RATIONALITY OF THE VARIETY OF MAXIMAL TORI

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In [SGA3, Exposé XIV], Grothendieck develops a theory of regular sections and Cartan subalgebras of the Lie algebras of smooth affine group schemes G over a base scheme S. As applications, Grothendieck deduces several results: rationality of the variety of maximal tori (Proposition 2.1) and unirationality of reductive groups (Proposition 2.4). Moreover, results about the Zariski-local lifting of maximal tori (Proposition 2.6) are important to Nisnevich's approach to the Grothendieck–Serre conjecture [Nis89].

The results of [SGA3, Exposé XIV] (like all of [SGA3]) are extremely general: for instance, they do not assume that S is the spectrum of a field or that G is reductive. In fact, it is crucial to assume as little as possible about S in order to facilitate functorial arguments. Moreover, throughout [SGA3] there are sometimes good reasons that one would not wish to assume that G is reductive: for instance, sometimes it is important to allow G to be a parabolic subgroup of a reductive group scheme or (for a more modern example) a pseudo-reductive or quasi-reductive group over an imperfect field.

However, in order to understand the theory developed in [SGA3, Exposé XIV], there is a compelling reason to restrict to the case that G is reductive: namely, in this case the arguments proving the main results can be condensed into five pages. The purpose of this note is to explain these arguments in a way which is more accessible than [SGA3], with the additional hope that the curious reader will find the arguments in [SGA3] easier to digest having seen a relatively simple case. That said, we will assume familiarity with scheme theory and some of the basic theory of reductive group schemes, for the latter of which an efficient reference is [Con14], especially Section 3.

1. Maximal tori and Cartan subalgebras

We aim to prove a result concerning the correspondence between maximal tori in semisimple group schemes of adjoint type and Cartan subalgebras of their Lie algebras. Before stating the theorem, we need to introduce some terminology. We begin with the case over a field. If $\mathfrak{g}$ is a Lie algebra over a ring A and $a \in \mathfrak{g}(A)$, let $\mathrm{Nil}(a) = \bigcup_{n \geq 1} \ker((\mathrm{ad}\, a)^n)$.

Definition 1.1. Let k be a field, and let $\mathfrak{g}$ be a Lie algebra of dimension n over k. Let $r = \min\{\dim \mathrm{Nil}(x) \colon x \in \mathfrak{g}(\overline{k})\}$. An element $a \in \mathfrak{g}(k)$ is *regular* if the characteristic polynomial of $\mathrm{ad}\, a$ is of the form $X^n + a_{n-1}X^{n-1} + \cdots + a_r X^r$, where $a_r \neq 0$.

A Lie subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is a *Cartan subalgebra* if there exists a regular element $a \in \mathfrak{g}(\overline{k})$ such that $\mathfrak{t}_{\overline{k}} = \mathrm{Nil}(a)$.

It is clear from the definition that the locus of regular elements of $\mathfrak{g}$ is open and dense.

Definition 1.2. [SGA3, Exposé XIV, Définitions 2.4 et 2.5] Let S be a scheme, and let $\mathfrak{g}$ be a finite locally free $\mathcal{O}_S$-Lie algebra. An $\mathcal{O}_S$-Lie subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is a *Cartan subalgebra* if $\mathfrak{t}$ is fpqc-locally a direct summand of $\mathfrak{g}$ (i.e., $\mathfrak{g}/\mathfrak{t}$ is flat) and, for all $s \in S$, the $k(s)$-subalgebra $\mathfrak{t}(k(s))$ of $\mathfrak{g}(k(s))$ is a Cartan subalgebra.

A section $a \in \mathfrak{g}(S)$ is *quasi-regular* if a_s is a regular element of $\mathfrak{g}(\overline{k(s)})$ for all $s \in S$; we say that a is *regular* if it is quasi-regular and contained in a Cartan subalgebra of $\mathfrak{g}$.

In Lemma 1.10, we will see that quasi-regular sections are automatically regular if G is of adjoint type. For more general G, these are distinct notions; see Remark 1.6.

Recall also the following definition.

Definition 1.3. If S is a scheme and G is a smooth affine S-group scheme, then a smooth closed S-subgroup scheme $T \subset G$ is a *maximal torus* if, for every geometric point $\overline{s}$ of S, the $k(\overline{s})$-subgroup $T_{\overline{s}} \subset G_{\overline{s}}$ is a maximal torus.

