SPECTRA AND STABLE HOMOTOPY THEORY

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Abstract. These are course notes for the class *Algebraic Topology II* taught by the author at Bonn University during the summer term 2026.

Contents

Introduction

These notes aim to develop into an introduction to the foundations of stable homotopy theory. One possible road towards 'spectral algebra' is an associative and commutative smash product on a good point-set level category of spectra, which lifts the well-known smash product pairing on the *homotopy category*. Between the mid 1990's until around 2010, this was the preferred approach, and it is the path we will take in these notes. In recent years, the higher categorical approach pioneered by Lurie [27, 28] has become increasingly popular and versatile. And as more and more foundations are being worked out in this language, the theory grew more and more powerful; in the long run, I expect the higher categorical approach to prevail. Still, higher categorical theory demands a substantial intellectual investment for learning the basic formalism, and a model based approach should continue to have its merits.

We begin with a quick historical review and attempt at a motivation. A much more comprehensive and detailed history and of the early days of stable homotopy theory with many more references can be found in May's [35]. The first construction of what is now called 'the stable homotopy category', including its symmetric monoidal smash product, is due to Boardman [4, 5] (unpublished); accounts of Boardman's construction appear in [55], [52] and [1, Part III] (Adams devotes more than 30 pages to the construction and formal properties of the smash product).

To illustrate the drastic simplification that occurred in the foundations in the mid-90s, let us draw an analogy with the algebraic context. Let R be a commutative ring and imagine for a moment that the notion of a chain complex (of R-modules) has not been discovered, but nevertheless various complicated constructions of the unbounded derived category $\mathcal{D}(R)$ of the ring R exist. Moreover, constructions of the *derived* tensor product on the *derived* category exist, but they are complicated and the proof that the derived tensor product is associative and commutative occupies 30 pages. In this situation, you could talk about objects A in the derived category together with morphisms $A \otimes^L_R A \longrightarrow A$, in the derived category, which are associative and unital, and possibly commutative, again in the derived category. This notion may

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be useful for some purposes, but it suffers from many defects – as one example, the category of modules (under derived tensor product in the derived category), does not in general form a triangulated category.

Now imagine that someone proposes the definition of a chain complex of R-modules and shows that by formally inverting the quasi-isomorphisms, one can construct the derived category. She also defines the tensor product of chain complexes and proves that tensoring with suitably nice (i.e., *homotopically projective*) complexes preserves quasi-isomorphisms. It immediately follows that the tensor product descends to an associative and commutative product on the derived category. What is even better, now one can suddenly consider differential graded algebras, a 'rigidified' version of the crude multiplication 'up-to-chain homotopy'. We would quickly discover that this notion is much more powerful and that differential graded algebras arise all over the place (while chain complexes with a multiplication which is merely associative up to chain homotopy seldom come up in nature).

Fortunately, this is not the historical course of development in homological algebra, but the development in stable homotopy theory was, in several aspects, as indicated above. In the stable homotopy category people could consider ring spectra 'up to homotopy', which are closely related to multiplicative cohomology theories. However, the need and usefulness of ring spectra with rigidified multiplications soon became apparent, and topologists developed different ways of dealing with them. One line of approach uses operads for the bookkeeping of the homotopies which encode all higher forms of associativity and commutativity, and this led to the notions of A_∞- respectively E_∞-ring spectra. Various notions of point-set level ring spectra had been used (which were only later recognized as the monoids in a symmetric monoidal model category). For example, the orthogonal ring spectra had appeared as $\mathscr{I}_*$*-prefunctors* in [34], the *functors with smash product* were introduced in [6] and symmetric ring spectra appeared as *FSPs defined on spheres* in [20, 2.7].

At this point it had become clear that many technicalities could be avoided if one had a smash product on a good point-set category of spectra which was associative and unital *before* passage to the homotopy category. For a long time no such category was known, and there was even evidence that it might not exist [26]. In retrospect, the modern spectra categories could maybe have been found earlier if Quillen's formalism of *model categories* [39] had been taken more seriously; from the model category perspective, one should not expect an intrinsically 'left adjoint' construction like a smash product to have a good homotopical behavior in general, and along with the search for a smash product, one should look for a compatible notion of cofibrations.

In the mid-90s, several categories of spectra with nice smash products were discovered, and simultaneously, model categories experienced a major renaissance. Around 1993, Elmendorf, Kriz, Mandell and May introduced the *S-modules* [16] and Jeff Smith gave the first talks about *symmetric spectra*; the details of the model structure were later worked out and written up by Hovey, Shipley and Smith [22]. In 1995, Lydakis [29] independently discovered and studied the smash product for $\mathbf{\Gamma}$-spaces (in the sense of Segal [46]), and a little later he developed model structures and smash product for *simplicial functors* [30]. Except for the *S-modules* of Elmendorf, Kriz, Mandell and May, all other known models for spectra with nice smash product have a very similar flavor; they all arise as categories of continuous (or simplicial), space-valued functors from a symmetric monoidal indexing category, and the smash product is a convolution product (defined as a left Kan extension), which had much earlier been studied by the category theorist Day [12]. This unifying context was made explicit by Mandell, May, Schwede and Shipley in [33], where another example, the *orthogonal spectra* were first worked out in detail. The different approaches to spectra categories with smash product have been generalized and adapted to equivariant homotopy theory [14, 31, 32] and motivic homotopy theory [15, 23, 24].

