A VERSION OF THE VALUATIVE CRITERION OF PROPERNESS

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If A is a noetherian local ring, then there are two simple ways to construct complete discrete valuation rings over A: first, if k is the residue field of A and k'/k is a finite field extension, then $k'[\![t]\!]$ is such an A-algebra. Second, if $\mathfrak{p}$ is a prime ideal of A such that $\dim A/\mathfrak{p} = 1$ and K is a finite field extension of the fraction field of $A/\mathfrak{p}$, then the normalization B_0 of $A/\mathfrak{p}$ in K is a Dedekind domain, and the completion of B_0 at a maximal ideal is a discrete valuation ring. Let D_A denote the class of discrete valuation rings B over A constructed in one of the above two ways.

The main aim of this note is to prove the following strengthened form of the valuative criterion of properness [EGA, II, Théorème 7.3.8], which shows, roughly speaking, that one need only consider discrete valuation rings of the type constructed above. The argument is very similar to the one given in [EGA, II, Corollaire 7.3.10(ii)] (which can also be found essentially in a MathOverflow answer of Jason Starr [hs] and a paper of Matthieu Romagny [Rom13, Lemma 4.1.1]), but I am not aware of any reference for this precise statement.

Proposition 1. *Let $(A, \mathfrak{m})$ be a noetherian local ring with residue field k, and let X be a finite type separated A-scheme. If $U \subset X$ is a dense open subscheme, then X is proper if and only if for every $B \in D_A$ with fraction field K, the natural map $X(B) \to X(K)$ contains $U(K)$ in its image.*

Corollary 2. *Let k be an algebraically closed field, and let $f\colon X \to Y$ be a separated morphism of finite type k-schemes. If $U \subset X$ is a dense open subscheme, then f is proper if and only if for every commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec} k(\!(t)\!) & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} k[\![t]\!] & \longrightarrow & Y \end{array}$$

there exists a lift $\operatorname{Spec} k[\![t]\!] \to X$ making the evident diagram commute.

Proof. If $y \in Y(k)$, then the local ring $A = \mathcal{O}_{Y,y}$ is a noetherian local ring with residue field k, and the Cohen structure theorem shows that D_A consists of k-algebras isomorphic to $k[\![t]\!]$. By Proposition 1, we see that $f^{-1}(\operatorname{Spec} A) \to \operatorname{Spec} A$ is proper, so by spreading out we conclude that f is proper. $\square$

Corollary 3. *Let S be a noetherian scheme such that every closed point of S has finite residue field, and let X be a finite type separated S-scheme. Then X is proper if and only if for every local field K with ring of integers B and every commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} B & \longrightarrow & S \end{array}$$

there exists a lift $\operatorname{Spec} B \to X$ making the diagram commute.

Proof. If $s \in S$ is a closed point, then the local ring $A = \mathcal{O}_{S,s}$ is a noetherian local ring with finite residue field k. If $B \in D_A$, then the residue field of B is a finite extension of k by construction, so every element of D_A is isomorphic to the ring of integers in a local field. By Proposition 1, it follows that $X \times_S \operatorname{Spec} \mathcal{O}_{S,s}$ is a proper $\mathcal{O}_{S,s}$-scheme, so by spreading out again we conclude that X is proper over S. $\square$

Proof of Proposition 1. The forward direction follows from the usual valuative criterion of properness, so we need only prove the backward direction. We may assume that X is nonempty. By Chow's lemma [EGA, II, Théorème 5.6.1], there exists a quasi-projective A-scheme Y which admits a projective A-morphism $\pi\colon Y \to X$, and we may choose π such that there is a dense open subscheme $V \subset X$ such that $\pi^{-1}(V) \to V$ is an isomorphism. By replacing U by $U \cap V$, we may assume that $U = V$. Note that for each $B \in D_A$ with fraction field K, one has the commutative diagram

$$\begin{tikzcd} Y(B) \arrow[r] \arrow[d] & Y(K) \arrow[d] \\ X(B) \arrow[r] & X(K) \end{tikzcd}$$

which shows, by properness of π, that the map $Y(B) \to Y(K)$ contains $\pi^{-1}(U)(K)$ in its image. Since X is separated, it is enough to show that Y is proper, so by passing from X to Y we may assume that X is quasi-projective over A. Passing to connected components, we may further assume that X is connected.