Note in particular that, if $S = \operatorname{Spec} k$ for a field k, then a *maximal torus* $T \subset G$ is a torus which is maximal after passing to *any* field extension k'/k, not just over k.[1]

Theorem 1.4. [SGA3, Exposé XIV, Théorèmes 3.9 et 3.18] *Let S be a scheme and let G be a semisimple S-group scheme of adjoint type. The function*

$$\{\textit{maximal tori of } G\} \to \{\textit{Cartan subalgebras of } \mathfrak{g}\}$$

given by $T \mapsto \operatorname{Lie} T$ is a bijection.

The key result in the proof of Theorem 1.4 is the following.

Proposition 1.5. *Let S be a scheme, and let G be a semisimple S-group scheme of adjoint type.*

(1) For any maximal torus T of G, the Lie algebra $\operatorname{Lie} T$ is a Cartan subalgebra of $\mathfrak{g}$.
(2) If S is the spectrum of an algebraically closed field then two Cartan subalgebras of $\mathfrak{g}$ are Ad_G-conjugate.
(3) If $a \in \mathfrak{g}(S)$ is quasi-regular and $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra, then the transporter $\operatorname{Transp}_G(a, \mathfrak{t})$ is representable and smooth.
(4) If $a \in \mathfrak{g}(S)$ is quasi-regular, then it is regular and lies in a unique Cartan subalgebra of $\mathfrak{g}$.
(5) If $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra, then the normalizer $N_G(\mathfrak{t})$ is smooth with Lie algebra $\mathfrak{t}$.

Using Proposition 1.5, we now prove the theorem.

Proof of Theorem 1.4. First, note that the map in the theorem is well-defined: this is just the statement of Proposition 1.5(1).

Next, the map is injective: if $H \subset G$ is a smooth closed subgroup with connected S-fibers such that $\mathfrak{t} := \operatorname{Lie} H$ is a Cartan subalgebra of $\mathfrak{g}$, then the adjoint action of H on $\mathfrak{g}$ normalizes $\mathfrak{t}$, so $H \subset N_G(\mathfrak{t})^0$, and this is an equality by Proposition 1.5(5) and the fibral isomorphism criterion. So a maximal torus is determined by its Lie algebra, i.e., the map is injective.

Finally, we show that the map is surjective. Let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan subalgebra, so $N := N_G(\mathfrak{t})$ is smooth by Proposition 1.5(5) and $H := N^0$ is smooth with connected S-fibers. To show that H is a maximal torus, we may work étale-locally to assume that there is some maximal torus $T \subset G$ over S. Let $\mathfrak{t} = \operatorname{Lie} T$, so that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ by Proposition 1.5(1). If $a \in \mathfrak{t}(S)$ is a regular section, which exists étale-locally on S, then $\operatorname{Transp}_G(\mathfrak{t}, \mathfrak{t}) = \operatorname{Transp}_G(a, \mathfrak{t})$ by Proposition 1.5(4), so the former is smooth by Proposition 1.5(3). Since $\operatorname{Transp}_G(\mathfrak{t}, \mathfrak{t})$ has nonempty geometric fibers by Proposition 1.5(2), it has a section étale-locally, i.e., by localizing we may assume there exists $g \in G(S)$ such that $\operatorname{Ad}(g)\mathfrak{t} \subset \mathfrak{t}$. Then $gTg^{-1} \subset H$ by the injectivity of the previous paragraph, so $gTg^{-1} = H$: indeed, the inclusion $gTg^{-1} \subset H$ of induces an isomorphism on Lie algebras by Proposition 1.5(5), so it is an isomorphism on fibers over S. Thus the fibral isomorphism criterion shows that H is a torus. $\square$

Thus we are reduced to proving Proposition 1.5, which we will do in the next section.

Remark 1.6. If G is not assumed to be of adjoint type, then Theorem 1.4 is false. For instance, if $G = \mathrm{SL}_2$ and S is the spectrum of a field of characteristic 2, then $\mathfrak{g}$ is a nilpotent Lie algebra, so the only Cartan subalgebra of $\mathfrak{g}$ is $\mathfrak{g}$ itself. However, see [SGA3, Exposé XIV, Théorème 3.9] for a generalization, yielding a correspondence between "subgroups of type (C)" and Cartan subalgebras, even for non-reductive G. Note also that SL_2 does not admit Cartan subalgebras in mixed characteristic $(0, 2)$, so although quasi-regular sections exist there is no regular section.

[1] In fact, these two conditions are equivalent, but that is not a priori clear.

