1 RHA, SS 13, Exercise Sheet 1

Due April 17 2013.

On this sheet C denotes universal constants which may change from line to line.

Exercise 1 :

Define the Fourier transform on $\mathbf{R}$ by

$$\widehat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-2\pi i \langle x, \xi \rangle} \, dx.$$

Define the (formally adjoint) operators A and A^* defined by

$$Af(x) = 2\pi x f(x) - f'(x)$$

$$A^* f(x) = 2\pi x f(x) + f'(x)$$

Consider the Gaussian function

$$g(x) = e^{-\pi x^2}$$

1. Calculate Ag, A^*g, and the commutator $[A, A^*]$.
2. Prove that the functions $A^n g$ for $n = 0, 1, \dots$ form an orthogonal set in $L^2(\mathbf{R})$. Determine the L^2 norm of these functions. (Hint: use the calculations under the previous item.
3. Prove that the functions $A^n g$ are eigenfunctions of the Fourier transform and determine the eigenvalues.
4. Prove that the functions $f_\xi(x) = e^{2\pi i x \xi} e^{-\pi x^2}$ are in the closed linear span in $L^2(\mathbf{R})$ of the above orthogonal set.
5. Prove that $L^2(\mathbf{R})$ is the closed linear span of the above orthogonal set and conclude that the Fourier transform is a unitary operator on $L^2(\mathbf{R})$.

Exercise 2:

Let the Hilbert transform of a function $f \in C_0^\infty(\mathbf{R})$ be defined as

$$Hf(x) := \lim_{\epsilon \to 0} \int_{[-\epsilon, \epsilon]^c} f(t) \frac{1}{x - t} \, dt$$

1. Prove that the limit exists for $f \in C_0^\infty$ and every x.

2. Prove that

$$Hf(x) = c \lim_{\epsilon\to 0} \int_{\epsilon}^{(1/\epsilon)} \int_{\mathbf{R}} f(t)(\frac{x-t}{s^3} e^{-\pi(\frac{x-t}{s})^2})\, dt\, ds$$

for some non-zero constant c, where the limit is in L^1 sense. Hint: show some scaling symmetry of the function

$$k(t) = \lim_{\epsilon\to 0} \int_{\epsilon}^{(1/\epsilon)} \frac{t}{s^3} e^{-\pi(\frac{t}{s})^2}\, ds$$

to identify this function up to scalar multiple. You do not have to evaluate this integral explicitly to solve this exercise.

3. Using the previous exercise and continuity of the Fourier transform in L^1 identify the Fourier transform of Hf as a nonzero scalar multiple of the Fourier transform of f times the signum function.

4. Conclude that the Hilbert transform can be extended to a bounded operator in L^2.

Exercise 3:

The purpose of this exercise is to show for every $f \in C_0^\infty$ and $\lambda > 0$ the bound

$$|\{x \in \mathbf{R} : |Hf(x)| > \lambda\}| \leq C\|f\|_1 \lambda^{-1}$$

1. An interval is called dyadic if it is of the form $[2^k n, 2^k(n+1))$ with integers k and n. Let $\mathbf{I}$ be the set of maximal dyadic intervals (maximal with respect to set inclusion) such that $\frac{1}{|I|}\int_I |f(x)|\, dx > \lambda$. Prove that these intervals are pairwise disjoint and that

$$\sum_{I\in\mathbf{I}} |I| \leq \lambda^{-1}\|f\|_1$$

Let $E = \bigcup_{I\in\mathbf{I}} 3I$ where $3I$ denotes the interval with the same center as I but three times the length. Prove $|E| \leq 3\lambda^{-1}\|f\|_1$

2. Split $f = g + b$ where $b = \sum_{I\in\mathbf{I}} b_I$ and b_I is supported on I and g is constant on each $I \in \mathbf{I}$. Show that $\|g\|_\infty \leq C\lambda$ and use the previously established L^2 bound for the Hilbert transform to conclude

$$|\{x \in E^c : |Hg(x)| \leq \lambda/2\}| \leq C\lambda^{-1}\|f\|_1$$

3. Show that

$$\|H(b_I)\|_{L^1((3I)^c)} \leq C\lambda|I|$$

(use the fact that b_I has mean zero, e.g. by some partial integration). Then conclude that

$$|\{x \in E^c : |Hf(x)| \leq \lambda/2\}| \leq C\lambda^{-1}\|f\|_1$$