening!