MISCELLANEOUS

SEAN COTNER

1. Openness of the strongly irreducible locus

I find the main theorem of this section very surprising. It answers a question from MathOverflow [FMB]. Throughout, let k be an algebraically closed field of characteristic 0, let Γ be a finitely generated group, and let G be a (possibly disconnected) reductive k-group.

Recall from the dynamic method [CGP15, §2.1] that if $\lambda\colon \mathbf{G}_{\mathrm{m}} \to G$ is a cocharacter, then there is a closed subgroup $P_G(\lambda)$ of G defined by

$$P_G(\lambda) = \{g \in G\colon \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\},$$

which we call an *R-parabolic subgroup* of G. Similarly, an *R-Levi subgroup* of $P_G(\lambda)$ is a closed k-subgroup of the form $Z_G(\mu)$ (the centralizer of μ), where μ is a cocharacter such that $P_G(\lambda) = P_G(\mu)$. Recall the following definitions from [BMR05, §6].

Definition 1.1. We say that a representation $\rho\colon \Gamma \to G$ is *G-irreducible* if $\rho(\Gamma)$ does not lie in any R-parabolic subgroup P of $G_{\overline{k}}$ such that $P \cap G^0_{\overline{k}} \subset G^0_{\overline{k}}$ is a strict inclusion.[1] Say that ρ is *strongly G-irreducible* if $\rho|_{\Gamma'}$ is G-irreducible for every finite index subgroup Γ' of Γ.

We say that ρ is *G-completely reducible* if, whenever $\rho(\Gamma)$ is contained in an R-parabolic subgroup P of G, it is contained in an R-Levi subgroup L of P.

One can make completely analogous definitions for closed k-subgroups of G.

Theorem 1.2. *The subset $\mathscr{H}_1$ of $\mathscr{H} := \underline{\mathrm{Hom}}(\Gamma, G)$ consisting of strongly irreducible representations is Zariski open.*

Remark 1.3. One reason this theorem is surprising is that it is false if k is of positive characteristic. For instance, let $G = \mathrm{GL}_2$, and let Γ be the free group on two generators x and y, so $\mathscr{H} \cong \mathrm{GL}_2 \times \mathrm{GL}_2$. We may assume that k is algebraically closed, so k contains $\overline{\mathbf{F}}_p$. Note that any homomorphism $\rho\colon \Gamma \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ has finite image, so it cannot be strongly irreducible. Thus the complement of the strongly irreducible locus in $\mathscr{H}$ contains the dense subset $\mathrm{GL}_2(\overline{\mathbf{F}}_p) \times \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, but it is not all of $\mathscr{H}$ and hence cannot be closed.

There are many different places in the following argument where the assumption that k is of characteristic 0 appears; we will point these out when they appear. Given $\rho\colon \Gamma \to G$, let $H_\rho = \overline{\rho(\Gamma)}^0$ (identity component of the Zariski closure of $\rho(\Gamma)$).

Lemma 1.4. *A representation ρ is strongly G-irreducible if and only if H_ρ is G^0-irreducible.*

Proof. It is elementary to check that if Γ' is a finite index subgroup of Γ, then $H_\rho = H_{\rho|_{\Gamma'}}$. Moreover, there exists some such Γ' such that $\rho(\Gamma') \subset H_\rho$. Thus the result follows from the definitions and the facts that ρ is G-irreducible if and only if $\overline{\rho(\Gamma)}$ is G-irreducible, and a closed k-subgroup $H \subset G^0$ is G-irreducible if and only if it is G^0-irreducible. $\square$

We will deduce Theorem 1.2 from the following finiteness result.

[1] This definition is non-standard, but it agrees with the usual definition if G is connected and removes uninteresting cases in which the conditions in the definition of *strongly G-irreducible* never hold.

Theorem 1.5. *There exist finitely many proper Zariski closed subgroups $M_1, \dots, M_n$ of G (independent of Γ) such that for each i, the identity component M_i^0 is not G-irreducible, and if $\rho\colon \Gamma \to G$ is not strongly irreducible then ρ factors through a $G(k)$-conjugate of M_i for some i.*

Proof that Theorem 1.5 implies Theorem 1.2. For each $1 \leq i \leq n$, the image $G \cdot \underline{\mathrm{Hom}}(\Gamma, M_i)$ of the action morphism $G \times \underline{\mathrm{Hom}}(\Gamma, M_i) \to \underline{\mathrm{Hom}}(\Gamma, G)$ is constructible by Chevalley's theorem. From the equality

$$\mathscr{H} - \mathscr{H}_1 = \bigcup_{i=1}^{m} G \cdot \underline{\mathrm{Hom}}(\Gamma, M_i),$$

we deduce that $\mathscr{H}_1$ is constructible.

If Γ' is a subgroup of Γ, let $\pi_{\Gamma'}\colon \underline{\mathrm{Hom}}(\Gamma, G) \to \underline{\mathrm{Hom}}(\Gamma', G)$ be the restriction map, and let $\mathscr{H}_{0,\Gamma'}$ be the (open) irreducible locus in $\underline{\mathrm{Hom}}(\Gamma', G)$. Then $\mathscr{H}_1 = \bigcap_{\Gamma' \subset \Gamma} \pi_{\Gamma'}^{-1}(\mathscr{H}_{0,\Gamma'})$, where Γ' ranges over all finite index subgroups of Γ. A constructible subset C of a variety which is an intersection of Zariski open subsets is automatically Zariski open: indeed, the latter condition implies that C is closed under generization. □

Lemma 1.6. *A representation ρ is strongly irreducible if and only if it is irreducible and the centralizer $Z_G(H_\rho)/Z(G^0)$ is finite. In this case, $H_\rho/Z(G^0)$ is semisimple.*

Proof. ($\Rightarrow$) Let Γ' be a finite index subgroup of Γ such that $H_\rho = \overline{\rho(\Gamma')}$. Thus $Z_G(H_\rho) = Z_G(\rho(\Gamma'))$ which is the stabilizer of ρ in the variety $\underline{\mathrm{Hom}}(\Gamma', G)$. By [BMR05, Proposition 3.12], $Z_G(H_\rho)$ is reductive. If $Z_G(H_\rho)/Z(G^0)$ is not finite, then $Z_G(H_\rho)$ contains a torus S which is not central in G^0. But then H_ρ is contained in the proper R-Levi subgroup $Z_G(S)$, contradicting the assumption of irreducibility. Moreover, H_ρ is reductive. The maximal central torus of H_ρ lies in $Z_G(H_\rho)$, so it must lie in $Z(G^0)$. Thus the image of H_ρ in $G/Z(G^0)$ is semisimple.

($\Leftarrow$) Under the assumptions, the group H_ρ is reductive, and it follows from [BMR05, Lemma 2.6] that H_ρ is G-completely reducible. Thus if H_ρ is not G-irreducible, it lies in a proper Levi subgroup. But then $Z_{G^0}(H_\rho)$ contains a non-central torus, contradicting finiteness of $Z_G(H_\rho)/Z(G^0)$. □

By Lemmas 1.4 and 1.6, if $\rho \in \mathscr{H} - \mathscr{H}_1$ then either

- (A) The representation ρ is not G-irreducible, or
- (B) The group H_ρ is reductive and $Z_G(H_\rho)^0/Z(G)$ is not semisimple, or
- (C) The groups $H_\rho/Z(G)$ and $Z_G(H_\rho)/Z(G)$ are semisimple, and $Z_G(H_\rho)/Z(G)$ is positive-dimensional.

We will describe the locus of $\rho \in \mathscr{H}_0$ satisfying one of the conditions (A), (B), and (C).

Note first that, by definition, (A) holds if and only if $\rho(\Gamma)$ lies in a proper R-parabolic subgroup of G. Recall that there are only finitely many $G(k)$-orbits of parabolic subgroups of G^0. If P is an R-parabolic subgroup of G, then $P \subset N_G(P^0)$, so in fact ρ factors through the normalizer of a parabolic subgroup of G^0.

If (B) holds, let S be the maximal central torus of $Z_G(H_\rho)^0$, and let $L = Z_{G^0}(S)$, a Levi subgroup of G^0. Since $\rho(\Gamma)$ normalizes H_ρ, it also normalizes $Z_G(H_\rho)^0$, and hence S. Thus ρ lies in the image of the conjugation map $G \times \underline{\mathrm{Hom}}(\Gamma, N_G(L)) \to \underline{\mathrm{Hom}}(\Gamma, G)$. Conversely, if ρ lies in this image, then (B) holds.

Now we deal with (C), which is the most interesting (and complicated) case. The main problem is that, although H_ρ lies a proper Levi subgroup L of G, it is not typically true that $\rho(\Gamma)$ normalizes any such L: for instance, if a subgroup Λ of $\mathrm{SL}_2(\mathbf{C})$ normalizes a nontrivial torus, then it is solvable, but $\mathrm{SL}_2(\mathbf{C})$ admits non-solvable finite subgroups.

Proposition 1.7. *For every integer $N \geq 1$, there are only finitely many $G(k)$-conjugacy classes of closed k-subgroups H of G containing $Z(G^0)$ such that $H^0/Z(G^0)$ is semisimple and $|\pi_0(H)| \leq N$.*

Proof. By passing from G to $G/Z(G^0)$, we may assume that G^0 is semisimple. First we handle the case of finite subgroups H. There are only finitely many finite groups of order $\leq N$ up to isomorphism, so we may fix one such group H. Note that $\underline{\mathrm{Hom}}(H, G)$ is a smooth affine k-scheme: indeed, by (e.g.) [SGA3$_{\text{I new}}$, Exposé III, Théorème 2.1(i)] the obstruction to the infinitesimal lifting criterion from $\underline{\mathrm{Hom}}(H, G) \to \operatorname{Spec} k$ lies in $\mathrm{H}^2(H, \mathfrak{g})$, which vanishes because H is finite and k is of characteristic 0. Furthermore, fixing a homomorphism $f\colon H \to G(k)$, there is an orbit map $\alpha_f\colon G \to \underline{\mathrm{Hom}}(H, G)$ through f. By (e.g.) [SGA3$_{\text{I new}}$, Exposé III, Théorème 2.1(ii)], the obstruction to smoothness of α_f lies in $\mathrm{H}^1(H, \mathfrak{g})$, which again vanishes. Thus α_f is smooth, and in particular it has open image. Thus every orbit is open, from which it follows that every orbit is also closed. Being affine, it follows that $\underline{\mathrm{Hom}}(H, G)$ is a *finite* union of clopen G-orbits. For each such orbit consisting of injective homomorphisms, choose a point ι, and note that $\{\iota(H)\}$ constitutes a system of representatives for the subgroups of G isomorphic to H.

Next we handle the case of connected semisimple subgroups H. By the classification of reductive groups by root data, there are only finitely many isomorphism classes of connected semisimple groups H of dimension $\leq \dim G$. Fix such an H. By [SGA3$_{\text{III new}}$, Exposé XXIV, Proposition 7.3.1], there is a smooth affine scheme $\underline{\mathrm{Hom}}(H, G)$ which parameterizes homomorphisms $f\colon H \to G$. For such f, there is an orbit map $\alpha_f\colon G \to \underline{\mathrm{Hom}}(H, G)$ given by $g \mapsto gfg^{-1}$, and because H is reductive and k is of characteristic 0, the same argument as in the case of finite groups shows that α_f is smooth, from which the result follows precisely as in the first paragraph.