There are already several good sources available which explain the stable homotopy category, starting with Adams' classic [1], and including [2, 41, 49]; these references do not focus on structured ring and module spectra, though. The monograph [16] by Elmendorf, Kriz, Mandell and May develops this theory in one of the competing frameworks, the S-modules, in detail. It has had a big impact and is widely used, for example because many standard tools can simply be quoted from that book. The theory of orthogonal spectra is by now also highly developed, but the results are spread over many research papers. The aim of these notes is to collect some basic facts about orthogonal spectra in one place, and use them to introduce the stable homotopy category as a tensor triangulated category. Needless to say that the tensor triangulated category stable homotopy category is only a shadow of the 'true' structure, i.e., the symmetric monoidal stable ∞-category of spectra...

Prerequisites. As a general principle, I assume knowledge of basic algebraic topology and unstable homotopy theory. On the other hand, no prior knowledge of *stable* homotopy theory is assumed. In particular, the eventual plan is to define the stable homotopy category using orthogonal spectra and develop its basic properties from scratch.

Conventions. Throughout this book, a *space* is a *compactly generated space* in the sense of [36], i.e., a k-space (also called *Kelley space*) that satisfies the weak Hausdorff condition. Two extensive resources with background material about compactly generated spaces are Section 7.9 of tom Dieck's textbook [53] and Appendix A of the author's book [43]. Two other influential – but unpublished – sources about compactly generated spaces are the Appendix A of Gaunce Lewis's thesis [25] and Neil Strickland's preprint [48]. We denote the category of compactly generated spaces and continuous maps by $\mathbf{T}$.

It will be convenient to define the n-sphere S^n as the one-point compactification of n-dimensional euclidean space $\mathbb{R}^n$, with the point at infinity as the basepoint. We will sometimes need to identify the 1-sphere with the space $[0,1]/\{0,1\}$, the quotient space of the unit interval with identified endpoints. The precise identifications do not matter, but for definiteness we fix a homeomorphism now. Our preferred homeomorphism is

$$\mathbf{t} \ : \ [0,1]/\{0,1\} \ \xrightarrow{\cong} \ S^1 \ , \quad x \longmapsto \frac{2x-1}{x(1-x)} \ .$$

Here the understanding is that the formula describes the function on the open interval $(0,1)$ (which is mapped homeomorphically to $\mathbb{R}$), and that the map extends continuously to the quotient space by sending the identified endpoints to the point at infinity in S^1.

The topological spaces we consider are usually pointed, and we use the notation $\pi_n(X)$ for the n-th homotopy group with respect to the distinguished basepoint, which we do not record in the notation.

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1. Sequential spectra and stable homotopy groups

Definition 1.1. A *sequential spectrum* consists of a sequence of pointed spaces X_n and continuous based maps $\sigma_n\colon S^1 \wedge X_n \longrightarrow X_{1+n}$ for $n \geq 0$. A *morphism* $f\colon X \longrightarrow Y$ of sequential spectra consists of based maps $f_n\colon X_n \longrightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{1+n} \circ \sigma_n = \sigma_n \circ (S^1 \wedge f_n)$ for all $n \geq 0$. We denote the category of sequential spectra by $\mathcal{S}p^{\mathbb{N}}$.

We refer to the space X_n as the *n-th level* of the sequential spectrum X.

Construction 1.2 (Stable homotopy groups)**.** Primary invariants of spectra are their homotopy groups: for $k \in \mathbb{Z}$, the *k-th homotopy group* of a sequential spectrum X is defined as the colimit

$$\pi_k(X) \ = \ \operatorname{colim}_n \, \pi_{n+k}(X_n)$$

taken over the *stabilization maps* defined as the composite

$$\pi_{n+k}(X_n) \ \xrightarrow{S^1 \wedge -} \ \pi_{1+n+k}\left(S^1 \wedge X_n\right) \ \xrightarrow{(\sigma_n)_*} \ \pi_{1+n+k}(X_{1+n}) \ .$$

If k is negative, then the colimit system is only defined for $n \geq -k$. For large enough n, the set $\pi_{n+k}(X_n)$ has a natural abelian group structure and the stabilization maps are homomorphisms, so the colimit $\pi_k(X)$ inherits a natural abelian group structure.

Example 1.3 (Sphere spectrum and suspension spectra)**.** The *sphere spectrum* $\mathbb{S}$ is given by $\mathbb{S}_n = S^n$, the n-sphere. Then structure maps

$$\sigma_n \ : \ S^1 \wedge S^n \ \longrightarrow \ S^{1+n}$$

are the canonical homeomorphisms.

