Introduction to Symplectic Topology

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What do we mean when we say "symplectic geometry", or what does a random person mean when he says he is a "symplectic geometor"? This name has been used to refer to many completely different areas of research in the mathematics community, though there does exist a group of people who claim they are really working on symplectic geometry, and didn't think other stuffs can be called this name at all.

This course, under the name "Introduction to Symplectic Topology", aims to clarify what we really mean by "Symplectic Geometry", or "Symplectic Topology", and to give audiences a sense of what symplectic geometors regard as an interesting symplectic-geometric problem.

During this fifteen-week (and a total of thrity lectures) course, we will first go through the main storyline of symplectic geometry, and then turn to three selected topics by the lecturer about some of the side stories. Unlike many other areas of research in mathematics, the "main storyline" of symplectic geometry ended quickly, and the current researches are like tangling vines growing out of the trunk. Any such finite selection of topics will be therefore always biased, and we hope the three topics selected can be of any help to the audiences.

Due to limitations of time, we are unfortunately unable to provide completely rigorous proofs of everything, leaving only sketches and ideas of proofs, where audiences are expected to fill in the remaining details themselves.

Oral Exam Patterns. During the lecture, there will be constantly practice questions popped out to the audiences, and they will be the final list of questions that will be (selectively) tested in the oral exam at the end of the course. There'll be two types of questions: basic ones, which consists of 50% of the total questions asked, which examinees have to answer all the questions asked; topic-based ones, which will be marked with a star ($\star$) and consists of the remaining 50% of the total questions asked, where examinees can choose which questions to answer before the exam. They will not be covered in the lecture, but references will be provided and the examinees are expected to learn the references and answer the questions correctly. Solutions to questions of this type will already be inside the references provided, and examinees only need to understand the given literatures.

CONTENTS

1. From Classical Mechanics

Classical mechanics focuses on motions of particles, where by Newton's second law, the accerleration of the motion of a particle is determined by the mass of the particle and **forces** imposing on it.

Before we move forward, let's make some conventions. Particles will always move on a smooth manifold M of dimension m, which is called the **configuration space** of the mechanical system. In the worst situation, forces imposing on a particle will depend on both the position of the particle and its velocity at the position, so it can be regarded as a smooth vector-valued function $F\colon TM \to \mathbb{R}^m$ on the tangent bundle TM of M. Assume that the particle has mass 1, and let $(\gamma, \tau)\colon (a, b) \to TM$ be any curve where particles are moving on, then Newton's second law can be reformulated into the following:

1.1 Proposition (Newton's Second Law).

$$\begin{cases} \dot{\gamma}(t) = \tau(t) \\ \dot{\tau}(t) = \dfrac{1}{m} F(\gamma(t), \tau(t)) \end{cases}$$

Recall that for ordinary differential equations $\dot{x}(t) = f(x(t))$ with f smooth, the existence and uniqueness of solutions are always guaranteed:

1.2 Theorem ([Lee13], page 664). Suppose $U \subseteq \mathbb{R}^n$ open and $f\colon U \to \mathbb{R}^n$ a smooth vector-valued function. For the initial value problem

$$\begin{aligned} &\dot{x}(t) = f(x(t)), \\ &x(t_0) = x_0 \end{aligned}$$

for arbitrary $t_0 \in \mathbb{R}$ and $x_0 \in U$, there exists an open interval $I \subseteq \mathbb{R}$ containing t_0 and an open subset $U_0 \subseteq U$ containing x_0 such that for each $x \in U_0$, there is a C^1-map $x\colon I \to U$ solving this initial value problem. Moreover, any two such solutions agree on their common domain.

If $\theta\colon I \times U_0 \to U$ is the map $\theta(t, y) = x(t)$, where $x(t)$ solves the differential equation with initial condition $x(0) = y$, then θ is smooth.

In other words, any motion of particles, given the force function $F\colon TM \to \mathbb{R}^m$, is completely determined by the position it starts with and the instantaneous velocity at that position.

1.3 Example (Harmonic Oscillator). If the force is given by Hooke's law, $F(x) = -kx$, where k is a positive constant, then Newton's law can be written as

$$\begin{cases} \dot{\gamma}(t) = \tau(t) \\ \dot{\tau}(t) = -\dfrac{k}{m} \gamma(t) \end{cases}$$

On the space $TM \cong \mathbb{R}^2$. The general solution of this system is $x(t) = a\cos(\omega t) + b\sin(\omega t)$, where $\omega := \sqrt{k/m}$ is the frequency of oscillation.

Physically, this system can be described as a mass on a spring, where the force is proportional to the distance x that the string is stretched from its equilibrium position. The minus sign in $-kx$ indicates that the force pulls the oscillator back toward equilibrium.

1.4 Example (The Kepler Problem). The classical Kepler problem describes the motion of a planet around a star under the influence of gravity. The configuration space is

$M = \mathbb{R}^3 \setminus \{0\}$, and the force function is given by Newton's law of universal gravitation:

$$F(q) = -\frac{GM}{|q|^3} q,$$

where G is the gravitational constant, M is the mass of the star, and q is the position vector of the planet relative to the star. The negative sign indicates that the force is attractive, pulling the planet towards the star. The solutions to this system are conic sections (ellipses, parabolas, or hyperbolas) depending on the total energy of the system. However, this system is not easily solved from directly from the ODEs, and we will revisit this example later when we introduce some geometric techniques.

(1a) Hamilton formalism. Determination of the force function $F\colon TM \to \mathbb{R}^m$ on the whole tangent bundle TM is not practical in general, and we hope to have a simpler and more handable way to determine the mechanical system. Let

$$L = \frac{1}{2} m|v|^2 - V(q,v)$$

be the **action** of the mechanical system, where $V\colon TM \to \mathbb{R}$ is called the **potential energy** of the system. Physicists observe that

1.5 Principle (The Principle of Least Action). Motions of particles should minimize the action $L\colon TM \to \mathbb{R}$, i.e. if $x = (q,v)\colon (a,b) \to TM$ is the curve of motion of a particle, then for any other curve $y = (q',v')\colon (a,b) \to TM$ with the same endpoints as x, we have

$$\int_a^b L\left(x(t)\right) dt \leq \int_a^b L\left(y(t)\right) dt.$$

The principle of least action determines for us the motion of particles, given by the following **Euler-Lagrange equation**

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial v_i} = 0, \quad i = 1, \cdots, m. \tag{1}$$

The **Hamilton formalism** is an equivalent way to write down the system of motion from not the action functional L, but from the energy function E, where

$$E(q(t), v(t)) = \frac{1}{2} m v(t)^2 + V(q(t), v(t)).$$

This function is conserved along the trajectories of the system, i.e. assume that $x = (q,v)\colon (a,b) \to TM$ is the curve of motion of a particle, we must have

$$\frac{d}{dt} E\left(x(t)\right) = \left\langle \frac{\partial E}{\partial q}, \dot{q}(t) \right\rangle + \left\langle \frac{\partial E}{\partial v}, \dot{v}(t) \right\rangle = 0.$$

In fact, similar to the principle of least action, the energy function will completely determine the motion of particles, if we replace velocity coordinates v by momentum coordinates $p = mv$. Moments should be regarded as the dual of velocities, and the energy function

$$E(q,p) = \frac{1}{2m} p^2 + V(q)\colon T^*M \to \mathbb{R}$$

can be regarded as a function defined on the **phase space** T^*M of the system, which geometrically is the cotangent bundle of the configuration space M. The system of motions will be the following systems of ordinary differential equations

$$\begin{cases} \dot{q}(t) = \frac{\partial E}{\partial p}(q(t), p(t)) \\ \dot{p}(t) = -\frac{\partial E}{\partial q}(q(t), p(t)) \end{cases} \tag{2}$$

This is called the **Hamilton's system**, and we can interchange between these two formalisms via **Legendre transform**.

(1b) Geometric structures of mechanics. The reason that we turn our attentions to the Hamilton formalism and phase spaces is that the system of motions can be described simply by a smooth function $H\colon T^*M \to \mathbb{R}$, where the system of motions can be derived from an intrinsic geometric structure of the phase space T^*M.

(1b-i) Symplectic manifolds. We start with some slightly more general definitions, introduced originally by Arnold [Ad78] to describe the intrinsic geometric structure of classical mechanics.

1.6 Definition. For a given vector space V, a **symplectic structure** on V is a non-degenerate skew-symmetric bilinear form $\Omega\colon \wedge^2 V \to \mathbb{R}$. Let M be a smooth manifold. A **symplectic form** on M is a closed and non-degenerate 2-form $\omega \in C^\infty(M, \wedge^2 T^*M)$, i.e. for each $p \in M$, the bilinear form $\omega(p)\colon T_pM \times T_pM \to \mathbb{R}$ is non-degenerate. A **symplectic manifold** is a pair (M, ω) where M is a smooth manifold and ω is a symplectic form on M.

It's immediate from the definition that a symplectic manifold has to be even-dimensional. Non-degeneracy of ω also induces an isomorphism $\omega\colon TM \to T^*M$, which provides a bijection between vector fields $X \in C^\infty(M, TM)$ and 1-forms $\alpha \in C^\infty(M, T^*M)$ on M.

1.7 Lemma. A vector field X on (M, ω) is called **symplectic** if $\mathcal{L}_X\omega = 0$; a vector field X is symplectic if and only if the symplectic dual $\alpha(-) := \omega(-, X)$ is a closed 1-form; a vector field X is called **hamiltonian** if the symplectic dual α is an exact 1-form, i.e. there exists a smooth function $H\colon M \to \mathbb{R}$ such that $\alpha = dH$. In this case, we call H the **hamiltonian** of the hamiltonian vector field X.

Given a smooth function $H\colon M \to \mathbb{R}$, the vector field X_H dual to dH is called the **hamiltonian vector field** of H.

Proof. By Cartan's magical formula we have

$$\mathcal{L}_X\omega = d(X \lrcorner \omega) + X \lrcorner d\omega.$$

As ω is closed, $\mathcal{L}_X\omega = d(X \lrcorner \omega) = d\alpha$ is zero iff $d\alpha = 0$, i.e. α is a closed 1-form. $\square$

Note that the flow ϕ_H^t of a hamiltonian vector field X_H provides a smooth family of symplectomorphisms $\phi_H^t\colon M \to M$ for $t \in \mathbb{R}$.

1.8 Definition. We write $\mathrm{Symp}(M, \omega)$ for the group of symplectomorphisms of (M, ω), and $\mathrm{Ham}(M, \omega)$ for symplectomorphisms of M which are time 1 flow of some one-parameter family of hamiltonian vector fields $\{X_{H_t}\}_{t \in [0,1]}$ for some time-dependent Hamiltonians $\{H_t\}_{0 \le t \le 1}$.

1.9 REMARK. Though we only verify that for a single Hamiltonian $H\colon M \to \mathbb{R}$, the hamiltonian vector field X_H integrates to a smooth family of symplectomorphisms $\phi_H^t\colon M \to M$, this is also true for time-dependent hamiltonians $H_t\colon M \times [0,1] \to \mathbb{R}$, where the flow $\phi_{H_t}^t\colon M \to M$ is defined as the family of diffeomorphisms such that $\frac{d}{dt}\phi_{H_t}^t = X_{H_t}$ for each t.

$\mathrm{Ham}(M,\omega)$ is not only a subset of $\mathrm{Symp}(M,\omega)$ but actually a normal subgroup, as we will prove in the next Proposition.

1.10 PROPOSITION. Let ϕ, ψ be Hamiltonian diffeomorphisms generated by time-dependent Hamiltonians H_t, G_t, then

1. The composition $\phi \circ \psi$ is a Hamiltonian diffeomorphism generated by $H_t + G_t \circ \phi^{-1}$.
2. ϕ^{-1} is a Hamiltonian diffeomorphism generated by $-H_t \circ \phi$.
3. For any symplectomorphism $\chi \in \mathrm{Symp}(M,\omega)$, the conjugation $\chi^{-1} \circ \phi \circ \chi$ is a Hamiltonian diffeomorphism generated by $H_t \circ \chi$.

Proof. We only prove the first statement as a proof pattern. The reader can verify the remainings themselves. We write ϕ^t and ψ^t for the Hamiltonian flows generated by the given Hamiltonians, then by taking derivatives, we have

$$\left.\frac{d}{dt}\right|_t (\phi^t \circ \psi^t) = X_{H_t} \circ \phi^t \circ \psi^t + (\phi^t)_* X_{G_t} \circ \psi^t = X_{H_t} \circ \phi^t \circ \psi^t + X_{G_t \circ \phi^{-1}} \circ \phi^t \circ \psi^t,$$

which concludes that $\phi \circ \psi$ is a Hamiltonian diffeomorphism generated by $H_t + G_t \circ \phi^{-1}$. □

(1b-ii) "Physical phase space". Symplectic manifolds should be regarded as generalizations of the notion of phase spaces introduced beforehand, where we don't require the manifold M to be diffeomorphic to the cotangent bundle of some other smooth manifolds. In this paragraph, we will focus on "physical" phase spaces, i.e. cotangent bundle of smooth manifolds, to see the role that symplectic structures play in classical mechanics.

1.11 EXAMPLE. Before introducing general symplectic structures on physical phase spaces, let's look at the simplest example of symplectic manifolds: the **symplectic space** $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$ where $\omega_{\mathrm{std}} := \sum_{i=1}^n dq_i \wedge dp_i$ is the standard symplectic 2-form on $\mathbb{R}^{2n}$ with coordinates $(q_1, p_1, \cdots, q_n, p_n)$, regarding (q_i) as the position coordinates and (p_i) as the momentum coordinates. Given any smooth function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$, we can compute that $dH = \sum_{i=1}^n \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i$, and hence the Hamiltonian vector field is given by

$$X_H = \sum_{i=1}^n \left(-\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} + \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} \right).$$

Therefore an integral curve $\gamma(t) = (q_1(t), p_1(t), \cdots, q_n(t), p_n(t))$ of $-X_H$ is a solution of the system of ODEs

$$\begin{cases} \dot{q}_i(t) = \partial_{p_i} H(q(t), p(t)), \\ \dot{p}_i(t) = -\partial_{q_i} H(q(t), p(t)), \end{cases}$$

with given initial condition $\gamma(0) = (q_1(0), p_1(0), \cdots, q_n(0), p_n(0))$. This is exactly the Hamilton's system we have described, and therefore describes the motion of particles in the mechanical system determined by H.

Now let's provide a general construction of symplectic structures on physical phase spaces, T^*M, for any smooth manifold M. We start with a canonical differential 1-form defined on cotangent bundles:

1.12 Definition. The **tautological 1-form** λ_{tau} on T^*M is the differential 1-form in $C^\infty(T^*M, T^*T^*M)$ such that for all differential form $\alpha \in C^\infty(M, T*M)$, regarded as a section of the cotangent bundle T^*M, we have $\alpha^*\lambda_{\text{tau}} = -\alpha$.

Unfolding the definition, we see for each pair $(q,p) \in T^*M$, we have $(\lambda_{\text{tau}})_{(q,p)} = -pdq$, and therefore $\omega_{\text{std}} \coloneqq d\lambda_{\text{tau}}$ defines a symplectic structure on T^*M.

1.13 Definition. In T^*M, there is a notable canonical vector field $Z \in C^\infty(T^*M, TT^*M)$, called the **Liouville vector field**, which is the dual vector field of the tautological 1-form λ_{tau}. By Cartan's magical formula we quickly see that $\mathcal{L}_Z\omega_{\text{std}} = \omega_{\text{std}}$, which is not symplectic but exponentially scales the symplectic form ω_{std}.

1.14 Example. Let's revisit the harmonic oscillator. The energy function/hamiltonian for the harmonic oscillator is given by

$$H(q,p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2 \colon T^*\mathbb{R} \to \mathbb{R},$$

and we can therefore write down the system of motions as

$$\begin{cases} \dot{q}(t) = \partial_p H(q(t), p(t)) = \frac{1}{m}p(t), \\ \dot{p}(t) = -\partial_q H(q(t), p(t)) = -kq(t). \end{cases}$$

This is exactly the same system of ODEs we have written down before, and therefore describes the same motion of particles in the harmonic oscillator system.

1.15 Example. Let's also look at the Kepler problem. The energy function/hamiltonian for the Kepler problem is given by

$$H(q,p) = \frac{1}{2m}p^2 - \frac{GM}{|q|} \colon T^*(\mathbb{R}^3 \setminus \{0\}) \to \mathbb{R},$$

and we can therefore write down the system of motions as

$$\begin{cases} \dot{q}(t) = \partial_p H(q(t), p(t)) = \frac{1}{m}p(t), \\ \dot{p}(t) = -\partial_q H(q(t), p(t)) = -\frac{GM}{|q(t)|^3}q(t). \end{cases}$$

This is exactly the same system of ODEs we have written down before, and therefore describes the same motion of particles in the Kepler problem.

(1b-iii) Integrable systems. From the previous discussions, it's clear that classical mechanical systems can be completely described by a smooth function $H \colon M \to \mathbb{R}$ on a given symplectic manifold (M, ω), where the symplectic structure ω plays the role of an "intrinsic mechanical structure" that turns a smooth function up to transition (initial energy) into a system of motions where the function describes the energy of the system. Moreover, if we allow initial conditions of the system of motions to vary on M, solving system of motions is equivalent to determining the hamiltonian flow $\phi_H^t \colon M \to M$ of the hamiltonian vector field X_H of H, and when X_H is complete (true if in particular M is compact), ϕ_H^t is a well-defined family of symplectomorphisms of M for all $t \in \mathbb{R}$.