Now consider the general case. Let $H_1, \ldots, H_m$ be representatives for the $G(k)$-orbits of connected semisimple k-subgroups of G, and for each i let $\Lambda_{i1}, \ldots, \Lambda_{in_i}$ be representatives for the $G(k)$-orbits of finite subgroups of $N_G(H_i)/H_i$ (a reductive group by [BMR05, Proposition 3.12]) of order $\leq N$. If H is a closed k-subgroup of G such that H^0 is semisimple and $|\pi_0(H)| \leq N$, then by conjugacy we may assume $H^0 = H_i$ for some i. Moreover, the image of H in $N_G(H_i)/H_i$ is of order $\leq N$, so after conjugating by an element of $N_G(H_i)(k)$ we may assume the image of H in $N_G(H_i)/H_i$ is equal to Λ_{ij} for some j. Since H is the preimage of Λ_{ij} in $N_G(H_i)$, we have proven the result. $\square$

Remark 1.8. The proof of Proposition 1.7 uses the fact that representations of reductive groups in characteristic 0 are semisimple to obtain cohomology vanishing results (both explicitly and implicitly, since smoothness of $\underline{\mathrm{Hom}}(H, G)$ relies on vanishing of $\mathrm{H}^2(H, \mathfrak{g})$). In fact, the result fails if k is of characteristic $p > 0$, even for irreducible subgroups of G. To see this, for each $n \geq 0$ let V_n be the irreducible SL_2-representation of highest weight n. By the Steinberg tensor product theorem [Jan03, 3.17], if $n \geq 1$ we have $V_{p^n+1} \cong V_1^{(p^n)} \otimes V_1$ for all n, where $V_1^{(p^n)}$ denotes the nth Frobenius twist of V_1. In particular, each V_{p^n+1} is 4-dimensional. Moreover, each map $\rho_n\colon\ \mathrm{SL}_2 \to \mathrm{SL}_4$ induced by V_{p^n+1} is monic, so the images of the ρ_n give a sequence of pairwise non-conjugate connected semisimple subgroups of SL_4 in characteristic p.

Proposition 1.9. *If H is a reductive k-group, then there is a positive integer N such that for every finite subgroup $\Lambda \subset H$ of order $\geq N$, there is a nontrivial torus $S \subset H^0$ normalized by Λ.*

Proof. If H^0 has a nontrivial central torus then this is clear, so assume H^0 is semisimple. By Jordan's theorem and finiteness of H/H^0, there is a constant N_0 such that every finite subgroup $\Lambda \subset H$ admits an abelian normal subgroup $\Lambda_0 \subset \Lambda \cap H^0$ of index $\leq N_0$. Then Λ normalizes $Z_H(\Lambda_0)^0$, so it suffices to show that if H is connected then $Z_H(\Lambda_0)^0$ contains a nontrivial central torus for all sufficiently large finite abelian groups $\Lambda_0 \subset H^0$.

From now on, we may and do assume H is connected. There is a constant N_1 such that if $t \in H(k)$ is semisimple then t^{N_1} is contained in a nontrivial central torus of $Z_H(t)^0$: indeed, let N_1 be the maximum of $|\pi_0(Z(H'))|$ where H' ranges over the (finitely many $G(k)$-orbits of) maximal rank connected reductive subgroups of H. Since t lies in $Z(Z_H(t)^0)$ and $Z_H(t)^0$ is a maximal rank reductive subgroup, we find that t^{N_1} lies in the maximal central torus of $Z_H(t)^0$.

Let T be a maximal torus of H, and let W be the Weyl group of (H,T). Let N_2 be the maximum of $|\pi_0(S^{W'})|+1$, where S ranges over all maximal central subtori of maximal rank connected reductive subgroups M of H containing T and W' ranges over all subgroups of W which stabilize S. Let $\Lambda_0 \subset H$ be a finite abelian subgroup containing an element t of order $\geq N_1N_2$. Let S be the maximal central torus of $Z_H(t)^0$, which is nontrivial since it contains t^{N_1} by the previous paragraph. Note that Λ_0 normalizes S, and $|\pi_0(S^{\Lambda_0})| < N_2$ by the above. Because S^{Λ_0} contains the order $\geq N_2$ element t^{N_1}, it follows that $\left(S^{\Lambda_0}\right)^0$ is nontrivial. This exhibits a nontrivial central torus of $Z_H(\Lambda_0)$ when Λ_0 contains an element of order $\geq N_1N_2$.