Let $\overline{X}$ be a projective A-scheme such that there exists an open embedding $i\colon X \to \overline{X}$ with dense image. It suffices to show that i is an isomorphism. Since $\overline{X} - X$ can be given the structure of a proper A-scheme, if it is nonempty then it admits a point lying over $\mathfrak{m}$; thus by the Nullstellensatz it suffices to show $X(\overline{k}) = \overline{X}(\overline{k})$.

Let $x_0 \in \overline{X}(\overline{k})$; we wish to show $x_0 \in X(\overline{k})$. First, suppose that x_0 lies in the closure of U_k in $\overline{X}_k$, and let $x_1 \in U(\overline{k})$ lie in the same irreducible component of $\overline{X}_k$ as x_0. By [Mum70, Lemma in the proof of the Theorem of the Cube], there exists a reduced irreducible curve $C \subset \overline{X}_{\overline{k}}$ such that $x_0, x_1 \in C(\overline{k})$. Let k'/k be a finite field extension such that $x_0, x_1 \in \overline{X}(k')$ and such that $C = C'_{\overline{k}}$ for a closed subscheme $C' \subset \overline{X}_{k'}$. Since x_1 is a point of $C' \cap U(k')$, it follows that C' does not lie entirely in $\overline{X}_{k'} - X_{k'}$. If η' is the generic point of C', then $\eta' \in U_{k'}$, so if B is the completion of the normalization of $\mathcal{O}_{C',x_0}$ at a maximal ideal and K is the fraction field of B, then $\eta' \in X(K)$ and thus $x_0 \in X(k')$ by our assumption.

From now on, we will assume that x_0 does not lie in the closure of U_k in $\overline{X}_k$. Since U is dense in X and X is dense in $\overline{X}$, it follows that U is dense in $\overline{X}$ and in particular $U \neq U_k$. Thus there is some point $x_1 \in U$ which specializes to x_0; let Z denote the Zariski closure of x_1 in $\overline{X}$, regarded as a finite type A-scheme with its reduced structure, so $x_0 \in Z$ is a closed point by assumption. By [EGA, IV$_3$, Proposition 10.5.7] the schemes $\operatorname{Spec} \mathcal{O}_{Z,x_0} - \operatorname{Spec}(\mathcal{O}_{Z,x_0}/\mathfrak{m}) \subset \operatorname{Spec} \mathcal{O}_{Z,x_0} - \{x_0\}$ are Jacobson, and by [EGA, IV$_3$, Corollaire 10.5.9] the closed points of $\operatorname{Spec} \mathcal{O}_{Z,x_0} - \{x_0\}$ correspond to the prime ideals $\mathfrak{q}$ of $\mathcal{O}_{Z,x_0}$ such that $\dim \mathcal{O}_{Z,x_0}/\mathfrak{q} = 1$. Choose a closed point η of $\operatorname{Spec} \mathcal{O}_{Z,x_0} - \operatorname{Spec}(\mathcal{O}_{Z,x_0}/\mathfrak{m})$ which lies in U (as exists since U contains a nonempty open subscheme of the Jacobson scheme $\operatorname{Spec} \mathcal{O}_{Z,x_0} - \operatorname{Spec}(\mathcal{O}_{Z,x_0}/\mathfrak{m})$), and let $\mathfrak{p}$ be the prime ideal of A that η lies over. The schematic closure Z_0 of η in Z is an irreducible scheme of finite type over $A/\mathfrak{p}$, and since x_0 is a closed point of Z_0 it follows that $\dim Z_0 = 1$. Since $Z_0 \to \operatorname{Spec} A/\mathfrak{p}$ is surjective, we see that Z_0 is quasi-finite over $A/\mathfrak{p}$. If η corresponds to the prime ideal $\mathfrak{q}$ of $\operatorname{Spec} \mathcal{O}_{Z,x_0}$, then $\operatorname{Spec} \mathcal{O}_{Z,x_0}/\mathfrak{q}$ is a locally closed subscheme of Z, and so by [EGA, IV$_3$, Corollaire 10.4.7] we have $\dim A/\mathfrak{p} = 1$. If B is the completion of the normalization of $\mathcal{O}_{Z,x_0}/\mathfrak{q}$ at a maximal ideal, then $B \in D_A$, so our assumption finally implies $x_0 \in X$. $\square$

References

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