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The Stable Homotopy Category Has a Unique Model at the Prime 2

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1. INTRODUCTION

The stable homotopy category has been extensively studied by algebraic topologists for a long time. For many applications it is convenient or even necessary to work with point set level models of spectra as opposed to working up-to-homotopy, and the outcome of a calculation might depend on the choice of model. In recent years many new models for the stable homotopy category have been constructed. It is especially useful to have the structure of a *closed model category* in the sense of Quillen [16] and many examples of spectra categories fit into this context [3, 7, 10, 13, 14, 17]. Moreover all known examples capture the "same homotopy theory" —in technical terms one speaks of *Quillen equivalent* model categories [9, Definition 1.3.12]. Hence not only the homotopy categories, but also higher order information such as Toda brackets, homotopy colimits, and homotopy types of function spaces coincide. In two Quillen equivalent model categories the answer to every homotopy theoretic question comes out the same.

In a model category one can pass to the associated *homotopy category* by formally inverting the class of weak equivalences. However, passage to the homotopy category loses information and in general the "homotopy theory" cannot be recovered from the homotopy category; see Sections 2.1 and 2.2 for two examples. In this paper we show that in contrast to the general case, the stable homotopy category completely determines the

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stable homotopy theory 2-locally. We prove a uniqueness theorem which says that there is essentially only one model category structure underlying the stable homotopy category of 2-local spectra—the stable homotopy category has no "exotic" models at the prime 2.

We call a pointed model category *stable* if it is cocomplete and the loop and suspension functors defined on its homotopy category are inverse equivalences. The homotopy category of a stable model category is naturally triangulated with suspension and cofibration sequences defining the shift operator and the distinguished triangles [9, Proposition 7.1.6].

MAIN THEOREM. *Let $\mathscr{C}$ be a stable model category. If the homotopy category of $\mathscr{C}$ and the 2-local homotopy category of spectra are equivalent as triangulated categories, then there exists a Quillen equivalence between $\mathscr{C}$ and the 2-local model category of spectra.*

In the main theorem, and throughout the paper, our reference model is the category of spectra in the sense of Bousfield and Friedlander [3, Sect. 2]. This is probably the simplest model category of spectra and its objects are sequences $\{X_n\}_{n \geqslant 0}$ of pointed simplicial sets together with maps $\Sigma X_n \to X_{n+1}$. Morphisms are given on every level and commute strictly with the structure maps. As weak equivalences we either use the *stable equivalences*, i.e., the morphisms which induce isomorphisms of stable homotopy groups, or the *p-local* stable equivalences, for p a prime, i.e., those morphisms which induce an isomorphism of stable homotopy groups tensored with $\mathbb{Z}_{(p)}$. For the details of these model structures see [3, Theorem 2.3] (in the "integral" case) and [19, 4.1] (in the p-local case). With this particular 2-local model the main theorem provides a single Quillen equivalence (as opposed to a chain) whose left adjoint has $\mathscr{C}$ as its target. Since any other of the standard 2-local model categories of spectra is Quillen equivalent to this specific one, it can be linked by a chain of Quillen equivalences to any stable model category $\mathscr{C}$ which satisfies the assumptions of the main theorem.

We prove a stronger form of the main theorem as Theorem 3.5 below. The stronger version says that already the subcategory of *finite* 2-local spectra determines the model category structure up to Quillen equivalence of model categories. In particular there is only one way to "complete" the homotopy category of finite 2-local spectra to a triangulated category with infinite coproducts—as long as some underlying model structure exists. This gives a partial answer to Margolis' Uniqueness Conjecture [15, Chap. 2, Sect. 1] for the stable homotopy category, see Corollary 3.7.

The proof of the main theorem relies on the following characterization of self-equivalences of the 2-local stable homotopy category:

THEOREM. *Let F be an exact endofunctor of the homotopy category of finite 2-local spectra. If F takes the 2-local sphere spectrum to itself (up to isomorphism), then F is a self-equivalence.*

This result is a combination of Propositions 3.1 and 3.2 below. To obtain the same conclusion at an odd prime, one has to assume in addition that the functor F does not annihilate the first p-torsion element in the stable homotopy groups of spheres, see Proposition 3.1 for the precise statement. The reason that the prime 2 behaves differently from the odd primes goes back to the "misbehavior" of the mod-2 Moore spectrum that its identity map has order 4. In other parts of stable homotopy theory this is often a nuisance; for us it is the key to why we can prove the uniqueness theorem for 2-local spectra. Currently we have no replacement for this part of the argument at odd primes; see also Remark 5.1.