If H embeds into GL_{N_3}, then for any finite abelian subgroup Λ_0 of H of order $\geq (N_1N_2)^{N_3}$ there is an element $t \in \Lambda_0$ of order $\geq N_1N_2$ (since finite abelian subgroups of GL_{N_3} are diagonalizable). By the previous paragraph, $Z_H(\Lambda_0)$ contains a nontrivial central torus. □

Remark 1.10. Proposition 1.9 is also false in positive characteristic: indeed, $\mathrm{SL}_{2,\overline{\mathbf{F}}_p}$ contains the finite subgroup $\mathrm{SL}_2(\mathbf{F}_{p^n})$ for each n, and this does not normalize any maximal torus. In the proof, the assumption on the characteristic of k only appeared in the use of Jordan's theorem. In particular, Proposition 1.9 remains true in positive characteristic if H is connected and Λ is abelian. We have made no attempt to optimize the constant N.

Proof of Theorem 1.5. By Proposition 1.7, there are finitely many closed subgroups $H_1, \ldots, H_m$ of G containing $Z(G^0)$ such that $Z_{G/Z(G^0)}(H_i/Z(G^0))$ is positive-dimensional for all i and such that if $H \subset G$ is any subgroup with these properties then it is $G(k)$-conjugate to some such H_i. If $\rho \in \mathscr{H}(k)$ satisfies (C) above, then a G-conjugate of ρ factors through $N_G(H_i)$ for some i, and the induced map $\Gamma \to N_G(H_i)/H_i$ has finite image.

Let $\overline{\rho}$ be the induced map $\Gamma \to G/Z(G^0)$, and let N be a positive integer which satisfies the condition of Proposition 1.9 for each group $N_{G/Z(G^0)}(\overline{H}_i)/\overline{H}_i$, where $\overline{H}_i = H_i/Z(G^0)$. If the image of $\overline{\rho}$ is of order $\geq N$, then by the proposition there is a nontrivial torus $S' \subset N_{G/Z(G^0)}(\overline{H}_i)/\overline{H}_i$ which is normalized by Γ. The map $Z_G(H_i)^0 \to N_{G/Z(G^0)}(\overline{H}_i)^0/\overline{H}_i$ is surjective with central kernel (because the group scheme $\mathrm{Aut}_{\overline{H}_i/k}/\overline{H}_i^{\mathrm{ad}}$ is étale), so $Z_G(H_i)^0$ admits a central torus strictly containing $Z(G^0)$ and ρ satisfies (B).

By Proposition 1.7, there are finitely many closed subgroups $\{H_{ij}\}$ of G such that $H_{ij}^0 = H_i$ and $|\pi_0(H_{ij})| < N \cdot |\pi_0(Z(G^0))|$ for all i,j, and such that any closed subgroup of G with these properties is $G(k)$-conjugate to H_{ij} for some i,j. By the previous paragraph, if ρ factors through $N_G(H_i)$, either ρ satisfies (B) or there is a $G(k)$-conjugate of ρ which factors through some H_{ij}. Conversely, if ρ factors through some H_{ij} then H_ρ is contained in $H_{ij}^0 = H_i$ and thus $\rho \notin \mathscr{H}_1$. This shows that if ρ is not strongly G-irreducible then either ρ factors through the normalizer of a parabolic subgroup of G^0, the normalizer of a Levi subgroup of G^0, or a $G(k)$-conjugate of one of the H_{ij}. Since none of these subgroups has G-irreducible identity component, we conclude. □

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