Solving this system of motions is however still difficult in general, as we need to consider all possible initial conditions, and the behaviour of the integral curve under different initial conditions can be drastically different. In this section we introduce one way to simplify the computations, which probably also appeared in some previous courses that audiences have taken, originally called **first integrals**.

1.16 Definition. Given a system of motions $\dot{\gamma}(t) = X_H(\gamma(t))$ on a symplectic manifold (M, ω), a smooth function $G\colon M \to \mathbb{R}$ is a **conserved quantity** if $\dfrac{d}{dt}G(\gamma(t)) = 0$ for all solutions $\gamma(t)$ of the system.

In particular, the Hamiltonian H itself is a conserved quantity. Using our symplectic geometric formulation, we have a more "geometric" description of conserved quantities. Recall that given two vector fields X, Y of a smooth manifold M, the **Lie bracket** $[X, Y]$ of X and Y is the vector field $[X, Y] = XY - YX$, regarded as an action on the space of smooth functions on M.

1.17 Definition. Let $G, K\colon M \to \mathbb{R}$ be smooth functions on a symplectic manifold (M, ω). We define the **Poisson bracket** of G and K to be the smooth function

$$\{G, K\} \coloneqq \omega(X_G, X_K).$$

1.18 Proposition. $X_{\{G,K\}} = -[X_G, X_K]$.

Proof. Note that

$$d\{G, K\} = d\left(X_K \lrcorner X_G \lrcorner \omega\right) = \mathcal{L}_{X_K}(dG) = \left.\frac{d}{dt}\right|_{t=0} \left(\phi^t_{X_K}\right)^* dG,$$

while on the other hand,

$$([X_G, X_K]) \lrcorner \omega = \left(\left.\frac{d}{dt}\right|_{t=0} X_{K \circ \phi^t_{X_G}}\right) \lrcorner \omega = \left.\frac{d}{dt}\right|_{t=0} \left(\phi^t_{X_G}\right)^* dK = -\left.\frac{d}{dt}\right|_{t=0} \left(\phi^t_{X_K}\right)^* dG,$$

and we conclude that $d\{G, K\} = -([X_G, X_K]) \lrcorner \omega$. □

1 Exercise. Show that G is a conserved quantity for H if and only if $\{G, H\} = 0$.

It's clear that

2 Exercise. Let G be a conserved quantity for H, then any integral curve of the hamiltonian vector field X_H lies on the level set $G^{-1}(c)$ for some real number c.

Therefore if we can find a conserved quantity G for H, we have a constraint for the system of motions determined by H: it has to lie on one of the real codimension 1 hypersurfaces $\{G(x) = c\}$. Moreover, note that the hamiltonian flow ϕ^t_G of G also preserves the level set $\{G = c\}$, we can take a further quotient of $\{G = c\}$ by identifying trajectories of X_G, which, when c is a regular value of G and ϕ^t_G acts freely, gives us a smooth manifold $M//_c G$ of dimension $2n - 2$, which inherits a symplectic structure from M. As the quotient by orbits of X_G preserves values of the hamiltonian H, the function $H\colon M \to \mathbb{R}$ also descends to a smooth function $\bar{H}\colon M//_c G \to \mathbb{R}$, and therefore we have reduced the problem of finding solutions to the system of motions on M to finding solutions to the system of motions on $M//_c G$, which is a symplectic manifold of 2-dimension lower. This process partially solves the equation in some sense, and if we can find n conserved quantities $G_1, \cdots, G_n$ such that $\{G_i, G_j\} = 0$ for all i, j, then we are able to reduce the system of motions to a system of motions on a manifold of dimension 0, and therefore obtain a complete description of the system via these conversed quantities.

1.19 Definition. Under this assumption and further assume that the hamiltonian flows $\phi^t_{G_i}$ of the convserved quantities are defined for all t, then we call the collection $\{G_1, \cdots, G_n\}$ of conserved quantities a **completely integrable system** on M.

Note that each $G_i \colon M \to \mathbb{R}$ provides a natural $\mathbb{R}$-action on M by the hamiltonian flow $\phi^t_{G_i} \colon M \to M$, and as $[X_{G_i}, X_{G_j}] = 0$ for all i, j, the n-tuple $(G_1, \cdots, G_n)$ induces a natural $\mathbb{R}^n$-action on M by the hamiltonian flows $x \mapsto \phi^{t_1}_{G_1} \circ \phi^{t_2}_{G_2} \circ \cdots \circ \phi^{t_n}_{G_n}(x)$ for $t = (t_1, \cdots, t_n) \in \mathbb{R}^n$ and $x \in M$. Moreover, the map $(G_1, \cdots, G_n) \colon M \to \mathbb{R}^n$ is $\mathbb{R}^n$-equivariant, where $\mathbb{R}^n$ acts trivially on the base.

1.20 Theorem (Little Arnold-Liouville Theorem). Let $(G_1, \cdots, G_n) \colon M \to \mathbb{R}^n$ be a completely integrable system and $\sigma \colon \mathbb{R}^n \supseteq U \to M$ a smooth map such that $(G_1, \cdots, G_n)\,\sigma(\vec{c}) = \vec{c}$, then all orbits of $\sigma(\vec{c})$ under the $\mathbb{R}^n$ action are diffeomorphic to $\mathbb{R}^k \times T^{n-k}$ for some $0 \leq k \leq n$, where $T^{n-k} \cong \mathbb{R}^{n-k}/\Lambda$ is a torus of dimension $n-k$ for some lattice $\mathbb{Z}^{n-k} \cong \Lambda \subseteq \mathbb{R}^{n-k}$.

Proof. It suffices to look at the stabilizer of the $\mathbb{R}^n$-action on a given point $\sigma(\vec{c})$. The existence of σ implies that the differential of the map $(G_1, \cdots, G_n)$ has full rank at $\sigma(\vec{c})$, so that the hamiltonian vector fields $\{X_{G_1}, \cdots, X_{G_n}\}$ have to be linearly independent near $\sigma(\vec{c})$. Therefore the image $U_\varepsilon(0)\sigma(\vec{c})$ of a very small open subset $0 \in U_\varepsilon(0) \subseteq \mathbb{R}^n$ under the $\mathbb{R}^n$-action has to be n-dimensional, hence an open neighbourhood of $\sigma(\vec{c})$ in $(G_1, \cdots, G_n)^{-1}(c)$ as the vector fields have to preserve the fibres. This implies that the stabilizer subgroup $\Lambda_{\sigma(\vec{c})}$ at $\sigma(\vec{c})$ is discrete. Using the time of the hamiltonian flow $\phi^{t_i}_{G_i}$ of G_i as coordinates on $\mathbb{R}^n$, we can identify the stabilizer subgroup $\Lambda_{\sigma(\vec{c})}$ with the **period lattice** $\Lambda_{\sigma(\vec{c})} = \left\{(t_1, \cdots, t_n) \in \mathbb{R}^n \middle| \phi^{t_1}_{G_1} \circ \cdots \circ \phi^{t_n}_{G_n}\sigma(\vec{c}) = \sigma(\vec{c})\right\} \subseteq \mathbb{R}^n$. This concludes the proof. □

1.21 Lemma. Furthermore, if c is a regular value of $\vec{G} = (G_1, \cdots, G_n)$, then all connected components of the fibre $\vec{G}^{-1}(c)$ are orbits of the $\mathbb{R}^n$-action.

Proof. If c is a regular value of $\vec{G}$, then all the points on $\vec{G}^{-1}(c)$ are regular points of $\vec{G}$, i.e. the differential $d\vec{G}$ has full rank. This implies that the $\mathbb{R}^n$-action on any point x of $\vec{G}^{-1}(c)$ provides an open neighbourhood of x in $\vec{G}^{-1}(c)$, so that an $\mathbb{R}^n$-orbit is open in $\vec{G}^{-1}(c)$. On the other hand, it's nor hard to check that an $\mathbb{R}^n$-orbit also has to be closed in $\vec{G}^{-1}(c)$, and therefore it has to coincide with connected components of $\vec{G}^{-1}(c)$. □

1.22 Example. Consider the harmonic oscillator, which under Hamiltonian formalism is given by the smooth function $H(q,p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2$ on the symplectic manifold $(T^*\mathbb{R}, \omega_{\mathrm{std}})$. Set for simplicity $m = k = 1$ so that $H(q,p) = \frac{1}{2}(p^2 + q^2)$, with corresponding hamiltonian vector field $X_H = -p\frac{\partial}{\partial q} + q\frac{\partial}{\partial p}$. H is clearly a conserved quantity of this system, and since $\dim_{\mathbb{R}} T^*\mathbb{R} = 2$, $H \colon T^*\mathbb{R} \to \mathbb{R}$ forms a completely integrable system. Note that $(0,0)$ is a critical point of H, so that the regular values of H lie on the open interval $(0, +\infty) \subseteq \mathbb{R}$, where fibres are of the form

$$H^{-1}(c) = \left\{(q,p) \in T^*\mathbb{R} \middle| q^2 + p^2 = 2c\right\} \cong \mathbb{S}^1.$$

The integral curve of X_H is computed in Example 1.3: they are of the form $\gamma(t) = (a\cos t + b\sin t, -a\sin t + b\cos t)$ for some chosen constant $a, b \in \mathbb{R}$. We assume further that $\gamma(0) = (\sqrt{2c}, 0)$ so that $a = \sqrt{2c}$ and $b = 0$. It's then clear that $\{\gamma(t)\}_{t\in\mathbb{R}}$ covers the whole fibre $H^{-1}(c)$, with relation $\gamma(2\pi) = \gamma(0)$, and hence the period lattice at $(\sqrt{2c}, 0)$ is $2\pi\mathbb{Z} \subseteq \mathbb{R}$. It's not hard to check that the period lattices at any point of $H^{-1}(c)$, $c > 0$, are all given by $2\pi\mathbb{Z}$.

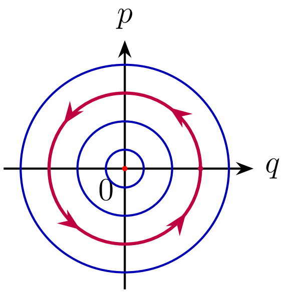

Figure 1: The phase space of the harmonic oscillator, where fibres of H are circles and the flow of X_H is periodic with period 2π.

As we can assign to each $x \in M$ a period lattice $\Lambda_x^{\vec{G}}$ in $\mathbb{R}^n$, patching all the lattices at each point of M provides us a subspace $\Lambda^{\vec{G}} \subseteq \mathbb{R}^n \times M$. This subspace is often what we refer to as the **period lattice** of the completely integrable system $\vec{G}\colon M \to \mathbb{R}^n$.

1.23 Definition. The period lattice $\Lambda^{\vec{G}}$ is **standard** if the projection $\Lambda^{\vec{G}} \to \mathbb{R}^n$ is the constant lattice $2\pi\mathbb{Z}^{n-k} \subseteq \mathbb{R}^n$.

Assume that fibres of $\vec{G}\colon M \to \mathbb{R}^n$ are all regular and connected, then Lemma 1.21 implies that the $\mathbb{R}^n$-action on M is transitive on each fibre, and hence the period lattice $\Lambda^{\vec{G}}$ descends to a subspace $\overline{\Lambda}^{\vec{G}} \subseteq \mathbb{R}^n \times B$ over $B \coloneqq \vec{G}(M) \subseteq \mathbb{R}^n$ whose fibre at each point $\vec{b} \in B$ is a lattice $\overline{\Lambda}_{\vec{b}}^{\vec{G}} \subseteq \mathbb{R}^n$.

2. FLEXIBILITY: MOSER'S ARGUMENT

The first classes of properties we saw about symplectic manifolds in history are closer to smooth topology, which we now call **flexibility** properties. Such properties were originally discovered by Moser [Mos65] for volume-preserving diffeomorphisms, whose arguments were easily adapted to symplectic manifolds. Several results concerning flexibility properties of symplectic maifolds were then quickly discovered, for instance neighbourhood theorems for symplectic and several special types of submanifolds of a given symplectic manifold, where we can arrange so that the symplectic structure in this neighbourhood is standard. As an outcome, there was a famous slogan which was probably known to many of the readers: **"Symplectic geometry has no local invariants"**, comparing to Riemannian manifolds where sectional curvatures obstruct us from establishing neighbourhood theorems.

In this section we will go through Moser's technique and prove several neighbourhood theorems for symplectic manifolds. Before going into the details, let's first have a look at some natural "non-physical" phase spaces, which arises from the study of complex manifolds.

(2a) Complex and almost-Kähler manifolds. Holomorphic geometry is not of major concern in this course, but the shadow of complex geometry will keep arising during the lecture, so let's spend a little bit of time to introduce some of the basic definitions. Readers who want to know more about complex geometry can go to D. Huybrechts [Huy05] for help.

2.1 DEFINITION. A **complex manifold** is defined very similarly to a smooth manifold, where we look at a topological space M so that it's locally diffeomorphic to $\mathbb{C}^n$ for some uniform n, and the transition function between different local charts are required to have complex linear differentials. Equivalently, they should all be holomorphic.

There is a special class of complex manifolds called **Kähler manifolds**, which are complex manifolds M equipped with a "good metric", i.e. a Riemannian metric g that is compatible with the complex structure J on M in the sense that $g(JX, JY) = g(X, Y)$ for all vector fields X, Y on M. The Riemannian metric g together with the complex structure J provides us a symplectic structure ω on M by the formula $\omega(X, Y) = g(JX, Y)$, which is usually referred to the **Kähler form** of the Kähler manifold (M, g, J).

2.2 EXAMPLE. The trivial example of the complex vector space $\mathbb{C}^n$ with the standard symplectic structure $\omega_{\text{std}} = \sum_{i=1}^{n} dx_i \wedge dy_i = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$ is clearly a Kähler manifold.

Let's look at a non-trivial example. Consider the quotient space

$$\mathbb{C}P^n := \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^\times,$$

where the group $\mathbb{C}^\times$ acts on $\mathbb{C}^{n+1} \setminus \{0\}$ by diagonal scalar multiplication. Recalling the proof that $\mathbb{R}P^n$ is a smooth manifold shows that $\mathbb{C}P^n$ is a complex manifold of dimension n, with canonical open cover

$$U_i = \{[z_0 : z_1 : \cdots : z_n] | z_i = 1\} \cong \mathbb{C}^n.$$

The symplectic structure on $\mathbb{C}P^n$ whose restriction to each U_i is given by

$$\omega_{\text{FS}}|_{U_i} := \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\left(1 + \sum_{j\neq i}|z_j|^2\right)$$

is a Kähler form on $\mathbb{C}P^n$, called the **Fubini-Study form** on $\mathbb{C}P^n$.

Let's look at the special case when $n = 1$. In this case we only have two charts $U_1 \cong \mathbb{C}_w$ and $U_0 \cong \mathbb{C}_z$ with

$$\omega_{FS}|_{U_0} = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\left(1+|w|^2\right) = \frac{\sqrt{-1}}{2}\partial\frac{wd\bar{w}}{1+|w|^2} = \frac{\sqrt{-1}}{2}\frac{dw\wedge d\bar{w}}{(1+|w|^2)^2},$$

and similarly

$$\omega_{FS}|_{U_1} = \frac{\sqrt{-1}}{2}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}.$$

Inside $U_0 \cap U_1 = \{[w : z] | w \neq 0, z \neq 0\} \cong \mathbb{C}^*$ where w is identified with z^{-1} under this identification, we have $\omega_{FS}|_{U_0}|_{U_0\cap U_1} = \frac{\sqrt{-1}}{2}\frac{dz\wedge d\bar{z}}{(1+|z|^{-2})^2}\frac{1}{|z|^4} = \frac{\sqrt{-1}}{2}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2} = \omega_{FS}|_{U_1}|_{U_0\cap U_1}$, so that ω_{FS} is well-defined on the whole $\mathbb{C}P^1 \cong \mathbb{S}^2$.

To compute the volume of $\mathbb{P}^1$, it suffices to restrict our attention to $U_0 \subseteq \mathbb{P}^1$ as the complement clearly has measure 0. We can then integrate directly to get

$$\int_{\mathbb{C}}\frac{\sqrt{-1}}{2}\frac{1}{(1+|z|^2)^2}dz\wedge d\bar{z} = \int_0^{2\pi}\int_0^{\infty}\frac{r}{(1+r^2)^2}drd\theta = \pi.$$

2.3 EXAMPLE. Let $M \hookrightarrow \mathbb{C}P^n$ be a smooth complex manifold holomorphically embedded into $\mathbb{C}P^n$, then the restriction of ω_{FS} to M induces a Kähler form on M, which we still call the **Fubini-Study form** on M.

There is then a natural question of comparing between symplectic and Kähler manifolds, and the answer is clear: there're much more symplectic manifolds than Kähler ones, which we will repeatly learn during the lecture. However, any symplectic manifold is automatically "almost Kähler", which we will discuss a little bit now.

2.4 DEFINITION. An **almost complex structure** on a smooth manifold M is a smooth vector bundle endomorphism $J\colon TM \to TM$ such that $J^2 = -\operatorname{Id}$. A symplectic structure ω on M is said to be **compatible** with the almost complex structure J if $\omega(JX, JY) = \omega(X, Y)$ for all vector fields X, Y on M, and $g(X, Y) := \omega(JX, Y)$ defines a Riemannian metric on M. In this case, we call the triple (M, ω, J) an **almost Kähler manifold**.