Every pointed space K gives rise to a *suspension spectrum* $\Sigma^\infty K$ with values

$$(\Sigma^\infty K)_n \;=\; S^n \wedge K \ .$$

The structure map $\sigma_n \colon S^1 \wedge S^n \wedge K \longrightarrow S^{1+n} \wedge K$ is the smash product of the canonical homeomorphism with K. For example, the sphere spectrum $\mathbb{S}$ is isomorphic to the suspension spectrum $\Sigma^\infty S^0$. A sequential spectrum X is isomorphic to a suspension spectrum (necessarily that of its zeroth space X_0) if and only if every structure map $\sigma_n \colon S^1 \wedge X_n \longrightarrow X_{1+n}$ is a homeomorphism. The homotopy group

$$\pi_k^{\mathrm{s}}(K) \;=\; \pi_k\left(\Sigma^\infty K\right) \;=\; \operatorname{colim}_n \pi_{n+k}(S^n \wedge K)$$

is called the k-th *stable homotopy group* of K. If K is a *well-pointed* based space, (i.e., the inclusion of the basepoint $\{k_0\} \longrightarrow K$ has the homotopy extension property in the category $\mathbf{T}$ of unbased spaces), then $S^n \wedge K$ is $(n-1)$-connected, see for example [53, Corollary 6.7.10]. So the groups $\pi_k(\Sigma^\infty K)$ vanish in negative dimensions, i.e., the suspension spectrum $\Sigma^\infty K$ is *connective.* For example, every space that admits a CW-structure is well-pointed for every choice of basepoint.

The homotopy group $\pi_k(\mathbb{S}) = \operatorname{colim}_n \pi_{n+k}(S^n)$ is called the *k-th stable homotopy group of spheres*, or the *k-th stable stem*, and will be denoted π_k^{s}. The group π_k^{s} is trivial for negative values of k. The degree of a self-map of a sphere provides an isomorphism $\pi_0^{\mathrm{s}} \cong \mathbb{Z}$. For $k \geq 1$, the homotopy group π_k^{s} is finite. This is a direct consequence of the Freudenthal's suspension theorem and Serre's calculation of the homotopy groups of spheres modulo torsion, which we recall without giving a proof.

Theorem 1.4 (Serre)**.** *Let $m > n \geq 1$. Then*

$$\pi_m(S^n) \cong \begin{cases} (\text{finite group}) \oplus \mathbb{Z} & \text{if } n \text{ is even and } m = 2n-1 \\ (\text{finite group}) & \text{else.} \end{cases}$$

Thus for $k \geq 1$, the stable stem $\pi_k^{\mathrm{s}} = \pi_k(\mathbb{S})$ is finite.

As a concrete example, we inspect the colimit system defining π_1^{s}, the first stable stem. Since the universal cover of S^1 is the real line, which is contractible, the theory of covering spaces shows that the groups $\pi_n S^1$ are trivial for $n \geq 2$. The Hopf map

$$\eta \;:\; S^3 \subseteq \mathbb{C}^2 \backslash \{0\} \xrightarrow{\;\text{proj}\;} \mathbb{C}\mathrm{P}^1 \cong S^2$$

is a locally trivial fiber bundle with fiber S^1, so it gives rise to a long exact sequence of homotopy groups. Since the fiber S^1 has no homotopy above dimension one, the group $\pi_3 S^2$ is free abelian of rank one, generated by the class of η. Here, and throughout the book, we identify the complex projective space $\mathbb{C}P^1$ with the 2-sphere S^2 via the homeomorphism from S^2 to $\mathbb{C}P^1$ that sends $(x, y) \in \mathbb{R}^2$ to $[x + iy, 1] \in \mathbb{C}P^1$ and the point at infinity in S^2 to the line $[1, 0]$.

By Freudenthal's suspension theorem the suspension homomorphism $- \wedge S^1 \colon \pi_3(S^2) \longrightarrow \pi_4(S^3)$ is surjective and from $\pi_4(S^3)$ on the suspension homomorphism is an isomorphism. So the first stable stem π_1^{s} is cyclic, generated by the image of η, and its order equals the order of the suspension of η. On the one hand, η itself is stably essential, since the Steenrod operation Sq^2 acts non-trivially on the mod-2 cohomology of the mapping cone of η, which is homeomorphic to $\mathbb{C}\mathrm{P}^2$.

