1 NLPDE I, WS 12/13, Exercise Sheet 4

Due Nov 6 in tutorial session. This sheet has 4 exercises

Exercise 1:

Let $h_{Q,i}$ be some Haar function on the cube Q in $\mathbf{R}^n$, let $1 > \epsilon > 0$ and let

$$g(x) = \int_{\mathbf{R}^n} h_{Q,i}(y)|x-y|^{\epsilon-n}\, dx$$

Estimate for any other Haar function $h_{Q',i'}$:

$$\int_{\mathbf{R}^n} h_{Q',i'}(x) g(x)\, dx$$

and deduce from that estimate for which β the function g is β - Hölder.

Exercise 2:

For a Haar function $h_{Q,i}$ in $\mathbf{R}^n$ define the j-th Riesz transform at a point x of continuity of $h_{Q,i}$ as

$$R_j h_{Q,i}(x) = \int_{\mathbf{R}^n \setminus B_\epsilon(x)} \frac{y_j}{|y|^{-n-1}} h_{Q,i}(x-y)\, dy$$

where ϵ is small enough so that $B_\epsilon(x)$ does not intersect the set of discontinuities of $h_{Q,i}$. Show that this definition is independent of the choice of ϵ. Prove pointwise estimates for $R_j h_{Q,i}(x)$ and show that this function is integrable.

Prove that

$$\int_{\mathbf{R}^n} (R_j h_{Q,i})(x) h_{Q',i'}(x)\, dx = -\int_{\mathbf{R}^n} h_{Q,i}(x)(R_j h_{Q',i'})(x)\, dx$$

Exercise 3:

For the j-th Riesz transform as above estimate

$$\int_{\mathbf{R}^n} (R_j h_{Q,i})(x) h_{Q',i'}(x)\, dx$$

for any two Haar functions $h_{Q,i}(x)$ and $h_{Q',i'}(x)$ in the case $l(Q) \le l(Q')$.

Exercise 4:

For a test function (smooth compactly supported function ϕ) define the j-th Riesz transform

$$R_j \phi(x) = \lim_{\epsilon \to 0} \int_{\mathbf{R}^n \setminus B_\epsilon(0)} \phi(x-y) \frac{y_j}{|y|^{n+1}}\, dy$$

Prove that this limit exists for every x and is smooth in x.