2.5 PROPOSITION. A symplectic manifold (M, ω) is always almost Kähler. In other words, there exists an almost complex structure J on M which is compatible with ω. Moreover, the space of such almost complex structures is contractible.

Proof. The proof roughly follows [Gro85]. For more details, see [AL94]. We first handle the local case when $M = \mathbb{R}^{2n}$ and $\omega = \omega_{\mathrm{std}}$ is the standard symplectic structure on $\mathbb{R}^{2n}$. In this case, a linear almost complex structure is a matrix $J \in \operatorname{End}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ satisfying $J^2 = -\operatorname{Id}$ and $J^T\omega_{\mathrm{std}}J = \omega_{\mathrm{std}}$. Write $\mathcal{J}(\mathbb{R}^{\nvDash\ltimes})$ to be the space of all complex structures on $\mathbb{R}^{2n}$, and let $J_0 = \omega_{\mathrm{std}}$ be the **standard complex structure**, then we have

2.6 LEMMA. The map $J \mapsto (J + J_0)^{-1} \circ (J - J_0)$ is a diffeomorphism from $\mathcal{J}(\mathbb{R}^{2n})$ onto the open unit ball in the vector space of symmetric matrices S such that $J_0S + SJ_0 = 0$. Here the topology is given by the operator norm

$$\|S\| = \inf\{c|\omega(Sx, J_0Sx) \leq c\omega(x, J_0x)\ \forall x\}.$$

This readily implies that the space of linear complex structures on $\mathbb{R}^{2n}$ is contractible.

For general manifold M, the tangent bundle $TM \to M$ is a symplectic vector bundle, meaning the symplectic structure ω_M provides a symplectic structure on each fibre T_pM for all $p \in M$. Therefore the space of complex structures $\mathcal{J}(T_pM)$ on the vector space T_pM is contractible. By taking the space of linear complex structures on each tangent space we get a fibre bundle $\mathcal{J}(M) \to M$ over M, whose fibres are contractible by Lemma 2.6.

Fibre bundles over a smooth manifold M with contractible fibres always admit global sections, and the space of sections are contractible. To show this, first note that M admits a CW decomposition, so that we can extend sections inductively from 0-skeleton to the top skeleton, by choosing arbitrary almost complex structures on each vertex, and for edges connecting vertices, choosing any path connecting the corresponding almost complex structures chosen on the vertex. We then see that obstructions to extending sections from k-skeleton to $(k+1)$-skeleton come from homotopy groups of $\mathcal{J}(\mathbb{R}^{2n})$, which are trivial since it's contractible. Therefore we always obtain a global continuous section of $\mathcal{J}(M)$, which can be perturbed to be smooth due to approximation theorems [Hir76, Theorem 2.2]. Isotopies between sections can be regarded as sections over the product manifold $M \times [0,1]$ with fixed boundary conditions, and the same argument applies. For more details on obstruction theory, see [Whi78]. □

Let's finish the proof of the lemma.

Proof of Lemma 2.6. This is proved by a direct verification. As J and J_0 are all complex structures, we have $\omega(x, (J+J_0)x) > 0$ if $x \neq 0$, so $J + J_0$ is invertible. Since $g(x,y) := \omega(x, Jy)$ defines a symmetric bilinear form, we must have $(\omega_{\text{std}}J)^T = \omega_{\text{std}}J$, while $\omega_{\text{std}} = J_0$ satisfies $J_0^{-1} = -J_0$. Therefore $(J+J_0)^{-1} \circ (J - J_0) = (J_0^{-1}J + \mathrm{Id})^{-1} \circ (J_0^{-1}J - \mathrm{Id})$ is symmetric. Write $S = (J+J_0)^{-1} \circ (J-J_0) = (J_0^{-1}J+\mathrm{Id})^{-1}(J_0^{-1}J-\mathrm{Id}) = (A+\mathrm{Id})^{-1}(A-\mathrm{Id})$, then the fact that $\|S\| < 1$ is equivalent to $\|Ax - x\|^2 < \|Ax + x\|^2$ for all $x \neq 0$. Let's check that

$$\|Ax + x\|^2 - \|Ax - x\|^2 = 4\omega(x, J_0Ax) = 4\omega(x, Jx) > 0.$$

Therefore the image should satisfy $\|S\| < 1$. To verify the last identity, note that

$$\omega(Sx, y) + \omega(x, Sy) = \frac{1}{2}\left(\omega((\mathrm{Id}+S)x, (\mathrm{Id}+S)y) - \omega((\mathrm{Id}-S)x, (\mathrm{Id}-S)y)\right),$$

where $\omega((\mathrm{Id}+S)x, (\mathrm{Id}+S)y) = \omega(2Ax, 2Ay) = 4\omega(x,y)$ and $\omega((\mathrm{Id}-S)x, (\mathrm{Id}-S)y) = \omega(2x, 2y) = 4\omega(x,y)$. Since this holds for all (x,y), by non-degeneracy of ω we have $SJ_0 + J_0S = 0$.

Finally, for each symmetric S with given conditions we know that $\mathrm{Id} - S$ is invertible so that

$$J = J_0(\mathrm{Id}+S) \circ (\mathrm{Id}-S)^{-1}$$

defines an endomorphism of $\mathbb{R}^{2n}$. We can check that $J^2 = -\mathrm{Id}$, $J^T\omega_{\text{std}}J = \omega_{\text{std}}$ and $(J\omega)^T = J\omega$ by direct computations, which we omit here. □

However, the converse is not true. These are far away from what we want to discuss in this course, so we will only give a brief overview. The existence of almost complex structures on a general smooth manifold of even dimension has topological obstructions, coming from the fact that with such an almost complex structure the tangent bundle

TM is a complex vector bundle, so that its **first Chern class** $c_1(TM) \in H^2(M;\mathbb{Z})$ is well-defined. When $\dim M = 4$, there is an identity

$$\langle c_1^2(TM), [M]\rangle = 2\chi(M) + 3\sigma(M),$$

where $\chi(M)$ is the Euler characteristic of M and $\sigma(M)$ is the signature of M. It was further proved that

2.7 THEOREM (Wu [Wu52]). Homotopy classes of almost complex structures on a smooth manifold M of dimension 4 are in one-to-one correspondence with the elements $c \in H^2(M;\mathbb{Z})$ lifting $w_2(TM) \in H^2(M;\mathbb{Z}/2)$ and such that $\langle c^2, [M]\rangle = 2\chi(M) + 3\sigma(M)$.

In the same paper of Wu [Wu52], he also constructed an almost complex structure on the 6-sphere $\mathbb{S}^6$, and showed that the only even-dimensional spheres that can have almost complex structures are $\mathbb{S}^2$ and $\mathbb{S}^6$.

3 EXERCISE. Show that $\mathbb{S}^6$ is not a symplectic manifold.

Hint: a symplectic structure is a priori a closed 2-form. Look at the de Rham cohomology of $\mathbb{S}^6$.

Therefore $\mathbb{S}^6$ is an example of an almost complex manifold that is not almost Kähler.

There're also obstructions from upgrading any almost Kähler manifold to a Kähler manifold, which is provided by the **Nijenhuis tensor** [NN57, Che55]

$$N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

for vector fields $X, Y \in C^\infty(M,TM)$, which can be regarded also as failures of J being parallel with respect to the Levi-Civita connection of the metric $g(X,Y) = \omega(JX,Y)$. The theorem of Newlander-Nirenberg [NN57] showed that whenever the Nijenhuis tensor vanishes, we can find an integrable complex structure on M making M into a Kähler manifold.

(2b) Moser's isotopy. Now we step into flexibility theorems for symplectic manifolds. The proof patterns are similar so we will only go through a few proofs and the remainings are left as exercises.

2.8 PROPOSITION. Let M be a smooth even-dimensional manifold and $Q \subseteq M$ a compact submanifold. Assume that ω_0 and ω_1 are two symplectic forms on M such that $\omega_0|_Q = \omega_1|_Q$ and $[\omega_0] = [\omega_1] \in H^2(M;\mathbb{R})$, then there exists open neighbourhoods $\mathrm{Nbd}_0(Q)$ and $\mathrm{Nbd}_1(Q)$ of Q in M and a diffeomorphism $\psi\colon \mathrm{Nbd}_0(Q) \to \mathrm{Nbd}_1(Q)$ such that $\psi^*\omega_1 = \omega_0$ and $\psi|_Q = \mathrm{Id}_Q$.

Proof. As Q is a closed smooth submanifold of M, the restriction $i^*TM \to Q$ of the tangent bundle of M to Q under the inclusion $i\colon Q \hookrightarrow M$ contains TQ as a subbundle, with quotient $N_MQ = i^*TM/TQ$. Pick a Riemannian metric on M so that N_MQ can be embedded back into i^*TM as the orthogonal complement of TQ.

Under the Riemannian metric, for each $q \in Q$, the exponential map [Cha93, Theorem I.3.2] $\exp_q\colon T_qM \to M$ determines a diffeomorphism from an open neighbourhood of q in T_qM to an open neighbourhood of q in M, whose restriction to T_qQ has image in Q. With local coordinate charts of M subordinate to the embedding $Q \hookrightarrow M$, we can see that $\exp^\perp\colon N_MQ \to M$ sending $(N_MQ)_q$ via $\exp_q$ to M is smooth. Since $\exp_q$

is a diffeomorphism onto its image in an open neighbourhood of q, we can also check that $\exp^{\perp}$ is a diffeomorphism from an open neighbourhood of $Q \hookrightarrow N_M Q$ to an open neighbourhood of $Q \hookrightarrow M$.

We can therefore pull back the symplectic structure ω_0 and ω_1 to an open neighbourhood of Q in $N_M Q$, which by abuse of notations still denoted by ω_0 and ω_1. Consider the family $\omega_t = t\omega_1 + (1-t)\omega_0$, then ω_t is a family of symplectic forms in an open neighbourhood of Q in $N_M Q$ with $[\omega_t] = [\omega_0] = [\omega_1]$ for all $t \in [0,1]$.

We look for family of diffeomorphisms $\{\psi_t \colon \mathrm{Nbd}_{N_M Q}(Q) \to N_M Q\}_{0 \le t \le 1}$ such that $\psi_0 = \mathrm{Id}$, $\psi_t|_Q = \mathrm{Id}$ and $\psi_t^* \omega_t = \omega_0$ for all $t \in [0,1]$. Differentiating the equation $\psi_t^*\omega_t = \omega_0$ with respect to t, we get

$$\psi_t^* \left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t \right) = 0,$$

where X_t is the derivative of the diffeomorphism ψ_t with respect to t, or in other words the vector field $d\psi_t(\partial/\partial t)$ regarding $\psi_t \colon \mathrm{Nbd}_{N_M Q}(Q) \times [0,1]_t \to N_M Q$ as a map on the product. Therefore we must have

$$\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t = 0.$$

By Cartan's magical formula, $\mathcal{L}_{X_t}\omega_t = d(X_t \lrcorner \omega_t) + X_t \lrcorner d\omega_t = d(X_t \lrcorner \omega_t) := d\sigma_t$ and $\frac{d}{dt}\omega_t = \omega_1 - \omega_0$. As $[\omega_1] = [\omega_0]$, the derivative $\dot\omega_t = \omega_1 - \omega_0$ is exact and hence this equation has a solution σ_t.

However, it's a problem whether this σ_t can be chosen to satisfy $\sigma_t|_Q = 0$, and we provide here a direct construction via **integration over fibres**. Consider a family of maps $\psi^t \colon N_M Q \times [0,1] \to M$ given by

$$\psi^t(q, v) = \exp_q^{\perp}(q, tv),$$

then for a chosen neighbourhood $\mathrm{Nbd}_{N_M Q}(Q)$, ψ^t is a diffeomorphism onto its image whenever $t \neq 0$, and ψ^0 contracts $\mathrm{Nbd}_{N_M Q}(Q)$ to Q. Up to identifying a neighbourhood of Q with an open subset of M, we also get $\psi^1 = \mathrm{Id}$. Therefore we can compute that $(\psi^1)^*(\dot\omega_t) = \dot\omega_t$ and $(\psi^0)^*\dot\omega_t = 0$. Cartan's magical formula implies that

$$\frac{d}{dt}(\psi^t)^*\dot\omega_t = (\psi^t)^*\left(\mathcal{L}_{X_t}\dot\omega_t\right) = d\left((\psi^t)^*(X_t \lrcorner \dot\omega_t)\right) = d\sigma_t,$$

where $\sigma_t := (\psi^t)^*(X_t \lrcorner \dot\omega_t)$ is a family of 1-forms determined by the vector field $X_t = \left(\frac{d}{dt}\psi^t\right) \circ \psi^{-t}$ is smooth in t. Though X_t is not defined at $t = 0$, but we can check directly that σ_t is still smooth for $t \in [0,1]$[1]. Now by Newton-Leibnitz formula,

$$\dot\omega_t = (\psi^1)^*\dot\omega_t - (\psi^0)^*\dot\omega_t = \int_0^1 \frac{d}{dt}(\psi^t)^*\dot\omega_t dt = \int_0^1 d\sigma_t dt = d\left(\int_0^1 \sigma_t dt\right) = d\sigma,$$

with $\sigma|_Q = 0$. Since ω_t is non-degenerate, σ induces a unique smooth vector field X_t so that $\sigma = X_t \lrcorner \omega_t$. As $\sigma|_Q = 0$, we have $X_t|_Q = 0$, and hence the flow $\phi_{X_t}^t$ of X_t restricts to identity on Q for all $t \in [0,1]$, with $\phi_{X_t}^0 = \mathrm{Id}$ and $(\phi_{X_t}^1)^*\omega_1 = \omega_0$ as discussed.

Note that the flow $\phi_{X_t}^1$ is not defined everywhere in $\mathrm{Nbd}_{N_M Q}(Q)$, but as $\phi_{X_t}^t(Q) = Q$, we can choose a smaller neighbourhood $\mathrm{Nbd}_0(Q)$ of Q in $N_M Q$ such that $\phi_{X_t}^1$ is well-defined, and write the image of $\phi_{X_t}^1$ as $\mathrm{Nbd}_1(Q)$. This is the desired diffeomorphism. □

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[1] We need a computation of differential of exponential maps along radial vectors, and the result is simple. See any Riemannian geometry books, e.g. [Cha93].

Set Q to be a point, then we get the famous **Darboux neighbourhood theorem**

2.9 THEOREM (Darboux). Let (M, ω) be a symplectic manifold and $p \in M$ any point, then there exists an open neighbourhood $p \in U \subseteq M$ of p in M, an open neighbourhood $0 \in U \subseteq \mathbb{R}^{2n}$ of the origin in $\mathbb{R}^{2n}$ and a diffeomorphism $\psi\colon U \to V$ such that $\psi^*\omega = \omega_{\mathrm{std}}$ and $\psi(p) = 0$.

4 EXERCISE. Prove Darboux' theorem directly without referring to Proposition 2.8.

This procedure of finding diffeomorphisms from constructing dual vector fields originated from Moser's work [Mos65], and this approach can be found in the great book of McDuff-Salamon [MS17]. Another application of Moser's argument shows that symplectic structures on compact symplectic manifolds are completely determined by their cohomology classes $H^2_{dR}(X; \mathbb{R})$.

2.10 THEOREM. Let M be a compact smooth manifold of dimension $2n$ and $\{\omega_t\}_{0\leq t\leq 1}$ be a smooth family of cohomologous symplectic forms on M, then there exists a family of diffeomorphisms $\{\psi^t\colon M \to M\}_{0\leq t\leq 1}$ such that $(\psi^t)^*\omega_t = \omega_0$ for all $t \in [0, 1]$.

Proof. We omit the procedure of running Moser's argument, and it suffices for us to check that there exists a smooth family of 1-forms σ_t such that $\dot{\omega}_t = d\sigma_t$. To achieve this, we consider an open cover of M where any of the finite intersections can be identified with an open subset of $\mathbb{R}^{2n}$ [BT82, Theorem 5.1], and pick a partition of unity subordinate to this open cover so that we can assume the two-forms $\dot{\omega}_t$ are supported in a compact subset in $\mathbb{R}^{2n}$. We can then pick a point p inside the support and run the integration over fibres as in Proposition 2.8 to get a smooth family of 1-forms σ_t such that $\dot{\omega}_t = d\sigma_t$.[2] For two such charts U, V we can find two a priori different families of 1-forms σ_t^U and σ_t^V, but they differ in $U \cap V$ by a family of exact 1-forms, so we can run the integration over fibres again to get a correction family of exact 1-forms σ_t^{UV} such that $\sigma_t^U - \sigma_t^V = \sigma_t^{UV}$, and therefore by adding this 1-form to one of the two families we get a family of 1-forms σ_t defined on the union $U \cup V$. We can therefore get a global 1-form σ_t defined on M by induction. □

(2c) Special submanifolds. After introducing the basics of Moser's argument, let's look at its effectivity by proving several neighbourhood theorems for certain special type of submanifolds of a given symplectic manifold (M, ω).

2.11 DEFINITION. A smooth submanifold $N \subseteq M$ of a symplectic manifold (M, ω) is **symplectic** if the restriction $\omega|_N$ is a symplectic structure on N.

A vector bundle $E \to N$ is a **symplectic vector bundle** if there exists a smooth family of non-degenerate skew-symmetric bilinear forms $\omega_p\colon E_p \times E_p \to \mathbb{R}$ for all $p \in N$.

