1 RHA, SS 13, Exercise Sheet 9

Due June 26. 2013.

Exercise 1:

Consider the curve $\gamma(t) = (t, t^k)$ on $\mathbf{R}^2$, with k an integer ≥ 2. If $k = 2$, its curvature vanishes nowhere; if $k > 2$, its curvature vanishes at the origin (and only there) of order $k - 2$. Consider the measure $d\mu$ defined by

$$\int_{\mathbf{R}^2} f d\mu = \int_{\mathbf{R}} f(t, t^k)\psi(t)dt,$$

where $\psi \in C_0^\infty(\mathbf{R})$ is such that $\psi(0) \neq 0$. Prove that:

(i) $|\widehat{d\mu}(\xi)| = O(|\xi|^{-1/k})$;

(ii) $|\widehat{d\mu}(0, \xi_2)| \geq c|\xi_2|^{-1/k}$ if ξ_2 is large.

Hint: For part (ii), consider the case when k is even and check that

$$\int_{-\infty}^{\infty} e^{i\lambda x^k} e^{-x^k} dx = c_\lambda (1 - i\lambda)^{-1/k}.$$

Exercise 2:

Let $d\mu$ denote the projection measure on the paraboloid $\mathbf{P} \subset \mathbf{R}^3$ given by

$$\mathbf{P} := \{(\xi, \tau) \in \mathbf{R}^2 \times \mathbf{R} : \tau = |\xi|^2\}.$$

Prove that there exists a constant $c_0 > 0$ such that

$$\mu * \mu(\xi, \tau) = c_0 \cdot \chi\Big(\tau \geq \frac{|\xi|^2}{2}\Big).$$

Determine the numerical value of c_0.

Exercise 3:

Let $d \geq 2$, and let $u = u(x, t)$ be a solution of the wave equation in $\mathbf{R}^{d+1}_{x,t}$,

$$\partial_t^2 u - \triangle_x u = 0,$$

subject to initial conditions $u|_{t=0} = 0$ and $\partial_t u|_{t=0} = f \in \mathcal{S}(\mathbf{R}^d)$. Prove that

$$\|u\|_{L^{\frac{2d+2}{d-1}}(\mathbf{R}^{d+1})} \lesssim_d \Big(\int_{\mathbf{R}^d} |\widehat{f}(\xi)|^2 \frac{d\xi}{|\xi|}\Big)^{1/2}.$$

Hint: Use the stationary phase method to establish an appropriate decay for $\widehat{\sigma_{\Gamma_0}}$, where

$$\Gamma_0 := \{(\xi, \tau) \in \mathbf{R}^{d+1} : 1 \leq \tau = |\xi| \leq 2\}$$

is a section of the light cone in $\mathbf{R}^{d+1}$. For $p = (2d+2)/(d+3)$, establish an $L^p \to L^2(\sigma_{\Gamma_0})$ restriction estimate for Γ_0. Rescale.