2. BACKGROUND AND RELATED RESULTS

2.1. *A Triangulated Category with Several Models*

In general, the triangulated homotopy category does not determine the Quillen equivalence type of a stable model category. As an example we consider the nth Morava K-theory spectrum $K(n)$ for a fixed prime p and some number $n > 0$. By a theorem of Robinson [18] this spectrum admits the structure of an A_∞-ring spectrum and so its module spectra form a stable model category. The ring of homotopy groups of $K(n)$ is the graded field $\mathbb{F}_p[v_n, v_n^{-1}]$ with v_n of dimension $2p^n - 2$. Hence the homotopy group functor establishes an equivalence between the homotopy category of $K(n)$-module spectra and the category of graded $\mathbb{F}_p[v_n, v_n^{-1}]$-modules.

Similarly the homology functor establishes an equivalence between the derived category of the graded field $\mathbb{F}_p[v_n, v_n^{-1}]$ and the category of graded $\mathbb{F}_p[v_n, v_n^{-1}]$-modules. This derived category arises from a model category structure on differential graded $\mathbb{F}_p[v_n, v_n^{-1}]$-modules with weak equivalences the quasi-isomorphisms. So the two stable model categories of $K(n)$-module spectra and dg-modules over $\mathbb{F}_p[v_n, v_n^{-1}]$ have equivalent triangulated homotopy categories. On the other hand they are not Quillen equivalent: if they were, then the homotopy types of the function spaces would agree [6, Proposition 5.4]. But for dg-modules all function spaces are products of Eilenberg–MacLane spaces, which is not the case for $K(n)$-modules.

2.2. *Franke's Algebraic Model for the $E(n)$-Local Stable Homotopy Category*

In [8], Franke constructs an exotic model for the homotopy category of $E(n)$-local spectra at a "large" prime. Earlier Bousfield [2] had given an algebraic description of the isomorphism classes of K-local spectra at an odd prime. However, Bousfield could not determine whether his algebraic model describes the morphisms between the spectra correctly. As one application of a general uniqueness theorem [8, Sect. 2.2, Theorem 5], Franke provides an algebraic derived category which is equivalent, as a triangulated category, to the homotopy category of K-local spectra; see [8, Sect. 3.1, Theorem 6]. Franke's uniqueness theorem applies more generally to the homotopy categories of $E(n)$-local spectra at a prime p whenever $n^2+n<2p-2$. Then he obtains an equivalence of triangulated categories between the homotopy category of an abelian model category and the homotopy category of $E(n)$-local spectra; see [8, Sect. 3.5, Theorem 10]. By the same reasoning as in Section 2.1, these two kinds of model categories are *not* Quillen equivalent; see also [8, Sect. 3.1, Remark 1].

We currently do not know whether there exist exotic models for the stable homotopy category at an odd prime. Via Proposition 3.1 this problem reduces to a question about the α_1-map. Franke's exotic equivalences are relevant for these considerations since the α_1-map survives K-localization.

2.3. *Reduction to "Exotic Sphere Spectra"*

Suppose that $\mathscr{C}$ is a stable model category and let Φ: Ho(Spectra) $\to$ Ho($\mathscr{C}$) be an equivalence of triangulated categories. Then the image of the sphere spectrum S^0 is a small weak generator (see [19, Definition 3.1]) of the homotopy category of $\mathscr{C}$. In [20], Shipley and the author associate to an object P of a stable model category a ring spectrum $\mathrm{End}_{\mathscr{C}}(P)$ called the *endomorphism ring spectrum*. The ring of homotopy groups of $\mathrm{End}_{\mathscr{C}}(P)$ is isomorphic to the graded ring of self-maps of P in the homotopy category of $\mathscr{C}$. If P is a small weak generator then we also show that the model category $\mathscr{C}$ is Quillen equivalent to the category of modules over the endomorphism ring spectrum $\mathrm{End}_{\mathscr{C}}(P)$.

The equivalence Φ: Ho(Spectra) $\to$ Ho($\mathscr{C}$) establishes an isomorphism between the ring π_*^s of stable homotopy groups of spheres and the graded ring of self-maps of $P=\Phi(S^0)$ in Ho($\mathscr{C}$); this in turn is isomorphic to the homotopy groups of the ring spectrum $\mathrm{End}_{\mathscr{C}}(P)$. Since Φ is an exact functor, the isomorphism between π_*^s and $\pi_*\,\mathrm{End}_{\mathscr{C}}(P)$ also preserves Toda brackets. Hence the endomorphism ring spectrum $\mathrm{End}_{\mathscr{C}}(P)$ looks very much like the sphere spectrum. If moreover the unit map $S^0 \to \mathrm{End}_{\mathscr{C}}(P)$ of the endomorphism ring spectrum is a stable equivalence, then the original model category $\mathscr{C}$ is Quillen equivalent to the category of spectra. In other words: the question whether there are exotic models for the stable homotopy category can be reduced to the question about the existence of "exotic sphere spectra," i.e., ring spectra which are not equivalent to the sphere spectrum, but whose derived category is equivalent to the stable homotopy category. While this reduction gives a better idea of what possible exotic models look like, we will not use the results of [20] here and rather prove our uniqueness theorem directly.