On the other hand, twice the suspension of η is null-homotopic. To see this we consider the commutative square

$$\begin{array}{ccccc} (x,y) & S^3 & \xrightarrow{\;\;\eta\;\;} & \mathbb{C}\mathrm{P}^1 & [x:y] \\ \big\downarrow & \big\downarrow & & \big\downarrow & \big\downarrow \\ (\bar{x},\bar{y}) & S^3 & \xrightarrow[\;\;\eta\;\;]{} & \mathbb{C}\mathrm{P}^1 & [\bar{x}:\bar{y}] \end{array}$$

in which the vertical maps are induced by complex conjugation in both coordinates of $\mathbb{C}^2$. The left vertical map has degree 1, so it is homotopic to the identity, whereas complex conjugation on $\mathbb{C}\mathrm{P}^1 \cong S^2$ has degree -1. So $(-1) \circ \eta$ is homotopic to η. Thus the suspension of η is homotopic to the suspension of $(-1) \circ \eta$, which by the following lemma is homotopic to the negative of $\eta \wedge S^1$.

Lemma 1.5. *Let Y be a based space, $m \geq 0$ and $f \colon S^m \longrightarrow S^m$ a continuous based map of degree k. Then for every homotopy class $x \in \pi_n(S^m \wedge Y)$ the classes $(f \wedge Y)_*(x)$ and $k \cdot x$ become equal in $\pi_{1+n}(S^{1+m} \wedge Y)$ after one suspension.*

Proof. Let $d_k \colon S^1 \longrightarrow S^1$ be any pointed map of degree k. Then the maps $S^1 \wedge f, d_k \wedge S^m \colon S^{1+m} \longrightarrow S^{1+m}$ have the same degree k, hence they are based homotopic. Suppose x is represented by $\varphi \colon S^n \longrightarrow S^m \wedge Y$. Then the suspension of $(f \wedge Y)_*(x)$ is represented by $(S^1 \wedge f \wedge Y) \circ (S^1 \wedge \varphi)$ which is homotopic to $(d_k \wedge S^m \wedge Y) \circ (S^1 \wedge \varphi) = (S^1 \wedge \varphi) \circ (d_k \wedge S^n)$. Precomposition with the degree k map $d_k \wedge S^n$ of S^{1+n} induces multiplication by k, so the last map represents the suspension of $k \cdot x$. $\square$

The conclusion of Lemma 1.5 does not in general hold without the extra suspension, i.e., $(f \wedge Y)_*(x)$ need not equal $k \cdot x$ in $\pi_n(S^m \wedge Y)$: as we showed above, $(-1) \circ \eta$ is homotopic to η, which is *not* homotopic to $-\eta$ since η generates the infinite cyclic group $\pi_3(S^2)$.

As far as we know, the stable homotopy groups of spheres don't follow any simple pattern. Much machinery of algebraic topology has been developed to calculate homotopy groups of spheres, both unstable and stable, but no one expects to ever get explicit formulae for all stable homotopy groups of spheres. The Adams spectral sequence based on mod-p cohomology and the Adams–Novikov spectral sequence based on MU (complex cobordism) or BP (the Brown–Peterson spectrum at a fixed prime p) are the most effective tools we have for explicit calculations as well as for discovering systematic phenomena.

Example 1.6 (Multiplication in the stable stems)**.** The stable stems $\pi_*^{\mathrm{s}} = \pi_*(\mathbb{S})$ form a graded commutative ring which acts on homotopy groups of every other spectrum X. We denote the action simply by a 'dot'

$$\cdot \ : \ \pi_k^{\mathrm{s}} \times \pi_l(X) \ \longrightarrow \ \pi_{k+l}(X) \ .$$

The definition is essentially straightforward, but there is one subtlety in showing that the product is well-defined. We let $f \colon S^{m+k} \longrightarrow S^m$ and $g \colon S^{n+l} \longrightarrow X_n$ represent classes in π_k^{s} and $\pi_l(X)$, respectively. We denote by $f \cdot g$ the composite

$$S^{m+k+n+l} \xrightarrow{f \wedge g} S^m \wedge X_n \xrightarrow{\sigma^m} X_{m+n}$$

and then define

$$[f] \cdot [g] \ = \ (-1)^{kn} \cdot [f \cdot g] \tag{1.7}$$

in the group $\pi_{k+l}(X)$.

We check that the multiplication is well-defined. If we replace $f \colon S^{m+k} \longrightarrow S^m$ by its suspension $S^1 \wedge f \colon S^{1+m+k} \longrightarrow S^{1+m}$, then

$$(S^1 \wedge f) \cdot g \ = \ \sigma^{1+m} \circ (S^1 \wedge f \wedge g) \ = \ \sigma_{m+n} \circ (S^1 \wedge \sigma^m) \circ (S^1 \wedge f \wedge g) \ = \ \sigma_{m+n} \circ (S^1 \wedge (f \cdot g)) \ .$$

