1 NLPDE I, WS 12/13, Exercise Sheet 9

Due Dec 18 in tutorial session. This sheet has 4 exercises

Exercise 1:

Let u be a positive smooth solution to the heat equation $\Delta u(x,t) = \partial_t u(x,t)$ in the open set $\mathbf{R}^n \times (0,\infty)$ such that for each fixed time t the functions the $u(.,t)$ and $\nabla u(.,t)$ are in $L^1(\mathbf{R}^n)$.

1) Prove that the energy

$$\int_{\mathbf{R}^n} u(x,t)\, dx$$

is constant in t

2) Prove that the entropy

$$\int_{\mathbf{R}^n} \log(u(x,t))u(x,t)\, dx$$

is monotone increasing in t.

Exercise 2:

Let $X \subset Y$ be two closed sets in $\mathbf{R}^n$ and let B be some linear space of continuous functions on Y such that every continuous function on X is the restriction of at least one function in B and there si a constnat C such that $\|f\|_{L^\infty(Y)} \leq C\|f\|_{L^\infty(x)}$ for all $f \in B$. 1) Prove that every continuous function on X is the restriction to X of a unique function in B.

2) Prove that for every $y \in Y$ there is a Borel measure μ on $\mathbf{R}^n$ with support in X such that for all $f \in B$ we have

$$\int_X f(x)\, d\mu(x) = f(y)$$

Exercise 3

Consider the elliptic operator in $\mathbf{R}^n$

$$Lu(x) = \sum_{i,j=1}^{n} a_{i,j}\partial_i\partial_j u(x) + \sum_{i=1}^{n} b_i\partial_i u(x)$$

for smoth functions $a_{i,j}$ and b_i where $(a_{i,j}(x))$ is a real symmetric matrix for every x with eigenvalues between c_1 and c_2 and $0 < c_1 < c_2$.

Let Ω be a bounded open set in $\mathbf{R}^n \times \mathbf{R}$ with boundary $\partial\Omega$. Let $(x_0, t_0) \in \Omega$ and let B be the set $\{(x,t) \in \partial\Omega : t \leq t_0\}$. Let u be a continuous function on the closure of Ω, smooth in Ω and satsifying $\partial_t u = L(u)$ in Ω.

Prove that

$$\|u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(S)}$$

where the latter norm is the sup norm of the restriction of the function to the set S. Prove any maximum principle that you may be using.

Exercise 4:

Let μ be a positive Borel measure on the real line $\mathbf{R}$ with total mass $\|\mu\| = 1$ and finite third moment

$$\int |x|^3 \, d\mu < \infty$$

Define the finite positive Borel measure μ_n by setting for bounded continuous function f

$$\int f(x)\, d\mu_n(x) = \int \int \dots \int f(\frac{x_1 + x_2 + \dots + x_n}{\sqrt{n}})\, d\mu(x_1) d\mu(x_2) \dots d\mu(x_2)$$

1) Under which additional assumption on μ can one translate and dilate μ so that we have the normalization $\int x d\mu(x) = 0$ and $\int x^2 \, d\mu(x) = 1$?

2) Calculate the Fourier transform of μ_n in terms of the Fourier transform of μ (Recall that the Fourier transform of μ is a continuous function).

3) Assuming the normalization 1) prove that for every compactly supported continuous f the limit

$$\lim_{n \to \infty} \int f(x) d\mu_n(x)$$

exists and equals $\int f(x)g(x)\, dx$ for some universal continuous function g. Determine the universal g. (Hint: express $\int f(x) d\mu_n(x)$ in terms of an integral involving the Fourier transforms of f and μ and use Lebesgue dominated convergence for that integral.)