2.4. *An Integral Uniqueness Result Assuming Additional Structure*

The homotopy category of a stable model category admits additional structure which one can take into account when proving a uniqueness result. The homotopy category of every model category admits an action of the homotopy category of simplicial sets [9, Theorem 4.3.4]. If the model category is stable, then this action induces an action of the graded ring π_*^s of stable homotopy groups of spheres; see [19, 2.4].

In [19] Shipley and the author show that with this extra structure the stable homotopy category determines the model category structure up to Quillen equivalence. More precisely we show that if $\mathscr{C}$ is a stable model category and if the homotopy category of $\mathscr{C}$ admits a π_*^s-*linear* equivalence to the homotopy category of spectra, then $\mathscr{C}$ is Quillen equivalent to the Bousfield–Friedlander stable model category of spectra. Hence the result of the present paper is a strengthening of the Uniqueness Theorem of [19], at least 2-locally. While [19] mainly depends on model category arguments, we have to use more information about the structure of the stable homotopy category here.

3. PROOF OF THE 2-LOCAL UNIQUENESS THEOREM

In this section we deduce our main theorem from other results which should be of independent interest. The first two results concern properties of the stable homotopy category. Proposition 3.1 is an elaboration on the idea that the stable homotopy groups of spheres are generated under "higher order Toda brackets" by the elements of Adams filtration one (see [5] for a precise formulation of this fact). For a prime p the *mod-p Adams filtration* of a map of spectra is the largest number n such that the map can be factored as a composite of n maps all of which induce the trivial map on mod-p cohomology. When the prime is understood we simply speak of the filtration of a map. Adams showed [1] that the only positive dimensional elements in $\pi_* S^0_{(2)}$ which have filtration one are multiples of the Hopf maps

η, ν and σ in dimensions 1, 3, and 7, respectively. For odd primes the only such elements are in the first non-trivial p-torsion homotopy group $\pi_{2p-3}S^0_{(p)}$, see [12, Theorem 1.2.1].

An *exact functor* between triangulated categories is an additive functor F which commutes with shift and preserves distinguished triangles. More precisely: F is endowed with a natural isomorphism $\iota_X\colon F(X[1]) \cong F(X)[1]$ such that for every distinguished triangle (homotopy cofibre sequence)

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

the sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\iota_X \circ F(h)} F(X)[1]$$

is again a distinguished triangle. An *equivalence of triangulated categories* is an equivalence of categories which is exact and whose inverse functor is also exact. In what follows square brackets $[-,-]$ denote morphisms in the homotopy category of spectra, possibly graded when decorated with a subscript.

PROPOSITION 3.1. *Let p be a prime number and let F be an exact endofunctor of the homotopy category of finite p-local spectra which takes the p-local sphere spectrum to itself (up to isomorphism). If every element of Adams filtration one in the graded endomorphism ring $[F(S^0_{(p)}), F(S^0_{(p)})]_*$ is in the image of F, then F is a self-equivalence.*

The next result says that the prime 2 is special because the Hopf maps are always taken care of. We do not know whether the analogue of the following proposition is true for odd primes; see also Remark 5.1.

PROPOSITION 3.2. *Let F be an exact endofunctor of the homotopy category of finite 2-local spectra which takes the 2-local sphere spectrum to itself (up to isomorphism). Then all maps of Adams filtration one in the graded endomorphism ring $[F(S^0_{(2)}), F(S^0_{(2)})]_*$ are in the image of F.*

We prove Propositions 3.1 and 3.2 in Sections 4 and 5 respectively.

In order to state the next auxiliary result we recall the notion of a *compactly generated* triangulated category. An object A of a triangulated category $\mathcal{T}$ is called *compact* (also called *small* or *finite*) if the representable functor $\mathcal{T}(A,-)$ preserves infinite coproducts. A full subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is *closed under extensions* if whenever two of the objects X, Y and Z in a distinguished triangle

$$X \to Y \to Z \to X[1]$$

belong to $\mathscr{S}$, the third object also belongs to $\mathscr{S}$ (this implies that $\mathscr{S}$ contains the zero objects and is closed under isomorphisms, finite sums and shift in either direction). A triangulated category $\mathscr{T}$ is *compactly generated* if $\mathscr{T}$ is the only subcategory which contains the compact objects and is closed under extensions and infinite coproducts. The compact objects of the stable homotopy category are precisely the finite spectra, and similarly for the p-local stable homotopy category. The stable homotopy category and its full subcategory of p-local spectra are compactly generated. However, there are triangulated categories in which the zero objects are the only small objects; see [11, Corollary B.13] for examples which are Bousfield localizations of the stable homotopy category.

