1 RHA, SS 13, Exercise Sheet 8

Due June 19. 2013.

Exercise 1:

The Beta function is defined for $\Re z > 0$ and $\Re w > 0$ by

$$B(z,w) = \int_0^1 (1-t)^{z-1} t^{w-1} dt.$$

(i) Prove that $B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$.

(ii) Show that the family of distributions $(j_z)_{z\in\mathbf{C}}$ defined in class satisfies

$$j_z * j_w = j_{z+w},$$

for every $z, w \in \mathbf{C}$.

Exercise 2:

Given $0 < \alpha < d$, for which pairs $1 \le p, q \le \infty$ is the fractional integral operator $f \mapsto I_\alpha(f) := f * |\cdot|^{\alpha-d}$ bounded from $L^p(\mathbf{R}^d)$ to $L^q(\mathbf{R}^d)$? A scaling argument shows that this is only possible if

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}.$$

The Hardy-Littlewood-Sobolev inequality shows that this condition is almost sufficient, and this is best possible. Indeed, show that $L^p \to L^q$ boundedness of the operator I_α fails at the two exceptional cases:

(i) $p = 1$ (which forces $q = d/(d-\alpha)$);

(ii) $q = \infty$ (and so $p = d/\alpha$).

Hint: For (i), use an approximation to the identity; for (ii), consider a function which locally looks like $f(x) = |x|^{-\alpha}(\log 1/|x|)^{C(\alpha,d)(1+\epsilon)}$ for sufficiently small $\epsilon > 0$.

Exercise 3:

Let K be a compact subset (with nonempty interior) of the paraboloid

$$\mathbf{P}^{d-1} := \Big\{\xi \in \mathbf{R}^d : \xi_d = \frac{1}{2}|\xi'|^2\Big\},$$

equipped with the pullback $d\sigma^*$ of the $(d-1)$-dimensional Lebesgue measure $d\xi'$ under the projection map $\xi \mapsto \xi'$. Show that a restriction estimate of the form

$$\Big\| \widehat{f}\Big|_K \Big\|_{L^q(d\sigma^*)} \lesssim_{p,q,K} \|f\|_{L^p(\mathbf{R}^d)}$$

is only possible if $\frac{d+1}{p'} \le \frac{d-1}{q}$. Verify that this necessary condition can be improved to $\frac{d+1}{p'} = \frac{d-1}{q}$ if one considers instead the *full* paraboloid $\mathbf{P}^{d-1}$.