1 NLPDE I, WS 12/13, Exercise Sheet 6

Due Nov 19 in tutorial session. This sheet has 4 exercises

Exercise 1:

Prove the following generalization of Schur's test:

Let $(a_{i,j})_{i,j\in\mathbf{N}}$ be an infinite matrix and assume that there are sequences $(p_i)_{i\in\mathbf{N}}$ and $(q_i)_{i\in\mathbf{N}}$ with positive entries such that for all i (resp. j)

$$\sum_j |a_{i,j}|p_j \leq Cq_i$$

$$\sum_i |a_{i,j}|q_i \leq Cp_j$$

Then the operator

$$(v_i)_{i\in\mathbf{N}} \to (\sum_j a_{i,j}v_j)_{j\in\mathbf{N}}$$

defines a bounded operator from $l^2(\mathbf{N})$ to itself. (Setting $p_i = q_j = 1$ for all i, j gives the Schur test proved in the lecture.)

Exercise 2

Let R_m be the m-the Riesz transform in $\mathbf{R}^n$, that is principal value convolution with $x_m/|x|^{n+1}$. Fix a dyadic grid and consider the Haar functions $h_{Q,i}$ for Q in the grid and $i = 1, \ldots, 2^{n-1}$ which take values $0, 1, -1$. Let $\tilde{h}_{Q,i} = |Q|^{-1/2}h_{Q,i}$ denote the L^2 normalized Haar function, so that the collection of $\tilde{h}_{Q,i}$ forms an orthonormal basis for $L^2(\mathbf{R}^n)$.

Estimate the matrix coefficients

$$(\mathbf{R}_m\tilde{h}_{Q,i}, \tilde{h}_{P,j})$$

(If $l(P) < l(Q)$ make a case distinction whether the distance of P to the singular set of $h_{Q,i}$ is smaller or larger than $(l(P)/l(Q))^\alpha l(Q)$ for some $0 < \alpha < 1$.)

Prove

$$\sum_{P,j}(\mathbf{R}_m\tilde{h}_{Q,i}, \tilde{h}_{P,j})\sqrt{|P|} \leq C\sqrt{|Q|}$$

By symmetry this also implies

$$\sum_{Q,i}(\mathbf{R}_m\tilde{h}_{Q,i}, \tilde{h}_{P,j})\sqrt{|Q|} \leq C\sqrt{|P|}$$

Use Schur's test as in the previous exercise to conclude that R_j is bounded in $L^2(\mathbf{R}^n)$ (a fact that could be proven differently for example using the Fourier transform.)

Exercise 3:

Define the Fourier transform of a Schwartz function f as

$$\mathcal{F}(f)(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i \xi x}\, dx$$

For $-n < s < 0$ consider the tempered distribution

$$\Lambda_s : f_s \to \int_{\mathbf{R}^n} f(x)|x|^s\, dx$$

Calculate the constant C_s in

$$\mathcal{F}(\Lambda_s) = C_s \Lambda_{-n-s}$$

by writing Λ_s as superposition of Gaussians. Express the result in terms of the Gamma function.

Exercise 4:

Let Q be a cube in $\mathbf{R}^n$ and let i be the imaginary unit and $s > 0$ and let $h_{Q,j}$ be some Haar function, normalized so as to take values $0, 1, -1$. Prove that

$$f(x) = \lim_{\epsilon > 0} \int_{\mathbf{R}^n \setminus B_\epsilon(0)} h_Q(x - y)|y|^{-n+is}\, dy$$

exists for almost all x and satisfies $\|f\|_1 \leq C|Q|$.