1 RHA, SS 13, Exercise Sheet 3

Due May 1. 2013.

Exercise 1:

Prove Bernstein's inequality in the following form.

Assume $1 \leq p \leq \infty$ and $f \in L^p(\mathbf{R}^n)$ and assume that the Fourier transform $\widehat{f}$ is supported in the ball $B(0, R)$ (of radius R about the origin 0).

Prove for any multi-index α

$$\|\partial^\alpha f\|_p \leq C_n R^{|\alpha|} \|f\|_p$$

with constnat $C - n$ depending only on n.

Hint: Convolve f with an appropriate function.

Exercise 2 :

Let $(M_N)_N$ be a family of sublinear operators. Assume M_N satisfies

$$|\{x : M_N f(x) \geq \lambda\}| \leq N\lambda^{-1}\|f\|_1$$

$$|\{x : M_N f(x) \geq \lambda\}| \leq \lambda^{-2}\|f\|_2$$

$$\|M_N f\|_\infty \leq \|f\|_\infty$$

1. Prove for $1 < p < 2$ an estimate of the form

$$\|f\|_p \leq C_p N^{\alpha(p)} \|f\|_p$$

(interpolation as in the last homework assignment) and discuss precisely the dependence of C_p and $\alpha(p)$ on p in your proof.

2. Interpolate this result with the L^∞ bound and obtain an estimate

$$\|f\|_2 \leq C(\epsilon, N)\|f\|_2$$

Choose ϵ depending on N so that you obtain a good bound (power of logarithm in N) on $C(\epsilon, N)$.

Exercise 3:

Let $\phi : \mathbf{R}^2 \to \mathbf{R}$ be a fixed function supported in $[1 - \epsilon^2, 1 + \epsilon^2] \times [-\epsilon, \epsilon]$ for some $\epsilon < 0.1$.

Assume that the radial (r in polar coordinates) and angular (θ in polar coordinatres) partial derivatives satisfy

$$\|\partial_r^\alpha \partial_\theta^\beta \phi\|_\infty \leq C_{\alpha,\beta}\epsilon^{-2\alpha-\beta}$$

for all $\alpha, \beta > 0$ (such as a typical Fefferman piece of the Bochner Riesz multiplier would). Prove that the ordinary partial derivatives satisfy

$$\|\partial_x^\alpha \partial_y^\beta \phi\|_\infty \leq C_{\alpha,\beta} \epsilon^{-2\alpha-\beta}$$

(As usual the constants $C_{\alpha,\beta}$ here may be different from the previous ones here, the improtant thing is they are independent of ϵ) Using your calculation, explain why we have chosen the particular scaling of the two sides of the support of ϕ.