The following lemma is well known and we give the easy proof at the end of Section 4.

LEMMA 3.3. *Let F be an exact functor between compactly generated triangulated categories with infinite coproducts. If F preserves coproducts and restricts to an equivalence between the full subcategories of compact objects, then F is an equivalence.*

Finally, we quote a result from [19] which is entirely model category theoretic. It roughly says that the model category of spectra is the "free stable model category on one object." Here spectra are understood in the sense of Bousfield and Friedlander, endowed with the stable model structure of [3, 2.3]. In particular the weak equivalences are those maps which induce isomorphisms of stable homotopy groups. The *p-local* model structure is the localization of the stable model structure of spectra in which the weak equivalences are the maps inducing an isomorphism of stable homotopy groups tensored with $\mathbb{Z}_{(p)}$; see [19, 4.1] for details. A *left Quillen functor* is a functor between model categories which has a right adjoint and which preserves cofibrations and acyclic cofibrations.

PROPOSITION 3.4 [19, Theorem 5.1]. *Let $\mathscr{C}$ be a stable model category and X a cofibrant and fibrant object of $\mathscr{C}$. Then there exists a left Quillen functor from the category of spectra to $\mathscr{C}$ which takes the sphere spectrum to X. If the endomorphism ring of X in the homotopy category of $\mathscr{C}$ is a $\mathbb{Z}_{(p)}$-algebra, then the functor is also a left Quillen functor with respect to the p-local stable model structure for spectra.*

The left Quillen functor provided by Proposition 3.4 is essentially uniquely determined by the object X; see part (4) of [19, Theorem 5.1] for the precise statement.

Now we can state and prove a uniqueness result which has the Main Theorem of the introduction as a special case. This version is stronger since the hypothesis only refers to the full subcategory of compact, or finite, objects in the triangulated homotopy category.

THEOREM 3.5. *Let $\mathscr{C}$ be a stable model category whose homotopy category is compactly generated. Suppose that the full subcategory of compact objects in the homotopy category of $\mathscr{C}$ and the homotopy category of finite 2-local spectra are equivalent as triangulated categories. Then there exists a Quillen equivalence between $\mathscr{C}$ and the 2-local model category of spectra, such that the left adjoint has $\mathscr{C}$ as its target.*

Proof. Let Φ be an equivalence of triangulated categories from the homotopy category of finite 2-local spectra to the compact objects in the homotopy category of $\mathscr{C}$. We choose a cofibrant and fibrant object X of $\mathscr{C}$ which is isomorphic to $\Phi(S^0_{(2)})$ in the homotopy category of $\mathscr{C}$. Proposition 3.4 yields a left Quillen functor, with respect to the 2-local stable model structure, from the category of spectra to $\mathscr{C}$ which takes the sphere spectrum to X. We denote the functor by $X \wedge -$. This left Quillen functor has an exact total left derived functor $X \wedge^L -$ on the level of homotopy categories (see [9, Proposition 6.4.1] or [16, I.4 Proposition 2]).

The derived functor $X \wedge^L -$ takes the localized sphere spectrum to the compact object X, hence it takes compact objects to compact objects; we denote by $(X \wedge^L -)|_{\text{small}}$ the restriction to finite 2-local spectra. The composite functor $F = \Phi^{-1} \circ (X \wedge^L -)|_{\text{small}}$ takes the 2-local sphere spectrum to itself, up to isomorphism, so by Propositions 3.1 and 3.2 it is a self-equivalence of the finite 2-local stable homotopy category. Since F and Φ^{-1} are equivalences of categories, so is $(X \wedge^L -)|_{\text{small}}$. By Lemma 3.3, the functor $X \wedge^L -$ is then an equivalence of categories, so the left Quillen functor $X \wedge -$ and its right adjoint are in fact a Quillen equivalence [9, Proposition 1.3.13]. ∎

Warning: The equivalence Φ takes the 2-local sphere spectrum to the object X, and the same is true for the left derived functor $X \wedge^L -$. If there was a natural transformation between Φ and $(X \wedge^L -)|_{\text{small}}$ which induces an isomorphism at $S^0_{(2)}$, then the natural transformation would be a natural isomorphism, so $X \wedge^L -$ would also be an equivalence on compact objects. However, there is no reason why such a natural transformation should exist, and so there is no a priori reason why $X \wedge^L -$ should be an equivalence.

In particular we do *not* claim that the left Quillen equivalence $X \wedge -$ lifts the triangulated equivalence Φ. Hence we leave open the question of whether there are exotic self-equivalences of the 2-local stable homotopy category, i.e., self-equivalences not induced from a Quillen equivalence (or what is the same: self-equivalences other than iterated (de-)suspensions).