Since the sign in the formula (1.7) does not change, the resulting stable class is independent of the representative f of the stable class in π_k^{s}. Independence of the representative for $\pi_l(X)$ is slightly more subtle. If we replace $g \colon S^{n+l} \longrightarrow X_n$ by the representative $\sigma_n \circ (S^1 \wedge g) \colon S^{1+n+l} \longrightarrow X_{1+n}$, then $f \cdot g$ gets replaced by the lower horizontal composite in the commutative diagram

$$\begin{tikzcd}
S^{1+m+k+n+l} \arrow[r, "S^1 \wedge f \wedge g"] \arrow[d, "\chi_{1,m+k} \wedge S^{n+l}"'] & S^{1+m} \wedge X_n \arrow[d, "\chi_{1,m} \wedge X_n"] & \\
S^{m+k+1+n+l} \arrow[r, "f \wedge S^1 \wedge g"] \arrow[rr, bend right, "f \cdot (\sigma_n \circ (S^1 \wedge g))"'] & S^{m+1} \wedge X_n \arrow[r, "\sigma^{m+1}"] & X_{m+1+n}
\end{tikzcd}$$

By Lemma 1.5 the map $\chi_{1,m} \wedge X_n$ induces multiplication by $(-1)^m$ on homotopy groups *after one suspension.* This cancels part of the sign $(-1)^{m+k}$ that is the effect of precomposition with the shuffle permutation

$\chi_{1,m+k}$ on the left. So in the colimit $\pi_{k+l}(X)$ we have

$$[f\cdot(\sigma_n\circ(S^1\wedge g))] \;=\; (-1)^k\cdot[\sigma^{m+1}(S^1\wedge f\wedge g)] \;=\; (-1)^k\cdot[f\cdot g]\ .$$

Since the dimension of $S^1\wedge g$ is one more than the dimension of g, the extra factor $(-1)^k$ makes sure that product $[f]\cdot[g]$ as defined in (1.7) is independent of the representative of the stable class $[g]$.

Now we verify that the dot product is biadditive. We only show the relation $x\cdot(y+y')=x\cdot y+x\cdot y'$, and additivity in x is similar. Suppose as before that $f\colon S^{m+k}\longrightarrow S^m$ and $g,g'\colon S^{n+l}\longrightarrow X_n$ represent classes in π^{s}_k and $\pi_l(X)$, respectively. Then the sum of g and g' in $\pi_{n+l}(X_n)$ is represented by the composite

$$S^{n+l}\xrightarrow{\text{pinch}} S^{n+l}\vee S^{n+l}\xrightarrow{g\vee g'} X_n\ .$$

In the square

$$\begin{tikzcd}
S^{m+n+k+l} \arrow[r, "S^m\wedge\chi_{n,k}\wedge S^l"] \arrow[dd, "\text{pinch}"'] & S^{m+k+n+l} \arrow[d, "\text{pinch}\wedge\mathrm{Id}"] \arrow[dr, bend left, "f\wedge(g+g')"] & \\
 & S^{m+k}\wedge(S^{n+l}\vee S^{n+l}) \arrow[r, "f\wedge(g\vee g')"] & S^m\wedge X_n \\
S^{m+n+k+l}\vee S^{m+n+k+l} \arrow[r, "(S^m\wedge\chi_{n,k}\wedge S^l)\vee(S^m\wedge\chi_{n,k}\wedge S^l)"'] & S^{m+k+n+l}\vee S^{m+k+n+l} \arrow[u, "\cong"'] \arrow[ur, bend right, "(f\wedge g)\vee(f\wedge g')"'] &
\end{tikzcd}$$

the right part commutes on the nose and the left square commutes up to homotopy. After composing with the iterated structure map $\sigma^m\colon S^m\wedge X_n\longrightarrow X_{m+n}$, the composite around the top of the diagram becomes $f\cdot(g+g')$, whereas the composite around the bottom represents $[f]\cdot[g]+[f]\cdot[g']$. This proves additivity of the dot product in the right variable.

If we specialize to $X=\mathbb{S}$ then the product provides a biadditive graded pairing $\cdot\colon\pi^{\mathrm{s}}_k\times\pi^{\mathrm{s}}_l\longrightarrow\pi^{\mathrm{s}}_{k+l}$ of the stable homotopy groups of spheres. We claim that for every sequential spectrum X the diagram

$$\begin{tikzcd}
\pi^{\mathrm{s}}_j\times\pi^{\mathrm{s}}_k\times\pi_l(X) \arrow[r, "\pi^{\mathrm{s}}_j\times\cdot"] \arrow[d, "\cdot\times\pi_l(X)"'] & \pi^{\mathrm{s}}_j\times\pi_{k+l}(X) \arrow[d, "\cdot"] \\
\pi^{\mathrm{s}}_{j+k}\times\pi_l(X) \arrow[r, "\cdot"'] & \pi_{j+k+l}(X)
\end{tikzcd}$$

commutes, so the product on the stable stems and the action on the homotopy groups of a spectrum are associative. After unraveling all the definitions, this associativity ultimately boils down to the equality

$$(-1)^{jm}\cdot(-1)^{(j+k)n} \;=\; (-1)^{kn}\cdot(-1)^{j(m+n)}$$

and commutativity of the square

$$\begin{tikzcd}
S^q\wedge S^m\wedge X_n \arrow[r, "S^q\wedge\sigma^m"] \arrow[d, "\cong\wedge X_n"'] & S^q\wedge X_{m+n} \arrow[d, "\sigma^q"] \\
S^{q+m}\wedge X_n \arrow[r, "\sigma^{q+m}"'] & X_{q+m+n}
\end{tikzcd}$$