On the other hand, a careful reading of the proof shows that the result remains true as long as there exists an fpqc cover $S' \to S$ and an S'-split maximal S'-torus $T' \subset G_{S'}$ such that

(1) $Z_{G_{S'}}(T')^0 = T'$,
(2) if α is a weight of T' on $\mathfrak{g}_{S'}$, then $\ker(\alpha)$ is smooth.

Although (2) is a standard fact when G is semisimple with trivial center, it seems to rely on a large part of the structure theory of reductive groups over algebraically closed fields.

1.1. **Proof of Proposition 1.5.** We proceed in order. Let T be a maximal S-torus of G and let $\mathfrak{t} = \operatorname{Lie} T$. Working étale-locally on S, we may assume that T is split, in which case there is a decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ into T-weight spaces. In particular, $\mathfrak{t}$ is a direct summand of $\mathfrak{g}$. So to prove (1), we can assume $S = \operatorname{Spec} k$ for an algebraically closed field k, in which case it suffices to prove the following lemma.

Lemma 1.7. *There exists a regular element $a \in \mathfrak{t}(k)$ with $\mathfrak{t} = \operatorname{Nil}(a)$. In particular, $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.*

Proof. If $\alpha \in \Phi$, then $\ker(\alpha)$ is a subtorus of T since G is of adjoint type. (One way to see this is to note that any choice of simple roots forms a basis for $X(T)$, and any root lies in a system of simple roots.) In particular, $\dim\ker(\mathrm{d}\alpha) = r-1$, so because k is infinite there is an element $a \in \mathfrak{t}(k)$ such that $\mathrm{d}\alpha(a) \neq 0$ for all roots α. Thus $\operatorname{Nil}(a) = \mathfrak{t}$, as desired. □

For (2), continue to assume $S = \operatorname{Spec} k$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\varphi\colon G \times \mathfrak{t} \to \mathfrak{g}$ be the map given by the adjoint action.

Lemma 1.8. *If $a \in \mathfrak{t}(k)$ is regular, then φ is smooth at (e, a). In particular, φ is dominant.*

Proof. Note that $\mathfrak{t} \subset \operatorname{Nil}(a)$ since $\mathfrak{t}$ is commutative. Moreover, the multiplicity of 0 as an eigenvalue of a is r by assumption, and by the discussion in the previous section we find $\mathfrak{t} = \operatorname{Nil}(a)$. Thus $\operatorname{ad}(a)_{\mathfrak{g}/\mathfrak{t}}$ is injective, and for dimension reasons it is also surjective. Now

$$\mathrm{d}\varphi_{(e,a)}(\xi, x) = [\xi, a] + x,$$

so $d\varphi_{(e,a)}$ is surjective and thus φ is smooth at (e, a). Dominance follows from the fact that smooth morphisms are topologically open. □

Now let $\mathfrak{t}'$ be another Cartan subalgebra of $\mathfrak{g}$. By the lemma above applied to $\mathfrak{t}'$, the adjoint map $\varphi' : G \times \mathfrak{t}' \to \mathfrak{g}$ is dominant, so there is some open subscheme $U \subset \mathfrak{g}$ such that every $x \in U(k)$ is $G(k)$-conjugate to an element of $\mathfrak{t}$ and to an element of $\mathfrak{t}'$. Since the regular locus is open and dense in $\mathfrak{g}$ by Lemma 1.7, there is some regular element $x \in U(k)$. Let $g, g' \in G(k)$ be such that $\operatorname{Ad}(g)x \in \mathfrak{t}(k)$ and $\operatorname{Ad}(g')x \in \mathfrak{t}'(k)$, and note that $\mathfrak{t} = \operatorname{Nil}(\operatorname{Ad}(g)x) = \operatorname{Ad}(g)\operatorname{Nil}(x)$ and $\mathfrak{t}' = \operatorname{Nil}(\operatorname{Ad}(g')x) = \operatorname{Ad}(g')\operatorname{Nil}(x)$ for dimension reasons. This proves (2).

Next we prove (3). Drop the assumption that $S = \operatorname{Spec} k$ and let $a \in \mathfrak{g}(S)$ be a quasi-regular section. The fact that $\operatorname{Transp}_G(a, \mathfrak{t})$ is representable comes from the Cartesian diagram

$$\begin{tikzcd} \operatorname{Transp}_G(a,\mathfrak{t}) \arrow[r] \arrow[d] & G \times \mathfrak{t} \arrow[d, "\varphi"] \\ S \arrow[r, "a"] & \mathfrak{g} \end{tikzcd}$$

where φ is the conjugation map as in the previous section.