Remark 3.6 [Margolis' Uniqueness Conjecture]. In "Spectra and the Steenrod algebra," H. R. Margolis introduces a set of axioms for a *stable homotopy category* [15, Chap. 2, Sect. 1]. The stable homotopy category of spectra satisfies the axioms, and Margolis conjectures [15, Chap. 2, Sect. 1] that this is the only model, i.e., that any category which satisfies the axioms is equivalent to the stable homotopy category.

As part of the structure Margolis requires a triangulation, infinite coproducts and that the whole category be generated by a single compact object. Furthermore, Margolis' Axiom 5 asks for an equivalence between the full subcategory of compact objects and the Spanier–Whitehead category of finite CW-complexes. So the Uniqueness Conjecture really concerns possible "completions" of the category of finite spectra to a triangulated category with infinite coproducts. Margolis also assumes the existence of a compatible symmetric monoidal smash product, but the smash product does not enter into our present considerations.

Margolis shows [15, Chap. 5, Theorem 19] that modulo phantom maps each model of his axioms is equivalent to the standard model. Moreover, Christensen and Strickland show in [4] that in any model the ideal of phantom maps is equivalent to the phantoms in the standard model.

One can consider a 2-primary analog of Margolis' Uniqueness Conjecture by modifying his Axiom 5 and instead requiring the full subcategory of compact objects in the stable homotopy category to be equivalent, as a triangulated category, to the homotopy category of finite 2-local spectra. Theorem 3.5 proves the following 2-primary analog of Margolis' Uniqueness Conjecture for stable homotopy categories with a model:

COROLLARY 3.7. *Suppose that $\mathcal{S}$ is a 2-primary stable homotopy category which is equivalent, as a triangulated category, to the homotopy category of a stable model category. Then $\mathcal{S}$ is triangulated equivalent to the homotopy category of 2-local spectra.*

Note that we do not assume any internal smash product on the model category, and the corollary does not give that the equivalence between $\mathcal{S}$ and the stable homotopy category of 2-local spectra preserves the smash products.

4. A CHARACTERIZATION OF SELF-EQUIVALENCES OF THE STABLE HOMOTOPY CATEGORY

In this section we prove Proposition 3.1. Throughout, p denotes any prime and F is an exact endofunctor of the homotopy category of finite p-local spectra. We assume further that F takes the p-local sphere spectrum to itself (up to isomorphism) and that all filtration one maps of positive dimension from $F(S^0_{(p)})$ to itself are in the image of F. We want to show that F is then a self-equivalence.

We first recall how the stable homotopy elements of Adams filtration at least two are characterized as those elements which are "decomposable" in a specific way; this is a well-known argument—it is, e.g., used by Cohen [5, Theorem 4.2] who attributes it to Adams.

LEMMA 4.1. *For a map* $\alpha: S^m_{(p)} \to S^0_{(p)}$ *with* $m \geqslant 1$ *the following two conditions are equivalent.*

(i) *The mod-p Adams filtration of* α *is at least two.*

(ii) *The map* α *factors through some finite p-local spectrum whose mod-p cohomology is concentrated in dimensions* 1 *through* $m-1$.

Proof. Both maps in a factorization as in (ii) are trivial on mod-p cohomology for dimensional reasons. So condition (ii) implies condition (i).

For the converse implication we denote by $\overline{H\mathbb{Z}_{(p)}}$ and $\overline{H\mathbb{F}_p}$ the fibers of the Hurewicz maps

$$S^0_{(p)} \to H\mathbb{Z}_{(p)} \qquad \text{and} \qquad S^0_{(p)} \to H\mathbb{F}_p$$

to the p-local and mod-p Eilenberg–MacLane spectra, respectively. We claim that the map $\alpha: S^m_{(p)} \to S^0_{(p)}$ lifts to a map $\bar{\alpha}: S^m_{(p)} \to \overline{H\mathbb{Z}_{(p)}}$ which induces the trivial map on integral spectrum homology. This implies that α factors through the $(m-1)$-skeleton of $\overline{H\mathbb{Z}_{(p)}}$, which is the desired finite p-local spectrum with cohomology concentrated in dimensions 1 through $m-1$.