In particular, a symplectic structure on M induces a symplectic structure on the tangent bundle TM, so the restriction $TM|_N$ of the tangent bundle of M to N is a symplectic vector bundle over N, whose subbundle $TN \hookrightarrow TM|_N$ is also symplectic.

2.12 DEFINITION. The symplectic orthogonal complement TN^ω of TN in $TM|_N$ is called the **symplectic conormal bundle** of N in M. It's clear that $(TN)^\omega$ is also a symplectic vector bundle over N.

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[2] Note that the 1-forms σ_t are timewise provided by integrations with respect to the exponential maps, and the smooth dependence on t can be checked directly from the formula.

Writte $\pi\colon TN^\omega \to N$ be the bundle map, and write $\omega_N = \omega|_N$ and $\Omega = \omega|_{TN^\omega}$ be the restricted symplectic structures from M and TM to N and TN^ω respectively, then we can check that $\pi^*\omega_N + \Omega$ defines a symplectic structure on the total space of the symplectic conormal bundle TN^ω, which we write as ω_{TN^ω}.

5 Exercise. Prove the following neighbourhood theorem: for $N \subseteq M$ a compact symplectic submanifold, there exists an open neighbourhood $\mathrm{Nbd}_M(N)$ of N in M and an open neighbourhood $\mathrm{Nbd}_{TN^\omega}(N)$ of the zero section $N \hookrightarrow TN^\omega$ in the symplectic conormal bundle TN^ω and a diffeomorphism $\psi\colon \mathrm{Nbd}_{TN^\omega}(N) \to \mathrm{Nbd}_M(N)$ such that $\psi|_N = \mathrm{Id}$ and $\psi^*\omega = \omega_{TN^\omega}$.

Despite symplectic submanifolds, there're some other special submanifolds that we are interested in.

2.13 Definition. A smooth submanifold $N \subseteq M$ of a symplectic manifold (M, ω) is **isotropic** if the restriction $\omega|_N$ is identically zero, or equivalently $TN \subseteq TN^\omega$.

A smooth submanifold $N \subseteq M$ of a symplectic manifold (M, ω) is **coisotropic** if $TN^\omega \subseteq TN$.

A smooth submanifold $N \subseteq M$ of a symplectic manifold (M, ω) is **Lagrangian** if $TN = TN^\omega$, or equivalently N is both isotropic and coisotropic.

6 Exercise. Show that an isotropic submanifold $N \subseteq M^{2n}$ has to have dimension at most n, and a coisotropic submanifold $N \subseteq M^{2n}$ has to have dimension at least n. In particular, a Lagrangian submanifold $N \subseteq M^{2n}$ has to be half-dimensional.

7 Exercise. Show that a real hypersurface $C \subseteq M$ of a symplectic manifold M is always coisotropic. Similarly, a one-dimensional submanifold $I \subseteq M$ is always isotropic.

2.14 Example. For any smooth manifold L, the "physical phase space" T^*L has natural Lagrangian submanifolds: the zero section $L \hookrightarrow T^*L$ and any cotangent fibres $T_q^*L \hookrightarrow T^*L$ for $q \in L$.

8 Exercise. Inside the cotangent bundle T^*L, find the equivalence condition for a graph

$$\Gamma(\alpha) := \{(x, \alpha_x) \in T^*L | x \in L\}$$

of a differential 1-form $\alpha \in C^\infty(L, T^*L)$ to be Lagrangian in T^*L.

2.15 Definition. Two Lagrangian submanifolds $L_0, L_1 \subseteq M$ are **Lagrangian isotopic** if there is a smooth family of Lagrangian submanifolds $\{L_t\}_{0 \le t \le 1}$ in M. They are furthermore **hamiltonian isotopic** if there exists a hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M, \omega)$ such that $\phi(L_0) = L_1$.

9 Exercise. Show that $\Gamma(\alpha)$, once being Lagrangian, is always Lagrangian isotopic to the zero section. Show that it is furthermore hamiltonian isotopic to the zero

section $L \hookrightarrow T^*L$ if and only if α is an exact 1-form.

10 Exercise. Show that for any compact Lagrangian submanifold $L \hookrightarrow M$ of a symplectic manifold (M, ω), there exists an open neighbourhood $\mathrm{Nbd}_M(L)$ of L in M and an open neighbourhood $\mathrm{Nbd}_{T^*L}(L)$ of the zero section $L \hookrightarrow T^*L$ in the cotangent bundle T^*L and a diffeomorphism $\psi\colon \mathrm{Nbd}_{T^*L}(L) \to \mathrm{Nbd}_M(L)$ such that $\psi|_L = \mathrm{Id}$ and $\psi^*\omega = \omega_{\mathrm{std}}$.

Hint. The difficult part is to identify the normal bundle of L to M with T^*L and find an embedding $T^*L \supseteq U \hookrightarrow M$ such that the pullback of ω restricts to ω_{std} along the zero section $L \hookrightarrow T^*L$. Pick an almost complex structure J on M compatible with ω, then we can check that $JTL \subseteq TM|_L$ is a Lagrangian subbundle of $TM|_L$ that is transversal to TL, and since J is an isomorphism of bundles, we see that $JTL \cong TL$. Let g_J be the Riemannian metric defined by ω and J, then g_J provides the identification of JTL with T^*L, the cotangent bundle of L. We write $\Phi\colon T^*L \to TL$ be the corresponding isomorphism, and via exponential map we define the map $\phi\colon T^*L \to M$ by

$$\phi(q, \alpha) = \exp_q(J_q \Phi_q(\alpha)).$$

By definition, for $v = v_0 + v_1$ under the identification $TT^*L \cong TL \oplus T^*L$ we have $d\phi_{(q,\alpha)}(v) = v_0 + J_q\Phi_q(v_1)$, and hence

$$\begin{aligned}\phi^*\omega_{(q,0)}(v, w) &= \omega_q(v_0 + J_q\Phi_q(v_1), w_0 + J_q\Phi_q(w_1)) = \omega_q(v_0, J_q\Phi_q(w_1)) - \omega_q(w_0, J_q\Phi_q(v_1)) \\ &= w_1(v_0) - w_0(v_1) = \omega_{\mathrm{std}}(v, w).\end{aligned}$$

This implies that $\phi^*\omega$ and ω_{std} are identified along L, and hence we can apply Moser's argument to finish the proof. □

For isotropic/coisotropic submanifolds, we need to provide a good description of the local model. Let $L \subseteq M$ be an isotropic submanifold, then $TL \subseteq TM$ has symplectic orthogonal complement

$$TL \subseteq (TL)^\omega,$$

whose quotient $(TL)^\omega/TL$ is a symplectic vector bundle over L with the inherited symplectic structure $\overline{\omega}$. Pick an almost complex structure J on TM compatible with ω and let g_J be the corresponding Riemannian metric, we then have direct sum decomposition

$$TM|_L = TL \oplus JTL \oplus E,$$

where $E = (TL \oplus JTL)^\omega$ is the symplectic vector bundle isomorphic to $(TL)^\omega/TL$ and JTL is an isotropic subbundle isomorphic to $TM/(TL)^\omega$. Under this identification, write $\omega_E = \omega|_E$ and identify JTL with T^*L via the metric g_J, we can construct a symplectic structure ω_J on $TM|_L/TL = T^*L \oplus E$ as follows: for any pair of tangent vectors $v = v_0 + v_1 + v_2$ and $w = w_0 + w_1 + w_2$ where $v_0, w_0 \in TL$, $v_1, w_1 \in T^*L$ and $v_2, w_2 \in E$, we define

$$\omega_J(v, w) = v_1(w_0) - w_0(v_1) + \omega_E(v_2, w_2).$$

Then we can check that ω_J is a symplectic structure on $TM|_L/TL$.

11 Exercise. Show that for any compact isotropic submanifold $L \hookrightarrow M$ of a symplectic manifold (M, ω), there exists an open neighbourhood $\mathrm{Nbd}_M(L)$ of L in M and an open neighbourhood $\mathrm{Nbd}_{TM|_L/TL}(L)$ of the zero section $L \hookrightarrow TM|_L/TL$ in the quotient bundle $TM|_L/TL$ and a diffeomorphism $\psi\colon \mathrm{Nbd}_{TM|_L/TL}(L) \to \mathrm{Nbd}_M(L)$ such that $\psi|_L = \mathrm{Id}$ and $\psi^*\omega = \omega_J$ for some chosen almost complex structure J on M.

12 Exercise. Set up the correct local model for a coisotropic submanifold $C \subseteq M$, state and proof the corresponding neighbourhood theorem for coisotropic submanifolds.

Hint: For those who want to directly see the whole procedure, check [Got82].

(2d) Arnold-Liouville theorems continued. The introduction of special submanifolds allows us to provide a better description of completely integrable systems. First of all, we have

2.16 Proposition. Let (M, ω) be a symplectic manifold with a Poisson commute collection of smooth functions $\{H_1, \cdots, H_k\}$. Assume that hamiltonian flows $\phi^t_{H_i}$ of H_i are defined for all $t \in \mathbb{R}$, giving a $\mathbb{R}^k$-action on M, then the orbits of the $\mathbb{R}^k$-action are isotropic and the level sets $\vec{H}^{-1}(c)$ for $c \in \mathbb{R}^k$ regular values are coisotropic.

Proof. The first claim follows from the commutativity and the second claim follows from from the fact that $T\vec{H}^{-1}(c)$ is the zero set in TM of the differential forms dH_i for $i = 1, \cdots, k$, which is the intersection of the symplectic orthogonal complements $\mathrm{span}\{X^{\omega}_{H_i}\}_{i=1}^k$ of the corresponding Hamiltonian vector fields. ☐

2.17 Corollary. In particular, for $\vec{H}\colon M \to \mathbb{R}^n$ a completely integrable system with regular fibres, the orbits of the $\mathbb{R}^n$-action are Lagrangian submanifolds of M.

Upon determining all the orbits of the completely integrable system, we want to understand how these orbits are patched together to form the symplectic manifold M. Locally around regular orbits, we have the following **Liouville coordinates**:

2.18 Proposition. Let $\vec{H}\colon M \to \mathbb{R}^n$ be a completely integrable system and let $b \in \mathbb{R}^n$ be a regular value of $\vec{H}$. Let $b \in U$ be an open neighbourhood of b in $\mathbb{R}^n$ and $\sigma\colon U \to M$ a smooth **Lagrangian section** of $\vec{H}$, i.e. $\vec{H} \circ \sigma = \mathrm{Id}_U$ and $\sigma(U)$ is a Lagrangian submanifold of M. Then the map

$$\Psi\colon U \times \mathbb{R}^n \to X, \quad \Psi(b, t) = \phi^{t_1}_{H_1} \circ \cdots \circ \phi^{t_n}_{H_n}(\sigma(b))$$

is both an immersion and a submersion such that $\Psi^*\omega = \sum_{i=1}^n db_i \wedge dt_i$, where $(b_1, \cdots, b_n)$ is the standard coordinate in $U \subseteq \mathbb{R}^n$.

Proof. This is basically a computation: write ∂_{b_i} and ∂_{t_i} to be the corresponding vector fields on $U \times \mathbb{R}^n$, then for $1 \leq i \leq n$ it's clear that $\Psi_*(\partial_{b_i})$ has image in $T\sigma(U)$, which is a Lagrangian submanifold and hence $\Psi^*\omega(\partial_{b_i}, \partial_{b_j}) = 0$ for all $1 \leq i, j \leq n$. As all $\mathbb{R}^n$-orbits are Lagrangian submanifolds, we also have $\Psi^*\omega(\partial_{t_i}, \partial_{t_j}) = 0$ for all $1 \leq i, j \leq n$. Therefore it suffices to determine $\Psi^*\omega(\partial_{b_i}, \partial_{t_j})$. By definition,

$$\omega(d\Psi(\partial_{b_i}), d\Psi(\partial_{t_j})) = \omega(d\phi^{\vec{t}}_{\vec{H}} \circ d\sigma(\partial b_i), d\phi^{\vec{t}}_{\vec{H}}(X_{H_j})) = dH_j \circ d\sigma(\partial b_i) = \delta_{ij}$$

using the fact that σ is a section of $\vec{H}$. We therefore conclude that $\Psi^*\omega = \sum_{i=1}^n db_i \wedge dt_i$. The remaining properties of Ψ follow immediately. $\square$

Under this symplectomorphism Ψ we get that $\Lambda^{\vec{H}}|_U = \Psi^{-1}(\sigma(U))$, and hence $\Lambda^{\vec{H}}$ is locally trivial over the submanifold of regular values of $\vec{H}$. However, it's still unclear whether the period over each point $b \in U$ is the same. This can be resolved by upgrading Liouville coordinates to the famous **action-angle coordinates**:

2.19 Proposition (Action-Angle Coordinates). Let $\vec{H}\colon M \to U \subseteq \mathbb{R}^n$ be a completely integrable system over a contractible open set U with only regular and connected fibres, then there is a local change of coordinates $\alpha\colon U \to V \subseteq \mathbb{R}^n$ such that $\alpha \circ \vec{H}$ has standard period lattice $\Lambda^{\alpha\circ H}$ and the map $\Psi\colon U \times \mathbb{R}^n \to M$ descends to a symplectomorphism $U \times \mathbb{R}^k \times (\mathbb{R}/2\pi\mathbb{Z})^{n-k} \to M$.

Proof. As U is contractible, we can find sections $\tau_i\colon U \to \Lambda^{\vec{H}}$ so that $2\pi\tau_1, \cdots, 2\pi\tau_k$ provide a basis of the period lattice $\Lambda^{\vec{H}}$ at each point of U. Let

$$T = \begin{pmatrix} \tau_1^1 & \tau_2^1 & \cdots & \tau_k^1 & 0 & \cdots & 0 \\ \tau_1^2 & \tau_2^2 & \cdots & \tau_k^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tau_1^k & \tau_2^k & \cdots & \tau_k^k & 0 & \cdots & 0 \\ \tau_1^{k+1} & \tau_2^{k+1} & \cdots & \tau_k^{k+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tau_1^n & \tau_2^n & \cdots & \tau_k^n & 0 & \cdots & 1 \end{pmatrix}$$

be a matrix-valued function on U where the first $k \times n$-entries are given by the sections τ_i and the remaining entries by the identity matrix. Note that $T_b(2\pi\mathbb{Z}^k) = \Lambda_b^{\vec{H}}$, so it suffices to show that T is induced from a change of coordinates $\alpha\colon U \to V$.

Write $\alpha = (\alpha_1, \cdots, \alpha_n)$, then for $\vec{G} = \alpha \circ \vec{H}$ we can compute that

$$d\vec{G} = d\alpha \circ d\vec{H} = \left(\sum_{j=1}^n \frac{\partial \alpha_i}{\partial b_j} db_j \left(dH_j \otimes \frac{\partial}{\partial b_j} \right) \right)_{i=1}^n = \left(\sum_{j=1}^n \frac{\partial \alpha_i}{\partial b_j} dH_j \right)_{i=1}^n ,$$

so by taking the symplectic dual we have

$$X_{G_i} = \sum_{j=1}^n \frac{\partial \alpha_i}{\partial b_j} X_{H_j}$$

for $1 \leq i \leq n$. Write $A = (\partial\alpha_i/\partial b_j)_{i,j}$, then by taking flows and using commutativity we have

$$\phi_{\vec{G}}^t = \phi_{\vec{H}}^{A^T t},$$

so that the period lattices of $\vec{H}$ and $\vec{G}$ are related by

$$A^T \Lambda_b^{\vec{G}} = \Lambda_{\alpha^{-1}(b)}^{\vec{H}}.$$

Now we want $\Lambda^{\vec{G}} = 2\pi\mathbb{Z}^k$ to be the standard period lattice, and from the previous discussion we look for a map α such that

$$\frac{\partial \alpha_i}{\partial b_j} = \begin{cases} \tau_i^j, & \text{if } 1 \leq i \leq k \\ 0, & \text{if } i > k, j \leq k \\ \delta_{ij}, & \text{if } i, j > k \end{cases}$$

As the space U is contractible, such a map α exists iff $\dfrac{\partial \tau_i^j}{\partial b_l} = \dfrac{\partial \tau_i^l}{\partial b_j}$ for all $1 \leq j, l \leq k$ and $1 \leq i \leq n$. Pick a Lagrangian section $\sigma\colon U \to M$ of $\vec{H}$ and use the Liouville coordinate $\Psi\colon V \to U$ so that $\Psi^*\omega = \sum_{i=1}^n db_i \wedge dt_i$ and $\Lambda^{\vec{H}} = \Psi^{-1}(\sigma(U))$. Note that $\Lambda^{\vec{H}} \subseteq \mathbb{R}^n \times U$ is a Lagrangian submanifold that is a cover over U, and is hence a disjoint union classified by the period maps $\tau_i\colon U \to \mathbb{R}^n$ for $1 \leq i \leq k$. The Lagrangian condition then implies that

$$0 = \omega\left(\frac{\partial}{\partial t_j} - \sum_{k=1}^n \frac{\partial \tau_i^j}{\partial b_k}\frac{\partial}{\partial b_k}, \frac{\partial}{\partial t_l} - \sum_{k=1}^n \frac{\partial \tau_i^l}{\partial b_k}\frac{\partial}{\partial b_k}\right) = \frac{\partial \tau_i^l}{\partial b_j} - \frac{\partial \tau_i^j}{\partial b_l},$$

which is exactly the desired condition, and therefore we conclude the proof. ☐

2.20 DEFINITION. The Liouville coordinates associated to the completely integrable system $\alpha \circ \vec{H}$ is called the **action-angle coordinate**, where $\alpha \circ \vec{H}\colon U \times \mathbb{R}^n \to U$ is the **action coordinate** while the projection $U \times \mathbb{R}^n \to \mathbb{R}^n$ is the **angle coordinate**.