Finally, the multiplication in the homotopy groups of spheres is commutative in the graded sense. Indeed, for representing maps $f\colon S^{m+k}\longrightarrow S^m$ and $g\colon S^{n+l}\longrightarrow S^n$ the square

$$\begin{tikzcd}
S^{m+k+n+l} \arrow[r, "f\wedge g"] \arrow[d, "\chi_{m+k,n+l}"'] & S^{m+n} \arrow[d, "\chi_{m,n}"] \\
S^{n+l+m+k} \arrow[r, "g\wedge f"'] & S^{n+m}
\end{tikzcd}$$

commutes. After one suspension, the two vertical coordinate permutations induce the signs $(-1)^{(m+k)(n+l)}$ and $(-1)^{mn}$, respectively, on homotopy groups. So in the stable group we have

$$[f]\cdot[g] \;=\; (-1)^{kn}\cdot[f\cdot g] \;=\; (-1)^{kl+lm}\cdot[g\cdot f] \;=\; (-1)^{kl}\cdot[g]\cdot[f]\ .$$

The following table gives the stable homotopy groups of spheres through dimension 8:

n	0	1	2	3	4	5	6	7	8
π_n^{s}	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$
generator	ι	η	η^2	ν			ν^2	σ	$\eta\sigma,\ \varepsilon$

Here ν and σ are the Hopf maps which arises unstably as fiber bundles $S^7 \longrightarrow S^4$ respectively $S^{15} \longrightarrow S^8$. The element ε in the 8-stem can be defined using Toda brackets as $\varepsilon = \eta\sigma + \langle \nu, \eta, \nu\rangle$. The table contains or determines all multiplicative relations in this range except for $\eta^3 = 12\nu$. A theorem of Nishida's [38] says that every homotopy element of positive dimension is nilpotent.

Example 1.8 (Eilenberg–Mac Lane spectra, sequential version)**.** For an abelian group A and $n \geq 0$, we let $K(A,n)$ be an Eilenberg–MacLane space of type (A,n), i.e., a pair $(K(A,n),\varphi_n)$ consisting of a based space admitting a CW-structure such that the homotopy group $\pi_k(K(A,n),*)$ is trivial for $k \neq n$, and an isomorphism $\varphi_n\colon \pi_n(K(A,n),*) \cong A$. Since $\Omega K(A,1+n)$ is also an Eilenberg–MacLane space of type (A,n), there is a continuous based map $\rho_n\colon K(A,n) \longrightarrow \Omega K(A,1+n)$ making the following diagram commute:

(1.9)
$$\begin{tikzcd}
\pi_n(K(A,n),*) \arrow[rr, "(\rho_n)_*"] \arrow[dr, "\cong", "\varphi_n"'] & & \pi_n(\Omega K(A,1+n),*) \arrow[r, "\cong"] & \pi_{1+n}(K(A,1+n),*) \arrow[dll, "\cong"', "\varphi_{1+n}"] \\
& A & &
\end{tikzcd}$$

The unnamed isomorphism is the adjunction isomorphism. Moreover, such a ρ_n is unique up to based homotopy, and it is a weak homotopy equivalence. We write $\sigma_n\colon S^1 \wedge K(A,n) \longrightarrow K(A,1+n)$ for the adjoint of the map ρ_n under the adjunction $(S^1 \wedge -, \Omega)$. Then the data

$$HA \;=\; \{K(A,n),\ \sigma_n\}_{n\geq 0}$$

form a sequential spectrum, an *Eilenberg–Mac Lane spectrum* for the group A. Because the group $\pi_{k+n}(K(A,n),*)$ is trivial for all $k \neq 0$ with $k+n \geq 0$, the stable homotopy group $\pi_k(HA)$ is trivial for $k \neq 0$. And the commutativity of the diagrams (1.9) guarantees that the isomorphisms $\varphi_n\colon \pi_n(K(A,n),*) \cong A$ assemble into an isomorphism

$$\varphi \;:\; \pi_0(HA) \xrightarrow{\;\cong\;} A\ .$$

We will later discuss a more refined version of the Eilenberg–MacLane spectra, one that does not depend on choices of Eilenberg–MacLane spaces, and that assembles into a lax symmetric monoidal functor from abelian groups (under tensor product) to orthogonal spectra (under smash product).