To establish the claim we argue as follows. Since the filtration of α is at least 2, there is a lift $\alpha': S^m_{(p)} \to \overline{H\mathbb{F}_p}$ of α which is trivial in mod-p cohomology. The map $\overline{H\mathbb{Z}_{(p)}} \to \overline{H\mathbb{F}_p}$ is an isomorphism on homotopy groups in positive dimensions, so the map α' in turn lifts to a map $\bar{\alpha}: S^m_{(p)} \to \overline{H\mathbb{Z}_{(p)}}$. Since $\overline{H\mathbb{Z}_{(p)}} \to \overline{H\mathbb{F}_p}$ is surjective in mod-p cohomology, the lift $\bar{\alpha}$ is again trivial in mod-p cohomology, hence also in mod-p homology. Since the integral spectrum homology of $\overline{H\mathbb{Z}_{(p)}}$ is killed by multiplication by p, the map

$$H_m(\overline{H\mathbb{Z}_{(p)}}, \mathbb{Z}) \to H_m(\overline{H\mathbb{Z}_{(p)}}, \mathbb{F}_p)$$

induced by reduction of coefficients, is injective. Hence $\bar{\alpha}$ is indeed trivial on integral spectrum homology. ∎

If K is a finite p-local spectrum we denote by $\beta(K)$ (resp. $\tau(K)$) the smallest (resp. largest) dimension in which the mod-p cohomology of K is non-trivial. As before square brackets $[-,-]$ denote morphisms in the homotopy category of spectra, possibly graded when decorated with a subscript.

LEMMA 4.2. *Suppose that the map of graded rings*

$$[S^0_{(p)}, S^0_{(p)}]_* \to [F(S^0_{(p)}), F(S^0_{(p)})]_*$$

induced by the functor F is an isomorphism below and including dimension n for some $n \geqslant 0$.

(1) *Let K and L be two finite p-local spectra. Then the map $F\colon [K, L] \to [F(K), F(L)]$ is an isomorphism if $\tau(K) - \beta(L) < n$ and an epimorphism if $\tau(K) - \beta(L) = n$.*

(2) *Let K be a finite p-local spectrum satisfying $\tau(K) - \beta(K) \leqslant n+1$. Then there exists a finite p-local spectrum K' with $\beta(K') \geqslant \beta(K)$ and $\tau(K') \leqslant \tau(K)$ and an isomorphism $K \cong F(K')$ in the homotopy category of spectra.*

(3) *Every map from $F(S^{n+1}_{(p)})$ to $F(S^0_{(p)})$ of Adams filtration at least two is in the image of F.*

Proof. For the course of the proof we omit the subscripts "(p)" from p-local sphere spectra, abbreviating $S^n_{(p)}$ to S^n.

(1) When K and L are localized sphere spectra, the claim holds by assumption. The general case is obtained by cell inductions for K and L.

We first prove the claim when L is a wedge of localized sphere spectra of a fixed dimension using induction on the total dimension of the mod-p cohomology of K. If H^*K is trivial, then K is contractible and the statement is true. Otherwise we can pinch off the top cells of K; i.e., we can choose a distinguished triangle

$$\bigvee_I S^{\tau(K)-1} \xrightarrow{\alpha} M \to K \to \bigvee_I S^{\tau(K)} \qquad (*)$$

with M a finite p-local spectrum with $\tau(M) < \tau(K)$ and with strictly smaller cohomology. Applying $[-, L]$ to the triangle $(*)$ gives a long exact sequence of abelian groups. The functor F preserves distinguished triangles, so applying $[-, F(L)]$ to the image sequence yields a similar exact sequence and F gives a map between the sequences. Using that L is a wedge of spheres and that the claim holds for M and ΣM by induction, the five lemma proves the statement for K and this special L.

Now we do a similar induction on the dimension of H^*L. This time we collapse the bottom cells of L; i.e., we embed L in a triangle

$$\bigvee_J S^{\beta(L)} \to L \to L' \to \bigvee_J S^{\beta(L)+1},$$

where the dimension of H^*L' is strictly smaller than that of L and $\beta(L') > \beta(L)$. By induction the claim holds for the spectra K and L', and using the five lemma and the previous paragraph we deduce it for K and L.

(2) We argue by induction on the difference $\tau(K)-\beta(K)$. If the cohomology of K is concentrated in at most one dimension, then K is equivalent to a (possibly empty) wedge of localized spheres and the statement is true. Otherwise there exists a distinguished triangle $(*)$ as in part (1) with M a finite p-local spectrum which satisfies $\tau(M) < \tau(K)$ and $\beta(M) = \beta(K)$. By induction there exists a finite spectrum M' with $\beta(M') \geqslant \beta(M)$, $\tau(M') \leqslant \tau(M)$ and an isomorphism between $F(M')$ and M. By part (1) the composite

$$F\left(\bigvee_I S^{\tau(K)-1}\right) \xrightarrow{\cong} \bigvee_I S^{\tau(K)-1} \xrightarrow{\alpha} M \xrightarrow{\cong} F(M')$$

is of the form $F(\alpha')$ for some $\alpha' \in [\bigvee_I S^{\tau(K)-1}, M']$. We let K' be some mapping cone of the map α', obtained by embedding α' in a distinguished triangle. Then we have the inequalities $\beta(K') \geqslant \beta(M') \geqslant \beta(M) = \beta(K)$ and $\tau(K') \leqslant \max\{\tau(M'), \tau(K)\} = \tau(K)$. We end up with a diagram