Lemma 1.9. *The transporter scheme $\operatorname{Transp}_G(a, \mathfrak{t})$ is smooth.*

Proof. Note that φ is smooth over quasi-regular sections of $\mathfrak{g}$: by the fibral flatness criterion and the fact that "smooth = flat with smooth fibers", it suffices to check this when $S = \operatorname{Spec} k$ for an algebraically closed field k. In this case, since φ is G-equivariant it suffices to prove smoothness at points (e, a) where $a \in \mathfrak{g}$ is regular. This was proved in Lemma 1.8. So the above Cartesian diagram implies the lemma. □

The proof of (4) is contained in the following lemma.

Lemma 1.10. *If S is a scheme, G is a semisimple S-group scheme of adjoint type with Lie algebra $\mathfrak{g}$, and $a \in \mathfrak{g}(S)$ is quasi-regular, then it is regular and there is precisely one Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing a, namely* $\mathrm{Nil}(a)$.

Proof. By the uniqueness claim, we may work locally to assume $S = \operatorname{Spec} A$ for a complete noetherian local ring A with algebraically closed residue field. By Proposition 1.5(3) and (2), if $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ (which exists by (1) and, e.g., [Con14, Corollary B.3.5]) then there is an element $g \in G(A)$ such that $\mathrm{Ad}(g)a \in \mathfrak{t}(A)$. In particular, a is regular. Moreover, if $a \in \mathfrak{t}'(A)$ for another Cartan subalgebra $\mathfrak{t}'$ of $\mathfrak{g}$, then because $\mathfrak{t}'$ is commutative we have $\mathfrak{t}' \subset \mathrm{Nil}(a)$. The formation of $\mathrm{Nil}(a)$ commutes with base change (by quasi-regularity), and the map $\mathfrak{t}' \otimes_A A/\mathfrak{m} \to \mathrm{Nil}(a) \otimes_A A/\mathfrak{m}$ is an isomorphism (again by quasi-regularity), so by Nakayama's lemma it follows that $\mathfrak{t}' = \mathrm{Nil}(a)$, proving uniqueness. □

Finally, we prove (5). We need to show that $N_G(\mathfrak{t})$ is smooth with Lie algebra $\mathfrak{t}$. To show smoothness of $N_G(\mathfrak{t})$, we may assume that S is the spectrum of an Artin local ring with algebraically closed residue field k. In this case $\mathfrak{t}$ admits a quasi-regular section a (by simply lifting a regular element on the special fiber), and $N_G(\mathfrak{t}) = \mathrm{Transp}_G(a, \mathfrak{t})$ since $\mathfrak{t} = \mathrm{Nil}(a)$. So $N_G(\mathfrak{t})$ is smooth by Lemma 1.9. Moreover, we have

$$\mathrm{Lie}\, N_G(\mathfrak{t}) = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{t}) \coloneqq \{x \in \mathfrak{g} \colon [x, \mathfrak{t}] \subset \mathfrak{t}\},$$

as one sees from the functorial definition of $N_G(\mathfrak{t})$. We are therefore reduced to the following lemma.

Lemma 1.11. *The Cartan subalgebra $\mathfrak{t}$ is equal to its own normalizer.*

Proof. Since S is the spectrum of a strictly henselian local ring, T is split and thus there is a weight space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. By Lemma 1.7, if $\mathfrak{t} = \mathrm{Nil}(a)$ for a regular section $a \in \mathfrak{t}(R)$, then $\mathrm{d}\alpha(a) \neq 0$ for all $\alpha \in \Phi$. In particular, if $x = x_t + \sum_{\alpha \in \Phi} x_\alpha$ for $x_t \in \mathfrak{t}(R)$ and $x_\alpha \in \mathfrak{g}_\alpha(R)$, then $[a, x] = \sum_{\alpha \in \Phi} \mathrm{d}\alpha(a) x_\alpha$, which lies in $\mathfrak{t}(R)$ if and only if $x_\alpha = 0$ for all $\alpha \in \Phi$. It follows that $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$. □

2. Some consequences of Theorem 1.4

In this section we record a few consequences of Theorem 1.4 whose proofs deserve to be better-known. Throughout this section, let k be a field and let G be a connected reductive k-group.

Below, we will use the fact (proved by Serre in the appendix to [SGA3, Exposé XIV] using classical methods) that the Lie algebra of a semisimple group of adjoint type G admits a regular element over any field k. If k is infinite, then this is obvious from Lemma 1.7, but if k is finite then it is *not* true in general that the Lie algebra of every maximal torus $T \subset G$ contains a regular element (see Remark 2.8). However, Serre shows explicitly that this is true if T is a Coxeter torus.