2.21 COROLLARY (Big Arnold-Liouville Theorem). Assume that $\vec{H}\colon M \to \mathbb{R}^n$ is a completely integrable system with regular and connected fibres, then all the fibres $\mathcal{O}$ are diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$ for some $0 \leq k \leq n$, and there is a neighbourhood of $\mathcal{O}$ in M that is symplectomorphic to $U \times \mathbb{T}^k \times \mathbb{R}^{n-k}$ where the symplectic structure is given by $\sum db_i \wedge dt_i$. Under this symplectomorphism, $\mathcal{O}$ is sent to the fibre $\{b\} \times \mathbb{T}^k \times \mathbb{R}^{n-k}$ for some $b \in U$.

When the fibres are all compact, we get a **Lagrangian torus fibration** $\vec{H}\colon M \to B$ over some subset $B \subseteq \mathbb{R}^n$.

(2e) Hamiltonian G-manifold. An alternative point of view toward the "dimension reduction" is to describe the quotient manifold as a "quotient by a symplectic group action", which we will introduce briefly in this section. Write $\mathrm{symp}(M, \omega)$ to be the space of symplectic vector fields on (M, ω), and $\mathrm{ham}(M, \omega)$ to be the space of hamiltonian vector fields. Directly from the definition 1.7 and the definition of de Rham cohomology we get that

2.22 LEMMA. There is a short exact sequence of vector spaces

$$0 \to \mathrm{ham}(M, \omega) \to \mathrm{symp}(M, \omega) \to H^1_{dR}(M; \mathbb{R}) \to 0,$$

and $\mathrm{ham}(M, \omega)$ can be identified with the quotient of space of smooth functions $C^\infty(M, \mathbb{R})$ by the constant functions.

2.23 DEFINITION. Let G be a Lie group. A **symplectic G-action** on a symplectic manifold (M, ω) is a smooth G-action $\rho\colon G \times M \to M$ such that $\rho_g^*\omega = \omega$ for all $g \in G$.

Looking at the infinitesimal action we can see that the Lie algebra $\mathfrak{g}$ of G maps to $\mathrm{symp}(M, \omega)$. Also recall that there is a natural **adjoint action** of G on $\mathfrak{g}$ given by

$$\mathrm{Ad}_g(\xi) \coloneqq \left.\frac{d}{dt}\right|_{t=0} g \exp(t\xi) g^{-1}$$

where $\exp(t\xi)$ is a one-parameter subgroup of G whose derivative at $t = 0$ is the infinitesimal vector field ξ.

2.24 DEFINITION. A symplectic G-action $\rho\colon G \times M \to M$ is **hamiltonian** if furthermore the infinitesimal action $\xi \mapsto X_\xi \in \mathrm{symp}(M, \omega)$ lies in $\mathrm{ham}(M, \omega)$ and admits a linear lift in $C^\infty(M, \mathbb{R})$, and this map is G-equivariant with respect to the adjoint action of G on $\mathfrak{g}$ and the pullback action of G on $C^\infty(M, \mathbb{R})$.

Therefore a Hamiltonian G-action on (M, ω) provides a G-equivariant linear map $\mathfrak{g} \to C^\infty(M, \mathbb{R})$, and note very unrigorously that there is an identification

$$\hom_{\mathbb{R}}(\mathfrak{g}, C^\infty(M, \mathbb{R})) = C^\infty(M, \hom_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})) = C^\infty(M, \mathfrak{g}^\vee),$$

so that a linear map $\mathfrak{g} \to C^\infty(M, \mathbb{R})$ should be identified with a smooth map $\mu_G \colon M \to \mathfrak{g}^\vee$, which we call the **moment map** for the hamiltonian G-action on M. Rigorously, it's defined as follows: for $\xi \in \mathfrak{g}$ and $x \in M$, we define

$$\langle \mu_G(x), \xi \rangle := H_\xi(x),$$

where $H_\xi \in C^\infty(M, \mathbb{R})$ is the image of ξ under the linear lift $\mathfrak{g} \to C^\infty(M, \mathbb{R})$.

2.25 Example. Consider the harmonic oscillator 1.3. In this case the assignment $\xi \mapsto H$ provides the hamiltonian action $\mathfrak{u}(1) \cong \mathbb{R} \to C^\infty(T^*\mathbb{R}, \mathbb{R})$ of the Lie algebra $\mathfrak{u}(1)$ of the unitary group $U(1) \cong \mathbb{S}^1$, which acts as the rotation on the phase space $T^*\mathbb{R} \cong \mathbb{R}^2$ counter-clockwise.

From now on, we assume that $G = T \cong \mathbb{T}^k$ is a torus. Write $\mathfrak{t}^\vee$ to be the dual Lie algebra of T.

2.26 Lemma. Let $c \in \mathfrak{t}^\vee$ be a regular value of the moment map $\mu_T \colon M \to \mathfrak{t}^\vee$, then the level set $\mu_T^{-1}(c)$ is a coisotropic submanifold of M.

Proof. Write the moment map in coordinates, this is essentially Proposition 2.16. ☐

Note that if $C \subseteq V$ is a coisotropic subspace of a symplectic vector space (V, Ω), then the restriction $\Omega|_C$ has non-trivial kernel $\ker \Omega \subseteq C$ so that Ω descends to the quotient space $\overline{\Omega} \colon \wedge^2 (C/\ker \Omega) \to \mathbb{R}$ which is again non-degenerate. $\ker \Omega$ is often called the **characteristic subspace** of C.

In particular, we know that orbits of T are isotropic submanifolds in $\mu_T^{-1}(c)$, and by dimension count we see that the characteristic subbundle of $T\mu_T^{-1}(c)$ is exactly the tangent bundle of T-orbits. Therefore if the quotient $\mu_T^{-1}(c)/T$ is a smooth manifold, the symplectic structure ω will descend to a symplectic structure on the quotient $\mu_T^{-1}(c)/T$.

Conditions for a quotient space M/G of a smooth manifold M by a Lie group G to be a smooth manifold are well-known [Lee13, Chapter 21]. We will recall some of the terminologies used there.

2.27 Definition. A Lie group action G on M is **free** if the stabilizer subgroup of G at each point of M is trivial, and is **proper** if the map $G \times M \to M \times M$ given by $(g, x) \mapsto (gx, x)$ is proper.

2.28 Theorem ([Lee13, Theorem 21.10]). Suppose G is a Lie group acting smoothly, freely and properly on a smooth manifold M, then the quotient space M/G is a topological manifold of dimension $\dim M - \dim G$ and admits a unique smooth structure such that the quotient map $M \to M/G$ is a smooth submersion.

Now it's an immediate corollary that

2.29 Theorem (Marsden-Weinstein-Meyer). Assume that T acts freely and properly on $\mu_T^{-1}(c)$, i.e. stabilizer subgroup of T at each point of $\mu_T^{-1}(c)$ is trivial and the orbits of T are embedded submanifolds of M, then the quotient space

$$M/\!/_c T = \mu_T^{-1}(c)/T$$

is a symplectic manifold of dimension $\dim M - 2\dim T$.

2.30 EXAMPLE. Consider the symplectic vector space $(\mathbb{C}^{n+1}, \omega_{std})$ regarded as a complex vector space of dimension $2n+2$. Consider a $T = U(1)$-action on $\mathbb{C}^{n+1}$ given in coordinates by

$$\theta.(z_1, \cdots, z_{n+1}) = (e^{i\theta} z_1, \cdots, e^{i\theta} z_{n+1}).$$

One can check that the moment map for T is given by $\mu_T(z_1, \cdots, z_{n+1}) = \frac{1}{2}\sum_{i=1}^{n+1} |z_i|^2$, which is regular over $\mathbb{R}_{>0}$ and has a critical value at 0 whose fibre is exactly one point. For any $c > 0$, $\mu_T^{-1}(c) = \{|z_1|^2 + \cdots + |z_{n+1}|^2 = 2c\}$ defines a sphere of radius $\sqrt{2c}$ in $\mathbb{C}^{n+1}$, and it's clear that the quotient $\mu_T^{-1}(c)/T$ is diffeomorphic to the complex projective space $\mathbb{C}P^n$. However, the symplectic structures on each $\mathbb{C}^{n+1}//_c T$ are different for different $c > 0$: if we rescale $z_i \mapsto \sqrt{c}^{-1} z_i$, then the corresponding symplectic structure scales by c, so that the quotient structure on $\mathbb{C}^{n+1}//_c T$ is c times the quotient symplectic structure on $\mathbb{C}^{n+1}//_1 T$.

2.31 DEFINITION. A group action $G \curvearrowright M$ is **effective** if the action homomorphism $G \to \mathrm{Diff}(M)$ is injective. A symplectic manifold (M, ω) is called **toric** if there is an effective hamiltonian T-action on M with $\dim T = \frac{1}{2}\dim M$.

Note that a basis for $\mu_T \colon M \to \mathfrak{t}^\vee$ provides a completely integrable system on M, and since an effective action has a dense open subset of free orbits, we can find an open subset $U \subseteq \mu_T(M)$ in the image of M under μ_T such that the restriction $\mu_T|_{\mu_T^{-1}(U)} \colon \mu_T^{-1}(U) \to U$ is a smooth Lagrangian torus fibration.

1 PROBLEM. Understand and explain the proof of the following theorem of Atiyah and Guillemin-Sternberg [Ati82, GS82]:

2.32 THEOREM. Let (M, ω) be a compact connected symplectic manifold with an effective hamiltonian T-action, then the image $\mu_T(M)$ of the moment map is a convex polytope in $\mathfrak{t}^\vee$, and the fibres of μ_T are connected.

Reference: [MS17, Theorem 5.5.1], [Aud91, Theorem 4.2.1].

3. Rigidity: Non-squeezing Theorem

In the previous discussions we have shown that symplectic manifolds share some similarities with general smooth topology: there're no local geometries, and their special submanifolds also have pretty good neighbourhood descriptions. Probably non-surprisingly, symplectic manifolds also have some regidities, which distinguishes symplectic topology from smooth topology.

Clearly as $[\omega] \in H^2_{dR}(M;\mathbb{R})$ has to be non-trivial for compact M, the **symplectic volume** $\mathrm{vol}_\omega(M) := \langle[\omega^n],[M]\rangle$ provides a first symplectic invariant that gives some rigidity to symplectic manifolds. For instance, one readily see that

3.1 Lemma. Assume that (M,ω_M) and (N,ω_N) are symplectic manifolds, possibly with boundary, such that $\mathrm{vol}_{\omega_M}(M) > \mathrm{vol}_{\omega_N}(N)$, then there cannot exist any symplectic embedding $M \hookrightarrow N$.

One can therefore ask the following question: is volume the only obstruction to the existence of symplectic embeddings? Let's take a look at the infinitesimal story: given a symplectic vector space $(V,\Omega) = (\mathbb{R}^{2n},\omega_{\mathrm{std}})$, the space of linear symplectomorphisms of V is the symplectic group $\mathrm{Sp}(V,\Omega)$ of matrices $A\colon V \to V$ such that $A^T\Omega A = \Omega$. As $\wedge^{top}\Omega$ is propotional to the determinant map $\det\colon V \to \mathbb{R}$, the group of and volume-preserving maps is exactly the group of matrices with $\det(A) = 1$, i.e. the special linear group $\mathrm{SL}(V)$.

3.2 Lemma. A matrix of the form

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A,B,C,D are $n \times n$ matrices is symplectic if and only if the following conditions hold:

$$A^TC = C^TA, \quad B^TD = D^TB, \quad A^TD - C^TB = \mathrm{Id}\,.$$

In particular, $\mathrm{Sp}(\mathbb{R}^2,\omega_{\mathrm{std}}) = \mathrm{SL}(2,\mathbb{R})$.

So it's clear that in high dimensions volume-preserving is not enough to guarantee the existence of symplectic embeddings. However, we want a non-trivial non-linear example where this can happen. This is the famous theorem of Gromov [Gro85], which we will state and prove in this lecture.

3.3 Theorem (Gromov). Consider the following submanifolds of $\mathbb{C}^n$: for $r > 0$ the open cylinder $Z_R = \mathbb{D}^2_R(0) \times \mathbb{C}^{n-1}$ and the open ball $D_r = \mathbb{D}^{2n}_r(0) \subseteq \mathbb{C}^n$, with inherited symplectic structure from ω_{std}. Then there exists a symplectic embedding $D_r \hookrightarrow Z_R$ if and only if $r \le R$.

13 Exercise. Show that there exists a volume-preserving embedding $D_r \hookrightarrow Z_R$ for any $r,R > 0$. *Hint: Consider linear maps, i.e. matrices.*

The inclusion $D_r \hookrightarrow Z_R$ gives the trivial symplectic embedding provided $r \le R$, but as $\mathrm{vol}(Z) = \infty$, there're no volume obstructions to the existence of symplectic embeddings $D_r \hookrightarrow Z_R$ when $r > R$. This means we cannot symplectically "squeeze" a ball to embed it into a cylinder of smaller radius, which gives the name **non-squeezing theorem**.

The proof of the non-squeezing theorem is highly non-trivial and it's the first time where the pseudo-holomorphic curve techniques were introduced in symplectic topology.

Although symplectic manifolds does not have local invariants, pseudo-holomorphic curves provide global invariants of symplectic manifolds that was proved extremely effective, and was extensively used later to derive a bunch of symplectic-geometric results. We will go to this later in the lecture.

(3a) Differential calculus on Riemann surfaces. Let's quickly recall some basic facts about Riemann surfaces. By **Riemann surface** we mean a smooth manifold Σ of real dimension 2, and in this lecture we will always assume a Riemann surface is closed, i.e. compact without boundary. An **area form** ω_Σ on Σ is a 2-form that is non-degenerate everywhere, which exists by partition of unity, and as any 2-forms on Σ are closed, it follows that Σ always admits symplectic structures, and therefore also admilts almost complex structures. Moreover, we can compute that the Nijenhuis tensor vanishes, and hence

3.4 Proposition. A Riemann surface Σ admits at least one complex structure j, and hence is a complex manifold.

This can be also shown without using Newlander-Nirenberg theorem. See for example [Che55].

Given an almost complex structure j on Σ, the complexfied tangent bundle $T_{\mathbb{C}}\Sigma \coloneqq T\Sigma \otimes_{\mathbb{R}} \mathbb{C}$ admits a natural splitting

$$T_{\mathbb{C}}\Sigma = T^{1,0}\Sigma \oplus T^{0,1}\Sigma$$

where $T^{1,0}\Sigma$ and $T^{0,1}\Sigma$ consist of vectors $v \in T_{\mathbb{C}}\Sigma_p$ such that $j_p(v) = iv$ and $j_p(v) = -iv$ respectively. Similarly, the cotangent bundle $T^*_{\mathbb{C}}\Sigma = T^*_{1,0}\Sigma \oplus T^*_{0,1}\Sigma$ also splits correspondingly.

3.5 Definition. Under this decomposition, we define

$$\mathcal{A}^{p,q}(\Sigma) \coloneqq C^\infty(\Sigma, \wedge^p(T^*_{1,0}\Sigma) \otimes_{\mathbb{C}} \wedge^q(T^*_{0,1}\Sigma)).$$

In particular, write $\Omega^n(\Sigma) \coloneqq C^\infty(\Sigma, \wedge^n T^*_{\mathbb{C}}\Sigma)$ to be the space of complex differential forms on Σ, then we get decompositions of vector spaces

$$\Omega^n(\Sigma) \cong \bigoplus_{p+q=n} \mathcal{A}^{p,q}(\Sigma).$$

Of course, $\dim_{\mathbb{R}} \Sigma = 2$, so it's non-trivial only when $n = 0, 1, 2$ and $0 \leq p+q \leq 2$, i.e. we have isomorphisms

$$\Omega^1(\Sigma) \cong \mathcal{A}^{1,0}(\Sigma) \oplus \mathcal{A}^{0,1}(\Sigma),\ \Omega^2(\Sigma) \cong \mathcal{A}^{1,1}(\Sigma)$$

as $\wedge^2(T^*_{1,0}\Sigma) = \wedge^2(T^*_{0,1}\Sigma) = 0$.

In local coordinates, let $\mathbb{C} \supseteq U \hookrightarrow \Sigma$ be an open subset of Σ with coordinate $z = x+iy$, then the complexified differential forms are of the form

$$f(x,y)dx + ig(x,y)dy,$$

where $f, g \colon U \to \mathbb{C}$ are smooth complex-valued functions. The standard complex structure $j|_U$ on U is given by $j^*(dx) = -dy$ and $j^*(dy) = dx$, so that $\mathcal{A}^{1,0}(U)$ is spanned by the complex-valued 1-form $dz = dx + idy$ as a module over the algebra $C^\infty(U, \mathbb{C})$, while $\mathcal{A}^{0,1}(U)$ is spanned by the complex-valued 1-form $d\bar{z} = dx - idy$.