$$\begin{array}{ccccccc}
\bigvee_I S^{\tau(K)-1} & \xrightarrow{\alpha} & M & \longrightarrow & K & \longrightarrow & \bigvee_I S^{\tau(K)} \\
\cong\downarrow & & \downarrow\cong & & \downarrow & & \downarrow\cong \\
F(\bigvee_I S^{\tau(K)-1}) & \xrightarrow{F(\alpha')} & F(M') & \to & F(K') & \to & F(\bigvee_I S^{\tau(K)})
\end{array}$$

in which both rows are distinguished triangles and the left square commutes. Hence we can choose a map $K \to F(K')$ which makes the entire diagram commute, and this map is the isomorphism we are looking for.

(3) Let $\alpha: F(S^{n+1}) \to F(S^0)$ be a map of filtration at least two. Since F preserves p-local spheres (up to isomorphism), Lemma 4.1 provides a finite p-local spectrum K with mod-p cohomology concentrated in dimensions 1 through n, and such that α factors through K. By part (2) of this lemma, K is isomorphic to $F(K')$ for some finite p-local spectrum K' with cohomology concentrated in dimensions 1 through n. So α factors through

$F(K')$ and by part (1), both maps in such a factorization are in the image of F. Hence the original map α is also in the image. ∎

Now we can give the

Proof of Proposition 3.1. Suppose F is an exact endofunctor of the homotopy category of finite p-local spectra which takes the p-local sphere spectrum to itself, up to isomorphism. Furthermore all filtration one maps of positive dimension from $F(S^0_{(p)})$ to itself are in the image of F. We claim that the map of graded rings $F\colon [S^0_{(p)}, S^0_{(p)}]_* \to [F(S^0_{(p)}), F(S^0_{(p)})]_*$ is an isomorphism. Since $F(S^0_{(p)})$ is isomorphic to $S^0_{(p)}$ the map is necessarily an isomorphism in non-positive dimensions, and in positive dimensions both sides of the map are finite groups of the same order. Suppose the claim was false and let $m > 0$ be the smallest dimension in which F is not bijective, hence not surjective. By Lemma 4.2 (3), any element not in the image has filtration one, which contradicts the assumptions. Hence the above map is indeed bijective in all dimensions.

So the hypothesis of Lemma 4.2 is satisfied for arbitrarily large n; conclusion (1) of that Lemma shows that F is full and faithful and conclusion (2) shows that F is surjective on isomorphism classes. Thus F is an equivalence of categories. ∎

It remains to prove Lemma 3.3 which allows us to detect an equivalence of triangulated categories on compact objects.

Proof of Lemma 3.3. Let $F\colon \mathcal{S} \to \mathcal{T}$ be an exact functor between compactly generated triangulated categories with infinite coproducts. Suppose that F preserves coproducts and restricts to an equivalence between the full subcategories of compact objects. We want to show that F itself is an equivalence.

We fix a compact object A of $\mathcal{S}$ and consider the full subcategory of $\mathcal{S}$ consisting of those Y for which the map

$$F\colon \mathcal{S}(A, Y) \to \mathcal{T}(F(A), F(Y))$$

is bijective. By assumption this subcategory contains all compact objects. Since F is exact, the subcategory is closed under extensions. Since A and $F(A)$ are compact and F preserves coproducts, this subcategory is also closed under coproducts. Since $\mathcal{S}$ is compactly generated, the map $F\colon \mathcal{S}(A, Y) \to \mathcal{T}(F(A), F(Y))$ is thus bijective for all compact A and arbitrary Y.

Similarly for arbitrary but fixed Y the full subcategory of $\mathcal{S}$ consisting of those X for which the map $F\colon \mathcal{S}(X, Y) \to \mathcal{T}(F(X), F(Y))$ is bijective is closed under extensions, coproducts and contains the compact objects.

Hence this subcategory coincides with $\mathscr{S}$ which means that F is full and faithful.

Now we consider the full subcategory of $\mathscr{T}$ of objects which are isomorphic to an object in the image of F. By assumption this category contains all compact objects, and it is closed under coproducts since these are preserved by F. We claim that this subcategory is also closed under extensions. Since $\mathscr{T}$ is compactly generated this shows that F is essentially surjective and hence an equivalence.