Example 1.10 (Smash products with and functions from spaces)**.** For a based space K, smashing with K and taking based mapping space from K are an adjoint functor pair

$$-\wedge K \;:\; \rightleftarrows \;:\; \mathrm{map}_*(K,-) \;=\; (-)^K$$

We can lift these functors to sequential spectra by applying them levelwise. More precisely, for a sequential spectrum X we define new sequential spectra $X \wedge K$ and $\mathrm{map}_*(K,X)$ by

$$(X\wedge K)_n \;=\; X_n \wedge K \qquad \text{and} \qquad \mathrm{map}_*(K,X)_n \;=\; \mathrm{map}_*(K,X_n)\ .$$

The structure maps do not interact with K: the n-structure map for $X \wedge K$ is

$$S^1\wedge(X\wedge K)_n \;=\; S^1\wedge X_n\wedge K \xrightarrow{\sigma_n\wedge K} X_{1+n}\wedge K \;=\; (X\wedge K)_{1+n}\ .$$

The n-structure map for $\mathrm{map}_*(K, X)$ is the composite

$$S^1 \wedge \mathrm{map}_*(K, X_n) \longrightarrow \mathrm{map}_*(K, S^1 \wedge X_n) \xrightarrow{\mathrm{map}_*(K,\sigma_n)} \mathrm{map}_*(K, X_{1+n})$$

where the first is an assembly map that sends $x \wedge f$ to the map sending $k \in K$ to $x \wedge f(k)$.

Just as the functors $- \wedge K$ and $\mathrm{map}_*(K, -)$ are adjoint on the level of based spaces, the two functors just introduced are an adjoint pair for sequential spectra. The adjunction unit $\eta \colon X \longrightarrow \mathrm{map}_*(K, X \wedge K)$ and counit $\epsilon \colon \mathrm{map}_*(K, X) \wedge K \longrightarrow X$ are defined levelwise as coevaluation and evaluation maps:

$$\begin{aligned} \eta_n \ &: \ X_n \longrightarrow \mathrm{map}_*(K, X_n \wedge K)\ , \quad && \eta_n(x)(k) = x \wedge k \\ \epsilon_n \ &: \ \mathrm{map}_*(K, X_n) \wedge K \longrightarrow X\ , \quad && \epsilon_n(f)(k) = f(k)\ . \end{aligned}$$

An important special case of this construction is when $K = S^1$ is a 1-sphere, i.e., the one-point compactification of $\mathbb{R}$. In this case we call $X \wedge S^1$ the *suspension* of X, and we call $\Omega X = \mathrm{map}_*(S^1, X)$ the *loop spectrum* of X. We obtain an adjunction between $- \wedge S^1$ and Ω as the special case $K = S^1$ of the previous adjunction.

Definition 1.11. A morphism $f \colon X \longrightarrow Y$ of sequential spectra is a *stable equivalence* if the induced map $\pi_k(f) \colon \pi_k(X) \longrightarrow \pi_k(Y)$ is an isomorphism for all integers k.

In Proposition 1.27 we prove that stable equivalences are closed under various constructions such as suspensions, loop, shift adjoint, wedges, and finite products.

We will develop some of the basic properties of homotopy groups for sequential spectra. We being by showing that looping and suspending a spectrum shifts the homotopy groups. The loop homomorphism starts from the isomorphism

$$\alpha \ : \ \pi_{n+k}(\Omega X_n) \ \cong \ \pi_{n+k+1}(X_n)$$

that is defined by the same adjunction as above, i.e., the class represented by a continuous based map $f \colon S^{n+k} \longrightarrow \Omega X_n$ is sent to the class of the map $\hat{f} \colon S^{n+k+1} \longrightarrow X_n$ given by $\hat{f}(s \wedge t) = f(s)(t)$, where $s \in S^{n+k}$ and $t \in S^1$. As n varies, these particular isomorphisms are compatible with stabilization maps, so they induce an isomorphism

$$\alpha \ : \ \pi_k(\Omega X) \xrightarrow{\cong} \pi_{k+1}(X)$$

on colimits.

The maps $- \wedge S^1 \colon \pi_{n+k}(X_n) \longrightarrow \pi_{n+k+1}(X_n \wedge S^1)$ given by smashing from the right with the identity of the circle are compatible with the stabilization process for the homotopy groups of X and $X \wedge S^1$, respectively, so upon passage to colimits they induce a natural map of homotopy groups

$$- \wedge S^1 \ : \ \pi_k(X) \longrightarrow \pi_{k+1}(X \wedge S^1)\ ,$$

which we call the *suspension homomorphism.*

As before we let $\eta \colon X \longrightarrow \Omega(X \wedge S^1)$ and $\epsilon \colon (\Omega X) \wedge S^1 \longrightarrow X$ denote the unit respectively counit of the adjunction. Then for every map $f \colon S^{n+k} \longrightarrow \Omega X_n$ we have $\hat{f} = \epsilon_n \circ (f \wedge S^1)$ and for every map $g \colon S^{n+k} \longrightarrow X_n$ we have $g \wedge S^1 = \widehat{\eta_n \circ g}$. This means that the two triangles

$$(1.12) \qquad \begin{tikzcd} \pi_k(\Omega X) \arrow[rr, "\alpha"] \arrow[dr, "- \wedge S^1"'] & & \pi_{k+1}(X) \\ & \pi_{k+1}((\Omega X) \wedge S^1) \arrow[ur, "\pi_{k+1}(\epsilon)"'] & \end{tikzcd} \qquad \begin{tikzcd} \pi_k(X) \arrow[rr, "- \wedge S^1"] \arrow[dr, "\pi_k(\eta)"'] & & \pi_{k+1}(X \wedge S^1) \\ & \pi_k(\Omega(X \wedge S^1)) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commute.