Let $\Pi_{p,q}\colon \Omega^{p+q}(\Sigma) \to \Omega^{p,q}(\Sigma)$ be the projection onto the (p,q)-component. The complexified differential $d_{\mathbb{C}} := d \otimes \mathrm{Id}_{\mathbb{C}}$ decomposes with respect to the decomposition of $\Omega^n(\Sigma)$ as $d^n_{\mathbb{C}} = \sum_{p+q=n+1} d^{p,q}$ with

$$d^{p,q} = \Pi_{p,q} \circ d^n_{\mathbb{C}} \colon \Omega^n(\Sigma) \to \Omega^{p,q}(\Sigma).$$

Write $\partial = d^{1,0}$ and $\bar\partial = d^{0,1}$.

3.6 Lemma. We have $\partial^2 = \bar\partial^2 = 0$ and $\partial\bar\partial = -\bar\partial\partial$, and ∂ and $\bar\partial$ satisfy the Leibniz rule with respect to the wedge product of forms.

Proof. They follow from the direct sum decomposition and the fact that $d^2_{\mathbb{C}} = 0$ and $d_{\mathbb{C}}$ satisfies the Leibniz rule. □

3.7 Proposition ([Huy05, Proposition 2.6.15]). $d_{\mathbb{C}} = \partial + \bar\partial$ if and only if j defines a complex structure on Σ.

14 Exercise. Show that any complex structure j on Σ is compatible with some symplectic structure ω_Σ on Σ. In other words, Σ is a Kähler manifold.

As $\partial^2 = \bar\partial^2 = 0$, the sequences

$$0 \to \mathcal{A}^{1,0}(\Sigma) \xrightarrow{\bar\partial} \mathcal{A}^{1,1}(\Sigma) \xrightarrow{\bar\partial} 0$$

and

$$0 \to \Omega^{0}(\Sigma) \xrightarrow{\bar\partial} \mathcal{A}^{0,1}(\Sigma) \xrightarrow{\bar\partial} \mathcal{A}^{0,2}(\Sigma) \xrightarrow{\bar\partial} 0$$

are complexes of $\mathbb{C}$-vector spaces, and therefore we can take cohomologies to get **Dolbeault cohomologies**

$$H^{p,q}_{\bar\partial}(\Sigma) := \frac{\ker \bar\partial|_{\mathcal{A}^{p,q}(\Sigma)}}{\operatorname{Im} \bar\partial|_{\mathcal{A}^{p,q-1}(\Sigma)}}.$$

Note that in local coordinates $\bar\partial f(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) d\bar z$, hence $\bar\partial f = 0$ iff f is holomorphic.

3.8 Definition. A map $f\colon M \to N$ is **holomorphic** if there exists open covers $\{U_i\}$ of M and $\{V_i\}$ of N by coordinate charts such that $f(U_i) \subseteq V_i$ and under these charts $f|_{U_i}\colon U_i \to V_i$ is given by a tuple of holomorphic functions.

f is a **biholomorphism** if f has an inverse $f^{-1}\colon N \to M$ that is also holomorphic. In this case we say M and N are **biholomorphic**. Write $\mathrm{Aut}(M)$ for the group of biholomorphisms of M.

Let's just recall a basic fact from complex analysis:

3.9 Theorem ([JS87, Theorem 2.1.1]). $\mathrm{Aut}(\mathbb{C}P^1) \cong \mathrm{PSL}(2,\mathbb{C})$, where $\mathrm{PSL}(2,\mathbb{C})$ is the quotient of the group $\mathrm{SL}(2,\mathbb{C})$ of invertible matrices with determinant 1 by its center $\{\pm\,\mathrm{Id}\}$. This isomorphism is explicitly given by the Möbius transformations. In particular, $\dim_{\mathbb{C}} \mathrm{Aut}(\mathbb{C}P^1) = 3$.

Recall from topology that a map $\phi\colon \widetilde{\Sigma} \to \Sigma$ is a **covering map** if fibres $\phi^{-1}(p)$ for each $p \in \Sigma$ is a disjoint union of finitely many points and ϕ is locally trivial, i.e. it induces a fibre bundle over Σ. A generalization to covering map in complex geometry is a **branched cover** $\phi\colon \widetilde{\Sigma} \to \Sigma$, which is a holomorphic map where there exists a finite number of points $p_i \in \Sigma$ such that $\phi^{-1}(\Sigma \setminus \{p_i\}) \to \Sigma \setminus \{p_i\}$ is a covering map, and ϕ is locally given by $z \mapsto z^k$ for some $k > 0$ near each point in $\phi^{-1}(p_i)$.

(3b) Moduli space of pseudo-holomorphic curves. Essential in the proof is to study the space of all pseudo-holomorphic maps $u\colon (\Sigma, j) \to (M, J)$ for some chosen compatible almost complex structure J on a symplectic manifold (M, ω), where (Σ, j) is a Riemann surface with a chosen complex structure j.

3.10 Definition. Let (Σ, j) be a closed Riemann surface with chosen complex structure j and (M, ω) be a symplectic manifold with chosen compatible almost complex structure J. A smooth map $u\colon \Sigma \to M$ is called **J-holomorphic** if it satisfies

$$du \circ j = J \circ du. \tag{3}$$

Equivalently, note that any $u\colon \Sigma \to M$ induces a vector bundle u^*TM over Σ, which is a complex vector bundle with complex structure u^*J. The differential $du\colon T\Sigma \to TM$ can then be understood as a section of the tensor bundle $T^*\Sigma \otimes_{\mathbb{C}} u^*TM$, and using local coordinates we can assume $u\colon \mathbb{C} \supseteq B \to U \subseteq \mathbb{C}^n$ is given by a tuple of smooth complex-valued functions $u = (u_1, \cdots, u_n)$ so that the differential is given by

$$du = \sum_{i=1}^{n} \left(\frac{\partial \operatorname{Re} u_i}{\partial z} \frac{\partial}{\partial x_i} \otimes dz + \frac{\partial \operatorname{Im} u_i}{\partial z} \frac{\partial}{\partial y_i} \otimes dz + \frac{\partial u_i}{\partial \bar{z}} \frac{\partial}{\partial x_i} \otimes d\bar{z} + \frac{\partial \operatorname{Im} u_i}{\partial \bar{z}} \frac{\partial}{\partial y_i} \otimes d\bar{z} \right),$$

so that (3) is equivalent to $\dfrac{\partial u_i}{\partial \bar{z}} = 0$ for all $1 \le i \le n$, i.e. $\bar{\partial}_J u = 0$. This description is closer to what we traditionally mean by J-holomorphic.

3.11 Definition. A J-holomorphic curve $u\colon \Sigma \to M$ is **simple** if there does not exist a J-holomorphic curve $v\colon S \to M$ and a branched cover $\phi\colon \Sigma \to S$ such that $u = v \circ \phi$.

An equivalent property for simpleness is the following local description: u is called **somewhere injective** if there exists $z \in \Sigma$ such that $du_z \neq 0$ and $u^{-1}(u(z)) = \{z\}$. This equivalence readily implies that

3.12 Proposition. The set of uninjective points of a simple J-holomorphic curve is discrete. Here uninjective points means $p \in \Sigma$ such that $du_p = 0$ or $u^{-1}(u(p)) \neq \{p\}$.

3.13 Definition. Fix $A \in H_2(M; \mathbb{Z})$, we define the moduli space of J-holomorphic curves of genus g in the homology class A to be

$$\widehat{\mathcal{M}}_g(M, J; A) := \{u\colon \Sigma_g \to M | \bar{\partial}_J u = 0,\ u \text{ is simple},\ u_*[\Sigma_g] = A\},$$

where Σ_g is a closed Riemann surface of genus g.

To understand $\widehat{\mathcal{M}}_g(M, J; A)$, the usual way in differential geometry is to find a larger space of maps, for instance we can look at the space of smooth maps $C^\infty(\Sigma_g, M)$, such that $\widehat{\mathcal{M}}_g(M, J; A) \subseteq C^\infty(\Sigma_g, M)$ is a subspace defined by some sections of some vector bundles $E \to C^\infty(\Sigma_g, M)$. In this case, intuitively we can think about $\widehat{\mathcal{M}}_g(M, J; A)$ as the zero locus of the section $\bar{\partial}_J$ of the **Fréchet bundle**

$$C^\infty(\Sigma_g, u^*TM) \to E \to C^\infty(\Sigma_g, M).$$

However, the metric structure on a Fréchet bundle is not good enough to make any meaningful conclusions, so we need to further enlarge the space of maps and sections, to allow non-smooth and even non-continuous maps, in order to “solve” this partial differential equation.

3.14 DEFINITION. A **Banach space** is a vector space V over $\mathbb{R}$ or $\mathbb{C}$ equipped with a norm $\|\cdot\|\colon V \to \mathbb{R}_{\geq 0}$ such that V is complete with respect to the metric induced by the norm. A linear map $f\colon V \to W$ of Banach spaces is **bounded** if there exists $C > 0$ such that $|f(v)| \leq C|v|$ for all $v \in V$.

A **Banach manifold** modelled on a Banach space V is a topological space M with an open cover $\{U_i\}_{i\in I}$ such that there exists embeddings $\phi_i\colon U_i \hookrightarrow V$ such that in the intersections $U_{ij} = U_i \cap U_j$ the transition maps $\phi_j \circ \phi_i^{-1}\colon \phi_i(U_{ij}) \to \phi_j(U_{ij})$ are homeomorphisms.

Note that with the Banach norm it's still possible to define differentials of maps between Banach spaces. Roughly speaking, the **differential** of a continuous map $f\colon V \to W$ at $0 \in V$ is a bounded linear map $df_p\colon V \to W$ such that

$$\lim_{t\to 0} \frac{|f(x+t) - f(x) - df_p(t)|}{\|t\|} = 0.$$

Using this, we are able to define smoothness of maps between Banach spaces, smoothness of Banach manifolds, and smoothness of maps between Banach manifolds. The notion of **Banach bundles** can also be similarly defined, where we require that locally it's homeomorphic to products of Banach spaces, and the transition maps are bounded linear on the fibre component. For more details, see [Lan62, Moo17].

The canonical enlargement of $C^\infty(\Sigma_g, M)$ so that the map spaces have good analysis properties is the so-called Sobolev spaces.

3.15 DEFINITION. Let $(\mathbb{R}^n, \mu)$ be a measure space and $(V, |\cdot|)$ a Banach space. For $k \in \mathbb{Z}_{\geq 0}$ and $p \in [1, \infty)$, the **Sobolev space** $W^{k,p}(\mathbb{R}^n, V)$ is defined to be the completion of $C^\infty(\mathbb{R}^n, V)$ with respect to the norm

$$\|f\|_{W^{k,p}} = \left(\sum_{j=0}^{k} \int_{\mathbb{R}^n} \|d^j f\|^p d\mu \right)^{1/p}.$$

This is a Banach space and in particular when $p = 2$ and the norm in V is induced by an inner product, it's a **Hilbert space** with the inner product given by

$$\langle f, g\rangle_{W^{k,2}} = \sum_{j=0}^{k} \int_{\mathbb{R}^n} \langle d^j f, d^j g\rangle d\mu.$$

As transition maps between coordinate charts of a smooth manifold M is smooth, by introducing a Riemannian metric g on M we can glue local Sobolev spaces to get a **Sobolev manifold** $W^{k,p}(N, M)$ of maps from a smooth manifold N to the Riemannian manifold M. This can be verified to be a C^1-Banach manifold whose tangent space at a point $u \in W^{k,p}(N, M)$ is the Banach space $W^{k,p}(N, u^*TM)$.

For a map $u\colon N \to M$, we can talk about the related topological invariants only when u is continuous, which requires us to look at (k, p) such that $W^{k,p}(\mathbb{R}^n, V) \subseteq C^0(\mathbb{R}^n, V)$. This can not be always achieved, but we have the following Sobolev embedding theorems:

3.16 THEOREM ([Eva98, Section 5.6]). Let U be an open subset of $\mathbb{R}^n$ and $u \in W^{k,p}(U, \mathbb{R})$. If $kp > n$, then $u \in C^0(U, \mathbb{R})$ and there exists $C > 0$ such that $\|u\|_{C^0} \leq C\|u\|_{W^{k,p}}$.

Back to our situation, we look at Sobolev manifold $W^{1,p}(\Sigma, M)$ where Σ is a compact Riemann surface of genus g and (M, ω) a symplectic manifold with chosen compatible almost complex structure J. When $p > 2$ the above Sobolev embedding theorem implies that $W^{1,p}(\Sigma, M) \subseteq C^0(\Sigma, M)$, so it makes sense to consider the submanifold

$$W^{1,p}(\Sigma, M; A) = \{u \colon W^{1,p}(\Sigma, M) | u_*[\Sigma] = A\}.$$

The tangent space at $u \in W^{1,p}(\Sigma, M; A)$ is then $W^{1,p}(\Sigma, u^*TM)$, and the section we are looking for is $\bar{\partial}_J \in C^1(W^{1,p}(\Sigma, M; A), TW^{1,p}(\Sigma, M; A) \otimes_{\mathbb{C}} T^{0,1}\Sigma)$, whose zero locus is a priori an enlargement of $\widehat{\mathcal{M}}_g(M, J; A)$. The problem now is to understand the space $\widehat{\mathcal{M}}_g^{1,p}(M, \omega, J; A) \coloneqq \bar{\partial}_J^{-1}(0)$ inside $W^{1,p}(\Sigma, M; A)$.

(3b-i) Transversality. Recall that given a smooth map $f \colon N \to M$ and a smooth submanifold $A \subseteq M$, we say f is **transversal** to A if for each $x \in f^{-1}(A)$ we have $T_{f(x)}M = T_{f(x)}A + df_x(T_xN)$. When f is transversal to A the preimage $f^{-1}(A)$ is a smooth submanifold of N with codimension equal to the codimension of A in M.

Now let $E \to M$ be a smooth finite-rank vector bundle over M, we know that the restriction $TE|_M \cong TM \oplus E$ admits a natural splitting into the tangent bundle TM of M and the vector bundle E. Let $s \colon M \to E$ be a section of E, then the transversality condition for $s^{-1}(0) \subseteq M$ to be a smooth submanifold of M of codimension $r = \operatorname{rank} E$ is equivalent to the projection $ds|_{s^{-1}(0)} \colon TM \to TE \twoheadrightarrow E$ being surjective for all $x \in s^{-1}(0)$.

In inifnite dimensions, in addition to the bundle map $TM \to E$ being surjective, we need to further require that the surjection admits a right inverse $E \to TM$, i.e. the composition $E \to TM \xrightarrow{ds} E$ is the identity map. This splitting provides a direct sum decomposition $TM \cong K \oplus E$ where $K = \ker(ds)|_{s^{-1}(0)}$ is the tangent bundle of the smooth manifold $s^{-1}(0)$.

For $\mathcal{E}^{1,p} = TW^{1,p}(\Sigma, M; A) \otimes_{\mathbb{C}} T^{0,1}\Sigma$ the aformentioned Banach bundle over $\mathcal{B}^{1,p} = W^{1,p}(\Sigma, M; A)$, we can compute the differential of $\bar{\partial}_J$ at $u \in W^{1,p}(\Sigma, M; A)$, which is also often called the **linearization** of $\bar{\partial}_J$ at u such that $\bar{\partial}_J(u) = 0$, as follows: recall that $TE|_M \cong TM \oplus E$ admits a splitting into TM and $\mathcal{E}^p$, where $d\bar{\partial}_J$ is identity on the first component, so it suffices to determine the projection $D\bar{\partial}_J \colon T\mathcal{B}^{1,p} \to \mathcal{E}^p$, whose restriction to $u \in \mathcal{B}^{1,p}$ is given by an operator $D_u \colon W^{1,p}(\Sigma, u^*TM) \to W^{1,p}(\Sigma, u^*TM \otimes T^{0,1}\Sigma)$.

3.17 DEFINITION. J is called **regular** for A and Σ if D_u is split surjective for every u with $\bar{\partial}_J u = 0$. We denote by

$$\mathcal{J}_c^{reg}(M, \omega; \Sigma, A)$$

the space of all compatible almost complex structures J for (M, ω) that is furthermore regular.

We will sketch but omit most details of the proof of the following theorem:

3.18 THEOREM ([MS12, Theorem 3.1.6]). Fix a compact Riemann surface $(\Sigma, j_\Sigma, d\operatorname{vol}_\Sigma)$ and a homology class $A \in H_2(M; \mathbb{Z})$.

(i) If $J \in \mathcal{J}_c^{reg}(M, \omega; \Sigma, A)$ then the space $\widehat{\mathcal{M}}_g^{1,p}(M, \omega, J; A)$ is a smooth manifold of dimension

$$\dim \widehat{\mathcal{M}}_g^{1,p}(M, \omega, J; A) = n(2 - 2g) + 2c_1(A)$$

carrying a natural orientation.

(ii) The set $\mathcal{J}_c^{reg}(M, \omega; \Sigma, A)$ is **residual** in $\mathcal{J}_c(M, \omega)$, i.e. it contains a countable intersection of open dense subsets of $\mathcal{J}$.

For a complex vector bundle $E \to M$ over some smooth manifold M, we can define its **determinant line bundle** $\det E = \wedge^r E \to M$, where r is the rank of E. For each map $u\colon \Sigma \to M$, we have the natural pull-back bundle $u^* \det E \to \Sigma$ which defines a complex line bundle over Σ. We define the element $c_1(A)$ to be the signed count of the points in a generic section of $u^* \det TM$. We can show that this does not depend on the choice of sections.