To prove the last claim we consider a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to X[1].$$

Since the subcategory under consideration is closed under isomorphism and shift in either direction we can assume that $X = F(X')$ and $Y = F(Y')$ are objects in the image of F. Since F is full there exists a map $f': X' \to Y'$ satisfying $F(f') = f$. We can then choose a mapping cone for the map f' and a compatible map from Z to $F(\mathrm{Cone}(f'))$ which is necessarily an isomorphism. ∎

5. TAKING CARE OF THE HOPF MAPS

In this final section we prove Proposition 3.2. Here F denotes an exact endofunctor of the homotopy category of 2-local spectra which takes the 2-local sphere spectrum to itself (up to isomorphism). We want to show that all maps of Adams filtration one in the graded endomorphism ring $[F(S^0_{(2)}), F(S^0_{(2)})]_*$ are in the image of F. We introduce a slight abuse of notation: after choosing an isomorphism between $F(S^0_{(2)})$ and the 2-local sphere spectrum we identify the ring $[F(S^0_{(2)}), F(S^0_{(2)})]_*$ with the ring of 2-local stable homotopy groups of spheres (this identification does not depend on the choice of isomorphism). With this convention we have to show that the Hopf maps η, ν, and σ are in the image of F in dimensions 1, 3, and, 7 respectively.

We start by showing that the map $F(\eta)$ is non-trivial. Since F is exact, both rows in the diagram

$$\begin{array}{ccccccc}
F(S^0_{(2)}) & \xrightarrow{\times 2} & F(S^0_{(2)}) & \to & F(M(2)) & \to & F(S^1_{(2)}) \\
\cong\downarrow & & \downarrow\cong & & \downarrow & & \downarrow\cong \\
S^0_{(2)} & \xrightarrow[\times 2]{} & S^0_{(2)} & \longrightarrow & M(2) & \longrightarrow & S^1_{(2)}
\end{array}$$

are distinguished triangles (here $M(2)$ denotes the mod-2 Moore spectrum). Since the left square commutes we can choose a map between $F(M(2))$ and

$M(2)$ making the entire diagram commute, and this map is necessarily an isomorphism.

The identity map of the Moore spectrum $M(2)$ has additive order 4. Since $F(M(2))$ is isomorphic to $M(2)$, its identity map also has order 4. Since F is additive, it does not annihilate the degree 2 map of the Moore spectrum. On the other hand this degree 2 map factors as the composite

$$M(2) \xrightarrow{\text{pinch}} S^1 \xrightarrow{\eta} S^0 \xrightarrow{\text{incl.}} M(2).$$

Hence $F(\eta)$ has to be non-zero, which forces $F(\eta) = \eta$.

Because of the relation $4\nu = \eta^3$ (see, e.g., [21, Theorem 14.1(i)]) we know that $4 \cdot F(\nu) = F(4\nu) = F(\eta^3) = \eta^3 = 4\nu$. Since ν generates the cyclic group $\pi_3 S^0_{(2)} \cong \mathbb{Z}/8$, we conclude that $F(\nu) = u \cdot \nu$ with u an odd integer. Hence ν is in the image of F.

For the last Hopf map σ we exploit the Toda bracket relation

$$8\sigma = \langle \nu, 8\iota, \nu \rangle$$

(see e.g. [21, Lemmas 5.13 and 5.14]) which holds without indeterminacy since the fourth stable homotopy group of spheres is trivial. Since F preserves triangles it takes threefold Toda brackets to threefold Toda brackets and we obtain

$$\begin{aligned} 8 \cdot F(\sigma) = F(8\sigma) = F(\langle \nu, 8\iota, \nu \rangle) &\subseteq \langle F(\nu), F(8\iota), F(\nu) \rangle \\ &= u^2 \cdot \langle \nu, 8\iota, \nu \rangle = 8u^2 \cdot \sigma. \end{aligned}$$

Since the latter Toda-bracket has no indeterminacy we conclude that $8 \cdot F(\sigma) = 8u^2 \cdot \sigma$. Again since σ generates the cyclic group $\pi_7 S^0_{(2)} \cong \mathbb{Z}/16$, we conclude that $F(\sigma)$ coincides with σ up to an odd integer, so σ is in the image of F. This finishes the proof of Proposition 3.2.

Remark 5.1. The fact that multiplication by 2 is non-trivial on the mod-2 Moore spectrum is equivalent to the fact that $M(2)$ does not admit a product; i.e, there is no map $M(2) \wedge M(2) \to M(2)$ in the stable homotopy category which splits the two inclusions $j \wedge \mathrm{id}, \mathrm{id} \wedge j : M(2) \to M(2) \wedge M(2)$, where $j\colon S^0 \to M(2)$ is the inclusion of the bottom cell.

For an odd prime p the Moore spectrum $M(p)$ admits a unique and commutative product, which is also associative for $p \geqslant 5$. Hence $M(p)$ is a ring spectrum in the stable homotopy category. However, the product on $M(p)$ can not be made associative up to "coherent higher homotopy", i.e., $M(p)$ does not admit the structure of an A_∞-ring spectrum. In fact the element $\alpha_1 \in \pi_{2p-3} S^0_{(p)}$ is the obstruction to p-th order homotopy associativity. One could try to use this to detect α_1 and to prove the odd primary version of Proposition 3.2.

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