Proposition 2.1. [SGA3, Exposé XIV, Théorème 6.1] *Let k be a field and let G be a connected reductive group over k. The variety $\mathscr{T}_G$ of maximal tori of G is rational.*

Proof. The natural map $\mathscr{T}_G \to \mathscr{T}_{G^{\mathrm{ad}}}$ is an isomorphism, so we may and do assume that G is of adjoint type. Note that Theorem 1.4 shows that, for a k-scheme S, the set $\mathscr{T}_G(S)$ is equal to the set of Cartan subalgebras $\mathfrak{t}$ of $\mathfrak{g}_S$. By [SGA3, Exposé XIV, Appendice], there exists a regular element $a \in \mathfrak{g}(k)$. Let $\mathfrak{t} = \mathrm{Nil}(a)$, and let M be a linear complement to $\mathfrak{t}$ in $\mathfrak{g}$. Let $\mathscr{T}_M$ be the open subfunctor of $\mathscr{T}_G$ consisting of those Cartan subalgebras $\mathfrak{t}'$ such that $\mathfrak{t}' \oplus M_S = \mathfrak{g}_S$ (i.e., $\mathfrak{t}' \to \mathfrak{g}_S/M_S$ is an isomorphism) and such that the unique element of $\mathfrak{t}' \cap (a_S + M_S)$ is regular. (The property of a map of vector bundles being an isomorphism is open because the kernel and cokernel both have closed support, and regularity is an open condition.) There is a natural inclusion $i\colon \mathscr{T}_M \to M$ (regarding M as an affine space) given by sending $\mathfrak{t}'$ to the unique element of $\mathfrak{t}' \cap (a_S + M_S)$. Moreover, the image of i consists of those sections $u' \in M(S)$ such that $a_S + u'$ is regular and the unique Cartan subalgebra of $\mathfrak{g}$ containing $a_S + u'$ is a linear complement to M. Thus $\mathscr{T}_M$ is an open subfunctor of M, so $\mathscr{T}_M$ is rational and thus $\mathscr{T}_G$ is rational. □

Corollary 2.2. [SGA3, Exposé XIV, Théorème 1.1] *There is a maximal k-torus $T \subset G$ (which remains maximal after any field extension k'/k).*

Proof. Note that $\mathscr{T}_G$ is a homogeneous space for G, so if k is finite then Lang's theorem shows that it has a rational point. If k is infinite, then this follows from Proposition 2.1. □

Remark 2.3. Corollary 2.2 is basic in the theory of algebraic groups. The proof that we give is conceptually rather complicated, as it depends on the structure theory of reductive groups over algebraically closed fields as well as the theory of tori over general base schemes, but it is not circular: the only serious elements of the relative theory we have used concern the fpqc-local existence and conjugacy of maximal tori (which relies principally on deformation theory for tori) and the representation theory of split tori. However, it is the "wrong" proof, in the sense that if one wishes to develop the theory of reductive groups over general fields before developing it over algebraically closed fields, it is good to know as early as possible that maximal tori exist.

Proposition 2.4. [SGA3, Exposé XIV, Corollaire 6.5] *Let k be a field and let G be a connected reductive group over k. Then G is unirational, i.e., there exists an open subscheme U of some affine k-space and a dominant k-morphism $U \to G$. In particular, if k is infinite then $G(k)$ is Zariski-dense in G.*

Proof. First, assume that G is a torus. Let k'/k be a finite separable field extension such that $G_{k'}$ is split (and in particular rational). Thus the Weil restriction $G' = \mathrm{R}_{k'/k} G_{k'}$ is rational. The norm morphism $N\colon G' \to G$ is surjective, so G is unirational.

Now consider the general case. It is enough to show that the fraction field of G is contained in a purely transcendental extension of k. Let Ω be the dense open subscheme of G consisting of regular semisimple elements (i.e., elements g whose connected centralizer is a maximal torus of G). Let $\mathscr{A} \to \mathscr{T}_G$ denote the universal torus over $\mathscr{T}_G$, and note that there is a natural open embedding $\Omega \subset \mathscr{A}$, so it suffices to show that $\mathscr{A}$ is unirational. If L is the fraction field of $\mathscr{T}_G$ and M is the fraction field of $\mathscr{A}$, then Proposition 2.1 shows that L is a purely transcendental extension of k, and the previous paragraph shows that M is contained in a purely transcendental extension of L, so we conclude. The final claim is a general fact about unirational varieties. □

Remark 2.5. If G is smooth affine but not necessarily reductive, then Proposition 2.1 (and therefore also Corollary 2.2) remains true, while Proposition 2.4 remains true if k is perfect, though it can fail for imperfect k. (See [SGA3, Exposé XIV, Remarque 6.11 a)].) We will give a brief sketch of the proofs, with the hope that this will give the interested reader some addded context for [SGA3, Exposé XIV].

