# 1 NLPDE I, WS 12/13, Exercise Sheet 8

**Due Dec 11 in tutorial session. This sheet has 3 exercises**

## Exercise 1:

Related to the space BMO in $\mathbf{R}^n$ is the sharp maximal function

$$f^\sharp(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy$$

where $f_Q$ is the average of $f$ over $Q$ and the supremum is taken over all cubes $Q$ which contain the point $x$. The dyadic sharp function $f_d^\sharp$ is defined analogously, with the supremum taken over all standard dyadic cubes. The Hardy Littlewood maximal function is defined as

$$Mf(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)| \, dy$$

with supremum over cubes containing $x$, and again it comes with a discrete version $M_d f$ with supremum taken over standard dyadic cubes.

1) Prove that for every $x$ we have $f^\sharp(x) \leq Mf(x)$.

2) Assume $f$ is non-negative. Show that an inequality uniformly for all $\lambda$ of the type

$$|\{x : Mf(x) \geq \lambda\}| \leq C|\{x : f^\sharp(x) \geq G\lambda\}| + c|\{x : Mf(x) \geq g\lambda\}|$$

for appropriate constants $G$, $g$, $C$, $c$ implies the inequality

$$\|Mf\|_p \leq K \|f^\sharp\|_p$$

for some constant $K$.

## Exercise 2

Prove the inequality in part 2 of the previous exercise in the case of the dyadic versions of sharp and maximal functions. Follow the following two steps:

1) Prove that

$$|\{x \in Q : M_d f(x) \geq \lambda\}| \leq c'|Q|$$

for any cube $Q$ which is a parent of a maximal cube contained in the set

$$\{x : M_d f(x) \geq g\lambda\}|$$

and which contains a point $x$ such that $f_d^\sharp(x) \leq G\lambda$.

2) Show that this implies

$$\|M_d f\|_p \leq K \|f_d^\sharp\|_p$$

via the second part of the previous exercise.

## Exercise 3:

Let $C : \mathbf{R}^n \to \mathbf{R}^{n\times n}$ be a measurable matrix valued function on $\mathbf{R}^n$ whose entries are real symmetric matrices with eigenvalues between $c_1$ and $c_2$ where $0 < c_1 < c_2$. Let $H$ be the Hilbert space (Sobolev space) of functions $f$ in $L^2(\mathbf{R}^n)$ such that $\int (1 + |\xi^2|)|\widehat{f}(\xi)|^2 \, d\xi < \infty$.

Prove that for every $h \in L^2(\mathbf{R}^n)$ and $\lambda > 0$ there is unique non-zero function $f \in H$ such that for all $g \in H$

$$\int \sum_{i,j=1}^{n} \partial_i \overline{g}(x) C_{i,j}(x) \partial_j f(x) \, dx + \lambda \int \overline{g}(x) f(x) \, dx = \int \overline{g}(x) h(x) \, dx$$

Hint: Define a modified Hilbert space, show that it is a Hilbert space, and use the Riesz representation theorem.