Sketch of Proof. Instead of studying D_u for fixed almost complex structure J, we consider the **universal moduli space**

$$\mathcal{U}^{k,p,\ell}(M;A) = \mathcal{B}^{k,p}(M;A) \times \mathcal{J}^\ell_\omega(M)$$

where $\mathcal{J}^\ell_\omega(M)$ is the completion of $\mathcal{J}_c(M,\omega)$ under the C^ℓ-metric. We are looking in this space at the family of moduli spaces $\widehat{\mathcal{M}}^{k,p,\ell}_g(M,\omega,\mathcal{J}^\ell_c;A) \xrightarrow{\pi} \mathcal{J}^\ell_c(M,\omega)$ parametrized by $\mathcal{J}^\ell_\omega(M)$. To prove (ii), it suffices for us to show that for generic choice of $J \in \mathcal{J}_c(M,\omega) \subseteq \mathcal{J}^\ell_\omega(M)$, the corresponding operator $D_{u,J}$ is split surjective for every $u \in \widehat{\mathcal{M}}^{k,p,\ell}_g(M,\omega,J;A)$.

An operator $D\colon V \to W$ between Banach spaces V and W is **Fredholm** if $\dim\ker D$ and $\dim\operatorname{coker} D$ are all finite-dimensional. When D is Fredholm, we define its **Fredholm index** by $\operatorname{Ind}(D) \coloneqq \dim\ker D - \dim\operatorname{coker} D$.

For maps $f\colon X \to Y$ between separable C^1-Banach manifolds with Fredholm differentials, Smale [Sma65] showed that the set $y \in Y^{reg}$ of values in Y where df is surjective is residual, which generalizes Sard's theorem to infinite dimensions.

In particular, we are now considering sections of the vector bundle $\mathcal{E}^{k-1,p} \to \mathcal{U}^{k,\ell,p}$ where the fibre at u is given by $W^{k-1,p}(\Sigma, u^*TM \otimes T^{1,0}\Sigma)$. The space $\mathcal{J}^\ell_\omega(M)$ is Banach with tangent space at J provided by bundle endomorphisms $Y\colon TM \to TM$ satisfying $YJ + JY = 0$ and $\omega(Yv,w) + \omega(v,Yw) = 0$ for all $q \in M$ and $v,w \in T_qM$. Write $s\colon \mathcal{U}^{k,\ell,p} \to \mathcal{E}^{k-1,p}$ to be the section, then we can compute that

$$Ds_{(u,J)}(\xi,Y) = D_{u,J}(\xi) + \frac{1}{2}Y(u)\circ du \circ j.$$

3.19 Proposition (Fredholm Property for Elliptic Operators). Fix a homology class $A \in H_2(M;\mathbb{Z})$, an integer $\ell \geq 2$, a real number $p > 2$, and an integer $1 \leq k \leq \ell$. Then the following holds.

(i) The operator $D_{u,J}$ is Fredholm with indices $\operatorname{Ind} D_{u,J} = n(2-2g) + 2c_1(A)$.

(ii) $Ds_{(u,J)}$ is surjective whenever $s(u,J) = 0$ and u is **somewhere injective**, i.e. there is $p \in \Sigma$ and $\delta > 0$ such that $d_M(u(p),u(\zeta)) \geq \delta d_\Sigma(z,\zeta)$ for all $\zeta \in \Sigma$.

(iii) $d\pi\colon (\xi,Y) \mapsto Y$ is Fredholm.

This Proposition then implies the theorem. □

Moreover, we can show that $\widehat{\mathcal{M}}^{k,p}_g(M,\omega,J;A)$ does not depend on the choice of the pair (k,p), by the following elliptic regularity result:

3.20 Proposition (Elliptic Regularity). Assume J is a smooth ω-compatible almost complex structure and $u\colon \Sigma \to M$ a J-holomorphic curve of class $W^{1,p}$ with $p > 2$, then u is smooth.

In particular, all maps in $\widehat{\mathcal{M}}_g^{k,p}(M,\omega,J;A)$ are smooth, and hence we can remove the superscripts and simply write $\widehat{\mathcal{M}}_g(M,\omega,J;A)$. This is a smooth manifold of dimension $n(2-2g)+2c_1(A)$. We mostly focus on the case when $g=0$, so that $\widehat{\mathcal{M}}_0(M,\omega,J;A)$ is a smooth manifold of dimension $2n+2c_1(A)$.

(3b-ii) Compactness. For the proof of the non-squeezing theorem, we need to consider the moduli space of curves with a marked point

$$\widehat{\mathcal{M}}_{g,1}(M,\omega,J;A)=\{(u,z)|u\in\widehat{\mathcal{M}}_0(M,\omega,J;A),z\in\Sigma\}\cong\widehat{\mathcal{M}}_g(M,\omega,J;A)\times\Sigma,$$

which is still a smooth manifold of dimension $n(2-2g)+2c_1(A)+2$. By taking the image of u under the given point z, we get an evaluation map

$$\mathrm{ev}\colon\widehat{\mathcal{M}}_{g,1}(M,\omega,J;A)\to M.$$

Note that $\mathrm{PGL}(2,\mathbb{C})$ acts on $\widehat{\mathcal{M}}_{g,1}(M,\omega,J;A)$ by reparametrization of the domain Σ, and hence does not affect the image of u. We want to consider the **unparametrized** version of the mouduli space

$$\mathcal{M}_{g,1}(M,\omega,J;A)\coloneqq\widehat{\mathcal{M}}_{g,1}(M,\omega,J;A)/\,\mathrm{PGL}(2,\mathbb{C}),$$

where elements in $\mathcal{M}_{g,1}(M,\omega,J;A)$ are just curves in M independent of the parametrization. As the image of the evaluation map does not change under the $\mathrm{PGL}(2,\mathbb{C})$-action, there is still a natural evaluation map $\mathrm{ev}\colon\mathcal{M}_{g,1}(M,\omega,J;A)\to M$, which can also be shown to be smooth.

Let $[D]\in H_*(M;\mathbb{Z})$ be a cycle represented by a closed subset D in M, then the preimage $\mathrm{ev}^{-1}(D)\subseteq\mathcal{M}_{g,1}(M,\omega,J;A)$ is the set of pairs (u,z) such that $u(z)\in D$. To get a better analysis of the preimage $\mathrm{ev}^{-1}(D)$, it'll be better if the moduli space $\mathcal{M}_{g,1}(M,\omega,J;A)$ is compact. However, $\mathcal{M}_{g,1}(M,\omega,J;A)$ is not compact in general, but geometrically we can provide a natural compactification of $\mathcal{M}_{g,1}(M,\omega,J;A)$ by adding the moduli space of, **cusp-curves** in the ancient language, or equivalently **stable curves** after Kontsevich [Kon95].

Let Σ be a compact Riemann surface of genus g with m marked points $z_1,\cdots,z_m$ and embedded disjoint simple closed curves $\gamma_1,\cdots,\gamma_k$ that do not pass through any marked points. A **cusp-curve** is the quotient space

$$\overline{\Sigma}\coloneqq\Sigma\big/\{x\sim y\Leftrightarrow\text{there exists a curve }\gamma_i\text{ so that }x,y\in\gamma_i\}$$

given by shrinking each γ_i to a point.

3.21 Theorem (Uhlenbeck-Gromov Compactness). Let $[u_i]\in\mathcal{M}_{g,m}(M,\omega,J;A)$ be a sequence of J-holomorphic curves with m marked points in the homology class A in a compact symplectic manifold (M,ω). Then there exists a subsequence $\{[u_{i_j}]\}$ that converges to a cusp-curve $u_\infty\colon\overline{\Sigma}\to M$, in an appropriate sense.

This theorem is intuitively easy to understand, while in practice difficult to formulate. A rigorous treatment can be found in [MS12, Chapters 4 and 5] via Gromov topology and stable maps. The rough idea is the following: firstly note that the **energy**

$$E(u)=\int_\Sigma u^*\omega=\omega(A)=\int_\Sigma|du|^2d\,\mathrm{vol}_\Sigma$$

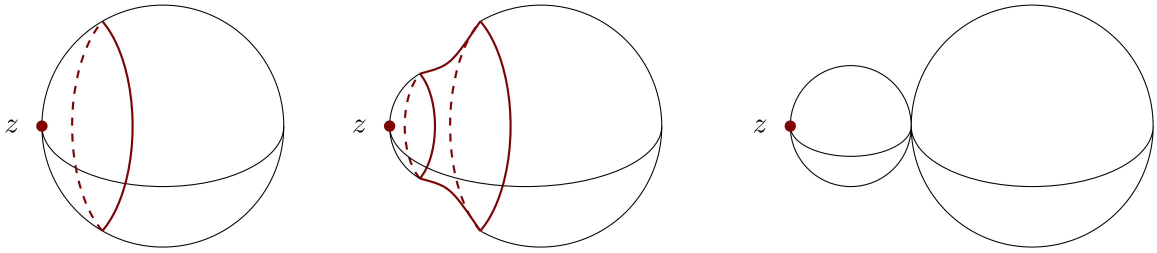
of $u \in \mathcal{M}_{g,m}(M,\omega,J;A)$ is uniformly bounded above. If furthermore $|du_i|^2$ is uniformly bounded over Σ, by compactness of M and the folowing theorem, $\{u_i\}$ has a subsequence that C^1, and after taking care of the regularity, in C^∞, to a J-holomorphic curve $u_\infty \colon \Sigma \to M$.

3.22 Theorem (Arzelà-Ascoli). Let Σ be a compact Hausdorff space and M a metric space. Assume that $\mathcal{F} \subseteq C(\Sigma, M)$ is a family of continuous maps. Then $\mathcal{F}$ is relatively compact in the compact-open topology of $C(\Sigma, M)$ if and only if it is

(i) (Equicontinuous) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_M(f(z), f(\zeta)) < \varepsilon$ for all $f \in \mathcal{F}$ and $z, \zeta \in \Sigma$ with $d_\Sigma(z,\zeta) < \delta$.

(ii) (Pointwise bounded) For every $z \in \Sigma$ the set $\{f(z) | f \in \mathcal{F}\}$ is bounded in M.

Therefore the problem only arises at points $z \in \Sigma$ where $|du_i(z)|^2$ is unbounded. For each such point, we can normalize the norm $|du_i|^2$ in a neighbourhood of z by rescaling, which looks like "bubbling off" the neighbourhood of z from Σ, and if we pass to the limit, we will gradually see a copy of $\mathbb{C}$ with z as the origin, where the theorem of Arzelà-Ascoli can be applied locally to provide a local C^∞-convergence to a J-holomorphic curve $u_\infty \colon \mathbb{C} \to M$ up to passing to subsequences. We can then show that u_∞ naturally extends to a map from $\mathbb{C}P^1$ to M, and this extended point is in the image of the limit of the remaining parts of the curve.



The pinching curve. The rescaling. Bubbling.

Figure 2: The bubbling procedure.

We write $\overline{\mathcal{M}}_{g,m}(M,\omega,J;A)$ to be the compactification of $\mathcal{M}_{g,m}(M,\omega,J;A)$ by adding all the possible limits of sequences of J-holomorphic curves. From Uhlenbeck-Gromov compactness, there will be two possible types of compactifications: cusp-curves and non-simple curves.

Suppose $g = 0$, then the cusp-curves are unions of $\mathbb{P}^1$s along several points, and if we represent each sphere by a vertex, and connect two vertices by an edge if the two spheres intersect at a point. When $g = 0$, cusp-curves that are limits of some holomorphic spheres are always represented by **trees**, i.e. graphs with no cycles, and we can compute the dimension of these moduli space of cusp-curves as follows. Regard intersection points as marked points on a sphere, so that we can write the corresponding moduli space by

$$\mathcal{M}_{0,m_i,e_i}(M,\omega,J;A_i)$$

where m_i is the number of marked points on this component, $\sum A_i = A$ and e_i is the number of points that intersect with other components. Let $T = (V, E, W)$ be the tree where V is the set of vertices, E is the set of edges and $W \colon V \to \mathbb{Z}_{\geq 0}$ assigns to each vertex the number of marked points on the corresponding component, then the moduli space $\mathcal{M}_T(M,\omega,J;A)$ of cusp-curves corresponding to T can be identified with

$$\mathcal{M}_T(M,\omega,J;A) \cong \left\{ (u_i)_{i\in V} \in \prod_{i\in V} \mathcal{M}_{0,W(i),e_i}(M,\omega,J;A_i) \,\middle|\, \begin{array}{l} \text{ev}_{e,i}(u_i)=\text{ev}_{e,j}(u_j) \text{ whenever} \\ e\in E \text{ is an edge connecting } i \text{ and } j. \end{array} \right\}$$

Up to further deforming J we can assume that the evaluation map $\text{ev}_{e,i}$ is transversal to $\text{ev}_{e,j}$, so that the dimension of $\mathcal{M}_T(M,\omega,J;A)$ is given by

$$\dim \mathcal{M}_T(M,\omega,J;A) = 2n|V| + 2m - 6|V| - 2(n-2)|E| + 2c_1(A).$$

As $|E| = |V| - 1$ for a tree, we have $\dim \mathcal{M}_T(M,\omega,J;A) = 2n + 2m - 4 + 2c_1(A) - 2|V|$. In particular, all non-trivial cusp-curves have codimension at least 2 in $\overline{\mathcal{M}}_{0,m}(M,\omega,J;A)$.

For non-simple curves appearing in the boundary of $\overline{\mathcal{M}}_{0,m}(M,\omega,J;A)$, one can replace it by simple ones: if $A = \sum_\alpha m_\alpha A_\alpha$ is a decomposition into some other curve classes, we can find simple cusp-curves in $\overline{\mathcal{M}}_{0,m}(M,\omega,J;\sum A_\alpha)$ having the same image as the non-simple ones. Note that this moduli space should have dimension

$$2n + 2m - 4 + 2\sum_\alpha c_1(A_\alpha) - 2|V|,$$

which has codimension ≥ 2 in $\overline{\mathcal{M}}_{0,m}(M,\omega,J;A)$ only when $c_1(A_\alpha) \geq 0$ for all α. This will hold if we impose the following condition on M:

3.23 DEFINITION. A symplectic manifold (M,ω) is **semipositive** if for every $A \in H_2(M;\mathbb{Z})$,

$$\omega(A) > 0, \quad c_1(A) \geq 3 - n \Rightarrow c_1(A) \geq 0.$$

Note that when $c_1(A) < 3-n$, the moduli space $\mathcal{M}_{0,0}(M,\omega,J;A)$ is empty, and hence this condition is enough to ensure the abovementioned codimension condition. Compactifying strata having codimension 2 implies that the fundamental cycle $\left[\overline{\mathcal{M}}_{0,m}(M,\omega,J;A)\right]^{vir} \in H_{2n+2m-6+2c_1(A)}\left(\overline{\mathcal{M}}_{0,m}(M,\omega,J;A);\mathbb{Z}\right)$ is well-defined.

(3c) Proof of the non-squeezing theorem. With the moduli theory of J-holomorphic curves established, we are ready to prove the non-squeezing theorem. Let's start with a lower bound for area of J-holomorphic curves in an Euclidean space.

(3c-i) Isoperimetric inequality for J-holomorphic curves. Let $\gamma\colon \mathbb{S}^1 \to V$ be a smooth loop in a symplectic vector space $(V = \mathbb{C}^n, \omega_{\text{std}}, J_{\text{std}})$, then we can define the **symplectic action** of γ by $\mathcal{A}(\gamma) = \frac{1}{2}\int_0^{2\pi} \omega(\dot\gamma(\theta),\gamma(\theta))d\theta$, the **length** of γ by $L(\gamma) := \int_0^{2\pi} |\dot\gamma(\theta)|d\theta$, and the **energy** of γ by $E(\gamma) := \frac{1}{2}\int_0^{2\pi} |\dot\gamma(\theta)|^2 d\theta$.