First, there is a version of Theorem 1.4 (see [SGA3, Exposé XIV, Théorème 3.9]) with "maximal tori" replaced by "subgroups of type (C)", i.e., smooth subgroup schemes with connected fibers whose Lie algebras are Cartan subalgebras of $\mathfrak{g}$. If $\mathscr{D}_G$ is the scheme of subgroups of type (C), then one shows by the same argument as in Proposition 2.1 that $\mathscr{D}_G$ is rational when $\mathfrak{g}$ admits a regular element (see [SGA3, Exposé XIV, Corollaire 2.17]).

Moreover, if $\mathscr{C}_G$ is the scheme of Cartan subgroups of G (i.e., centralizers of maximal tori), then there is a natural morphism $\mathscr{C}_G \to \mathscr{D}_G$ defined in [SGA3, Exposé XIV, §5] sending a Cartan subgroup to the unique subgroup of type (C) containing it. In [SGA3, Exposé XIV, Théorème 6.1], it is shown by induction on $\dim G$ that the generic fiber of $\mathscr{C}_G \to \mathscr{D}_G$ is rational, proving the analogue of Proposition 2.1. There is also a natural functor $\mathscr{T}_G \to \mathscr{C}_G$, given by sending T to

$Z_G(T)$, which is shown to be an isomorphism in [SGA3, Exposé XII, Théorème 3.2]. (See also the simpler [SGA3, Exposé XII, Proposition 1.13].) Thus $\mathscr{T}_G$ is rational, i.e., Proposition 2.1 holds.

Finally, we establish the Zariski-semi-local existence of maximal tori of G.

Proposition 2.6. *Let G be a reductive group scheme over an affine semi-local base scheme S with closed points $s_1, \ldots, s_n$. If T_i is a maximal $k(s_i)$-subtorus of G_{s_i} for all i such that each $\mathfrak{t}_i := \operatorname{Lie} T_i$ contains a regular element over $k(s_i)$, then there exists a maximal S-torus $T \subset G$ such that $T_{s_i} = T_i$ for all i.*

Proof. As usual, we may assume that G is of adjoint type. Let $a_0 \in \prod_{i=1}^n \mathfrak{t}_i(k(s_i))$. Since $\mathfrak{g}$ is an affine space, there exists an element $a \in \mathfrak{g}(S)$ whose image in $\mathfrak{g}\left(\prod_{i=1}^n k(s_i)\right)$ is equal to a_0, and since the regular locus in $\mathfrak{g}$ is open it follows that a has regular fibers. By Lemma 1.10 and Theorem 1.4, there is a unique maximal S-torus T of G such that $a \in \mathfrak{t}(S)$. This implies $T_{s_i} = T_i$ for all i, as desired. $\square$

Note that the hypothesis on the existence of regular elements in Proposition 2.6 always holds if each $k(s_i)$ is infinite, so in this case all maximal tori lift Zariski-semi-locally. In the general case, we still have the following weaker result.

Corollary 2.7. *Let G be a reductive group scheme over a semi-local base scheme S. Then G admits a maximal S-torus.*

Proof. By [SGA3, Exposé XIV, Appendice], there is a regular element in $\mathfrak{g}\left(\prod_{i=1}^n k(s_i)\right)$, so the result follows from Proposition 2.6. $\square$

Remark 2.8. Unfortunately, if k is a field and G is a semisimple k-group of adjoint type with maximal k-torus T, then $\mathfrak{t}(k)$ does not necessarily contain a regular element. For example, let $G = \mathrm{PGL}_3$ and let T be the diagonal torus in G. In this case, an element $\operatorname{diag}(a, b, c)$ of $\mathfrak{t}(k)$ is regular if and only if a, b, and c are pairwise distinct. If $k = \mathbf{F}_2$, then there is no such element. For this reason, it would be interesting to know whether Proposition 2.6 can fail without the assumption that each $\mathfrak{t}_i$ contains a regular element.

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