3.24 LEMMA ([MS12, Lemma 4.4.4]). Under the above hypotheses, we have the inequality

$$|\mathcal{A}(\gamma)| \leq \frac{1}{4\pi} L(\gamma)^2.$$

Proof. Using the vector version of Euler's identity, we have $e^{k\theta J}=\cos(k\theta)\operatorname{Id}+\sin(k\theta)J$, so that for given vectors $v,w\in V$ we have

$$\begin{aligned}\omega(e^{k\theta J}v,e^{m\theta J}w)&=\cos(k\theta)\cos(m\theta)\omega(v,w)+\sin(k\theta)\sin(m\theta)\omega(v,w)\\&\quad+\cos(k\theta)\sin(m\theta)\omega(v,Jw)+\sin(k\theta)\cos(m\theta)\omega(Jv,w)\\&=\cos((k-m)\theta)\omega(v,w)+\sin((k-m)\theta)\omega(v,Jw).\end{aligned}$$

Integrate θ over the interval $[0,2\pi]$ we get

$$\int_0^{2\pi}\omega(e^{k\theta J}v,e^{m\theta J}w)d\theta=\begin{cases}2\pi\omega(v,w), & k=m,\\ 0, & \text{otherwise.}\end{cases}$$

For any smooth $\gamma\colon\mathbb{S}^1\to V$, consider its Fourier series expansion

$$\gamma(\theta)=\sum_{k=0}^{\infty}e^{k\theta J}v_k,$$

so that $\dot\gamma(\theta)=\sum_{k=0}^{\infty}ke^{k\theta J}Jv_k$. We can then compute that

$$|\mathcal{A}(\gamma)|=\frac{1}{2}\left|\int_0^{2\pi}\omega(\dot\gamma(\theta),\gamma(\theta))d\theta\right|=\sum_{k=0}^{\infty}k\pi|v_k|^2\le\sum_{k=0}^{\infty}k^2\pi|v_k|^2=E(\gamma).$$

Suppose that γ is immersed, then we can reparametrize γ by its arc-length so that $|\dot\gamma(\theta)|=L(\theta)/2\pi$, then we can compute that

$$E(\gamma)=\frac{1}{2}\int_0^{2\pi}|\dot\gamma(\theta)|^2d\theta=\frac{1}{2}\int_0^{2\pi}\frac{L(\gamma)^2}{4\pi^2}d\theta=\frac{1}{4\pi}L(\gamma)^2.$$

This finishes the proof whenever γ is immersed. For general γ, choose a sequence $\{a_\nu\}_\nu$ so that $a_\nu\to0$ as $\nu\to\infty$ and $a_\nu\neq e^{-\theta J}J\dot\gamma(\theta)$ for each ν, and define $\gamma_\nu(\theta)=\gamma(\theta)+e^{\theta J}a_\nu$, then γ_ν is immersed for each ν with $\gamma_\nu\rightrightarrows\gamma$ C^∞-uniformly, and hence $\mathcal{A}(\gamma_\nu)\to\mathcal{A}(\gamma)$ and $L(\gamma_\nu)\to L(\gamma)$ as $\nu\to\infty$, so that the inequality also holds for general γ. ☐

3.25 COROLLARY. Let $u\colon S\to V$ be a J-holomorphic curve with $u(0)=0$ where S is a not necessarily compact Riemann surface, then for each $\varepsilon>0$ we have

$$\int_{S\cap B_\varepsilon(0)}u^*\omega\ge\pi\varepsilon^2.$$

Proof. Let

$$A(\varepsilon)=\int_{S\cap B_\varepsilon(0)}u^*\omega=\int_0^{2\pi}\omega\left(du(V),du(jV)\right)d\theta$$

be the symplectic area of u in the ball $B_\varepsilon(0)$, where V is a unit vector field on S. We can choose V such that near $\partial(S\cap B_\varepsilon(0))$ the vector $du(V)$ is the projection of the radial vector field $r\partial/\partial r$ on $\mathbb{C}^n$ to the tangent space of $u(S)$, so that $A(\varepsilon)=\mathcal{A}(\partial(S\cap B_\varepsilon(0)))$. On the other hand, coarea formula implies that

$$\frac{d}{d\varepsilon}A(\varepsilon)=\int_0^{2\pi}\frac{\omega(du(V),du(jV))}{|du(V)|}d\theta\ge\int_0^{2\pi}\frac{\omega(\dot\gamma_\varepsilon(\theta),\gamma_\varepsilon(\theta))}{|\dot\gamma_\varepsilon(\theta)|}d\theta=L(\gamma_\varepsilon).$$

Therefore we have

$$\frac{\dot{A}(\varepsilon)}{\sqrt{A(\varepsilon)}} \geq \frac{L(\gamma_\varepsilon)}{\sqrt{A(\varepsilon)}} \geq 2\sqrt{\pi}.$$

By taking integration with respect to ε we get that $A(\varepsilon) \geq \pi\varepsilon^2$ for each $\varepsilon > 0$ such that $S \cap \partial B_\varepsilon(0) \neq \emptyset$. ☐

For general not necessarily linear almost complex structures J in $\mathbb{C}^n$, we cannot achieve isoperimetric inequality with such sharp constant, but there is still a weaker version, which we will introduce here without a proof.

3.26 LEMMA (Monotonicity Lemma). Let $u\colon \Sigma \to M$ be a J-holomorphic curve in an almost Kähler manifold (M, ω, J), then there are constants c_0 and $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$ and if $x \in \Sigma$ is such that $u(\Sigma) \cap \overline{B}_\varepsilon(u(x))$ is compact with boundary contained in the boundary $\partial B_\varepsilon(u(x))$

$$A(S \cap \overline{B}_\varepsilon(u(x))) \geq \frac{\pi}{1 + c_0\varepsilon}\varepsilon^2.$$

(3c-ii) Holomorphic spheres in aspherical manifolds. For the sake of applications, we only look at a very special type of compact symplectic manifolds where the moduli space of holomorphic curves behave very nicely.

3.27 DEFINITION. A compact symplectic manifold (M, ω) is called **symplectically aspherical** if $\omega(A) = 0$ for all $A \in H_2(M; \mathbb{Z})$ that can be represented by a J-holomorphic sphere $u\colon \mathbb{S}^2 \to M$ for some ω-compatible almost complex structure J.

The monotonicity lemma then implies that

3.28 COROLLARY. For symplectically aspherical manifolds (M, ω, J), there does not exist non-constant J-holomorphic spheres $u\colon \mathbb{S}^2 \to M$.

3.29 EXAMPLE. Let $M = \mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ be a $2n$-dimensional torus with symplectic structure $\omega = \sum_{i=1}^{n} d\theta_i \wedge d\varphi_i$ where θ_i and φ_i are the angle coordinates in $\mathbb{R}^{2n}$. Then we can verify that M is symplectically aspherical as follows. Note that $q\colon \mathbb{R}^{2n} \to M$ is a universal cover of M, and since $\mathbb{S}^2$ is simply connected, any smooth map $u\colon \mathbb{S}^2 \to M$ can be lifted to a smooth map $\tilde{u}\colon \mathbb{S}^2 \to \mathbb{R}^{2n}$. Now we must have $u_*[\mathbb{S}^2] = q_*\tilde{u}_*[\mathbb{S}^2] = 0$ as $H^2(\mathbb{R}^{2n}; \mathbb{Z}) = 0$, so we conclude that $\omega(u_*[\mathbb{S}^2]) = 0$.

On the other hand, for J-holomorphic curves $u\colon \mathbb{S}^2 \to (\mathbb{P}^1, J_{\mathrm{std}})$, we can show using Riemann mapping theorem that any non-constant holomorphic curve u has to be a multiple cover of $\mathbb{P}^1$, and therefore simple ones has to be biholomorphisms. This implies that $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1, \omega_{\mathrm{FS}}, J_{\mathrm{std}}; A) \cong \mathbb{P}^1$ is already compact.

Recall the open charts of $\mathbb{P}^1$ in Example 2.2: we have $U_0 = \{[z_0 : z_1] \in \mathbb{P}^1 | z_0 \neq 0\} \cong \mathbb{C}_z$ and $U_1 = \{[z_0 : z_1] \in \mathbb{P}^1 | z_1 \neq 0\} \cong \mathbb{C}_w$, where the transition map is given by $w = 1/z = \bar{z}/|z|^2$. We consider the holomorphic tangent bundle $T\mathbb{P}^1$ of $\mathbb{P}^1$, whose restriction to U_0 is given by $\mathbb{C}_z \times \mathbb{C}\partial/\partial z$ and to U_1 is given by $\mathbb{C}_w \times \mathbb{C}\partial/\partial w$. The transition map of $T\mathbb{P}^1$ can be computed as follows:

$$w_*\left(\frac{\partial}{\partial z}\right) = \frac{dw}{dz}\frac{\partial}{\partial w} = -\frac{1}{z^2}\frac{\partial}{\partial w} = -w^2\frac{\partial}{\partial w}.$$

In particular, we can find a section $s\colon \mathbb{P}^1 \to T\mathbb{P}^1$ defined by $s(z) = z\partial/\partial z$ on U_0 and $s(w) = -w\partial/\partial w$ on U_1, which has two zeroes 0 and ∞, both non-degenerate. Therefore

s is a section of $T\mathbb{P}^1$ transverse to the zero section with exactly two zeroes, which shows that $c_1(A) = 2$ where $A = [\mathrm{pt}]$ is the point class of $\mathbb{P}^1$ generating $H_0(\mathbb{P}^1; \mathbb{Z})$. By dimension formula, we have

$$\dim \mathcal{M}_{0,1}(\mathbb{P}^1, \omega_{\mathrm{FS}}, J_{\mathrm{std}}; A) = 2 + 2 - 6 + 2c_1(A) = 2,$$

and the previous discussion implies that the fibre $\mathrm{ev}^{-1}(p)$ of the evaluation map

$$\mathrm{ev}\colon \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1, \omega_{\mathrm{FS}}, J_{\mathrm{std}}; A) \to \mathbb{P}^1$$

consists of exactly one point, and hence represents the class $[\mathrm{pt}] \in H_0(\overline{\mathcal{M}}(\mathbb{P}^1, \omega_{FS}, J_{\mathrm{std}}); \mathbb{Z})$.

3.30 PROPOSITION. Let $M = \mathbb{P}^1 \times N$ be a product of $\mathbb{P}^1$ and a symplectically aspherical manifold N, where the symplectic structure is given by $\omega_M = \pi_1^*\omega_{\mathbb{P}^1} + \pi_2^*\omega_N$. Then for any regular ω_M-compatible almost complex structure $J \in \mathcal{J}^{reg}_{\omega_M}(M)$ and any point $p \in M$, there exists at least one simple J-holomorphic sphere passing through p.

Proof. It suffices to show that the class $[\mathrm{ev}_J^{-1}(p)]$ does not vanish in $H_*\left(\overline{\mathcal{M}}_{0,1}(M, \omega_M, J; [p]); \mathbb{Z}\right)$ for each regular J. Any two compatible almost complex structures on M can be connected by a smooth path $\{J_t\}_{t\in[0,1]}$ of ω_M-compatible almost complex structures, though this family might pass through some non-regular almost complex structures. Nontheless, we can consider the moduli space of $\{J_t\}_{t\in[0,1]}$-holomorphic curves

$$\overline{\mathcal{M}}_{0,1}(M, \omega_M, \{J_t\}; [p]) = \{(t, [u]) | t \in [0,1], [u] \in \overline{\mathcal{M}}_{0,1}(M, \omega_M, J_t; [p])\},$$

which by forgetting $[u]$ admits a natural projection to $[0,1]_t$.

3.31 DEFINITION. Fix a compact Riemann surface $(\Sigma, j, d\,\mathrm{vol}_\Sigma)$ and a homology class $A \in H_2(M; \mathbb{Z})$. Let $J_0, J_1 \in \mathcal{J}^{reg}_{\omega_M}(M)$ be two regular ω_M-compatible almost complex structures. A homotopy $[0,1] \to \mathcal{J}_{\omega_M}(M)$, $\lambda \mapsto J_\lambda$ from J_0 to J_1 is called **regular** (for A and Σ) if

$$\Omega^{0,1}(\Sigma, u^*TM) = \mathrm{im}\, D_{J_\lambda, u} + \mathbb{R}v_\lambda$$

for every $(\lambda, u) \in \mathcal{M}_{g,0}(M, \omega_M, \{J_\lambda\}; A)$, where $v_\lambda = (\partial_\lambda J_\lambda) du \circ j_\Sigma$ is the image in $\Omega^{0,1}(\Sigma, u^*TM)$ of the tangent vector to the path $\{J_\lambda\}$. We write $\mathcal{J}^{reg}_{\omega_M}(M, A; J_0, J_1)$ for the space of all regular homotopies.

3.32 PROPOSITION (Regularity in families). Let $J_0, J_1 \in \mathcal{J}^{reg}_{\omega_M}(M)$ be two regular ω_M-compatible almost complex structures. Fix a compact Riemann surface $(\Sigma, j, d\,\mathrm{vol}_\Sigma)$ and $A \in H_2(M; \mathbb{Z})$.

(i) If $\{J_\lambda\}_{\lambda\in[0,1]}$ is regular then $\mathcal{M}_{g,0}(M, \omega_M, \{J_\lambda\}; A)$ is a smooth orientable manifold with boundary $\mathcal{M}_{g,0}(M, \omega_M, J_0; A) \sqcup \mathcal{M}_{g,0}(M, \omega_M, J_1; A)$. The boundary orientation agrees with the orientation of $\mathcal{M}_{g,0}(M, \omega_M, J_1; A)$ and is opposite to the orientation of $\mathcal{M}_{g,0}(M, \omega_M, J_0; A)$.

(ii) The set $\mathcal{J}^{reg}_{\omega_M}(M, A; J_0, J_1)$ is residual in the space of all smooth homotopies from J_0 to J_1.

The codimension 2 property of moduli spaces of cusp-curves for each J_i implies that the cobordism relation holds after passing to the compactification. Moreover, by further restricting to residual subsets we can assume that the evaluation map

$$\mathrm{ev}\colon \overline{\mathcal{M}}_{0,1}(M, \omega_M, \{J_\lambda\}; [p]) \to M$$

is transversal to p, so that $\mathrm{ev}^{-1}(p) \in \overline{\mathcal{M}}_{0,1}(M, \omega_M, \{J_\lambda\}; [p])$ is a smooth compact 1-dimensional manifold with boundary $\mathrm{ev}_{J_0}^{-1}(p) \sqcup \mathrm{ev}_{J_1}^{-1}(p)$, which implies that $[\mathrm{ev}_{J_0}^{-1}(p)] = [\mathrm{ev}_{J_1}^{-1}(p)]$ in $H_0(\overline{\mathcal{M}}_{0,1}(M, \omega_M, J_i; [p]); \mathbb{Z})$.

This allows us to consider special ω_M-compatible almost complex structures on $M = \mathbb{P}^1 \times N$. For instance, we can choose a split almost complex structure $J = J_{\mathbb{P}^1} \times J_N$ for some chosen regular almost complex structure $J_N \in \mathcal{J}_{\omega_N}^{reg}(N)$. Using the projection $\pi_1 \colon M \to \mathbb{P}^1$ and $\pi_2 \colon M \to N$ and the fact that N is symplectically aspherical we can show that all non-constant J-holomorphic spheres in M must be contained in the fibres of π_2, representing the class $[\mathbb{P}^1] \times [\mathrm{pt}]$. The previous discussions for $\mathbb{P}^1$ then helps us to conclude that $[\mathrm{ev}_J^{-1}(p)] \neq 0$ in $H_0(\overline{\mathcal{M}}_{0,1}(M, \omega_M, J; [p]); \mathbb{Z})$, which finishes the proof. □

Proof of theorem 3.3. Assume that there exists a symplectic embedding $i \colon D_r \hookrightarrow Z_R$, by shrinking the image of D_r if necessarily we can assume that the image is compact in Z_R, so that we can compactify Z_R into the product symplectic manifold $\mathbb{P}^1 \times \mathbb{T}^{2n-2}$ where the symplectic structure on $\mathbb{P}^1$ is of the form $(R + \varepsilon)\omega_{FS}$ for some $\varepsilon > 0$. As $\mathbb{T}^{2n-2}$ is symplectically aspherical by example 3.29, we can apply proposition 3.30 to conclude that there exists a J-holomorphic sphere $u \colon S \to M$ in $\mathbb{P}^1 \times \mathbb{T}^{2n-2}$ representing the class $[\mathbb{P}^1] \times [\mathrm{pt}]$ passing through any point of $\mathbb{P}^1 \times \mathbb{T}^{2n-2}$. This curve then clearly satisfy the identity

$$\int_S u^*\omega_M = \omega\left[\mathbb{P}^1 \times \{\mathrm{pt}\}\right] = (R + \varepsilon)^2 \pi.$$

Let us pick $p = i(0) \in i(D_r)$ be the image of the origin, so that u has non-trivial intersection with $i(D_r)$. The pull-back $i^{-1}(u(S))$ still defines a i^*J-holomorphic curve in D_r passing through 0, and we know that if i^*J is linear, monotonicity lemma implies that

$$\pi r^2 \leq \int_{i^{-1}(u(S))} i^*\omega \leq \int_S u^*\omega_M = (R + \varepsilon)^2 \pi,$$

which implies that $r \leq R$. We cannot prove that all such pull-backs are linear, but we can start with a linear almost complex structure J_0 on D_r, and consider an extension of $i_* J_0$ to $\mathbb{P}^1 \times \mathbb{T}^{2n-2}$, which still exists due to connectedness and contractibility of $\mathcal{J}_{\omega_M}(M)$ 2.5. The final step is to show that the space of almost complex structures with given behaviour on $i(D_r)$ is still big enough to give generic regularity, which can be achieved by keeping track of the proof of the regularity theorem, using the following two facts. This finishes the proof.

(i) The curve $u(S)$ cannot be completely contained in the image $i(D_r)$;

(ii) The proof of regularity requires the existence of almost complex structures with prescribed behaviour in a small neighbourhood of an injective point of u, which can be found in the complement of $i(D_r)$. □

2 PROBLEM. A symplectic 4-manifold is said to be **minimal** if it does not contain any symplectically embedded spheres with negative self-intersection number. Classify all compact minimal symplectic 4-manifolds containing a symplectically embedded sphere with non-negative self-intersection number.

Reference: [McD90]

3 PROBLEM. Consider the symplectic manifold $(\mathbb{P}^1 \times \mathbb{P}^1, a\omega_{FS} \oplus b\omega_{FS})$ where $a, b > 0$ are not necessarily equal. Compute $H^* \left(\mathrm{Symp}_0(\mathbb{P}^1 \times \mathbb{P}^1, a\omega_{FS} \oplus b\omega_{FS}); \mathbb{Z}\right)$.

Reference: [Abr98]

(4c) Morse homology.

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