Proposition 1.13. *Let X be a sequential spectrum.*

(i) *The loop and suspension homomorphisms*

$$\alpha \ : \ \pi_k(\Omega X) \longrightarrow \pi_{k+1}(X) \qquad \textit{and} \qquad - \wedge S^1 \ : \ \pi_k(X) \longrightarrow \pi_{k+1}(X \wedge S^1)$$

are isomorphisms of homotopy groups.

(ii) *The unit* $\eta\colon X \longrightarrow \Omega(X \wedge S^1)$ *and counit* $\epsilon\colon (\Omega X) \wedge S^1 \longrightarrow X$ *of the adjunction are stable equivalences.*
(iii) *For every continuous based map* $h\colon S^m \longrightarrow S^m$*, the morphism of sequential spectra* $X \wedge h\colon X \wedge S^m \longrightarrow X \wedge S^m$ *induces multiplication by the degree of* h *on all homotopy groups.*

Proof. (i) We already argued that the loop homomorphism α on homotopy groups is bijective since it is the colimit of compatible bijections. The case of the suspension homomorphism $- \wedge S^1$ is slightly more involved. We show injectivity first. Let $f\colon S^{n+k} \longrightarrow X_n$ represent an element in the kernel of the suspension homomorphism. By stabilizing, if necessary, we can assume that the suspension $f \wedge S^1\colon S^{n+k+1} \longrightarrow X_n \wedge S^1$ is nullhomotopic. Then $\sigma_n \circ \tau \circ (f \wedge S^1)\colon S^{n+k+1} \longrightarrow X_{n+1}$ is also nullhomotopic, where $\tau\colon X_n \wedge S^1 \cong S^1 \wedge X_n$ is the twist homeomorphism. The maps $\sigma_n \circ \tau \circ (f \wedge S^1)$ and $\sigma_n \circ (S^1 \wedge f)$, the stabilization of f, only differ by a coordinate permutation of the source sphere, hence the stabilization of f is nullhomotopic. So f represents the trivial element in $\pi_k(X)$, which shows that the suspension homomorphism is injective.

It remains to show that the suspension homomorphism is surjective. Let $g\colon S^{n+k+1} \longrightarrow X_n \wedge S^1$ be a map which represents a class in $\pi_{k+1}(X \wedge S^1)$. We consider the map $f = \sigma_n \circ \tau \circ g\colon S^{n+k+1} \longrightarrow X_{1+n}$ where τ is again the twist homeomorphism. We claim that $(-1)^{n+k} \cdot (f \wedge S^1)\colon S^{n+k+1+1} \longrightarrow X_{1+n} \wedge S^1$ represents the same class as g in $\pi_{k+1}(X \wedge S^1)$. To see this, we contemplate the diagram:

$$\begin{tikzcd}
S^{1+n+k+1} \arrow[d, "\chi_{1,n+k}\wedge S^1"'] \arrow[rr, bend left, "S^1\wedge g"] & & \\
S^{n+k+1+1} \arrow[r, "g\wedge S^1"] \arrow[drr, bend right, "f\wedge S^1"'] & X_n \wedge S^1 \wedge S^1 \arrow[r, "\tau\wedge S^1"] & S^1 \wedge X_n \wedge S^1 \arrow[d, "\sigma_n\wedge S^1"] \\
& & X_{1+n} \wedge S^1
\end{tikzcd}$$

The composite through the upper right is the stabilization of g, and the composite through the lower left represents $(-1)^{n+k} \cdot (f \wedge S^1)$. However, the upper triangle does *not* commute! The failure to commutativity are the involutions of $S^{1+n+k+1}$ and $S^1 \wedge X_n \wedge S^1$ which interchange the outer two sphere coordinates in each case. This coordinate change in the source induces multiplication by -1; the coordinate change in the target is a map of degree -1, so after a single suspension it also induces multiplication by -1 on homotopy groups (see Lemma 1.5). Altogether this shows that the upper triangle commutes up to homotopy *after one suspension*, and so the suspension map on homotopy groups is also surjective.

(ii) Since loops and suspension homomorphism are bijective and the triangles (1.12) commute, the unit and counit of the adjunction are stable equivalences.

(iii) Because the iterated suspension homomorphism

$$- \wedge S^m \ :\ \pi_k(X) \ \longrightarrow\ \pi_{k+m}(X \wedge S^m)$$

is bijective by part (i), every class in $\pi_{k+m}(X \wedge S^m)$ has a representative of the form $f \wedge S^m\colon S^{n+k+m} \longrightarrow X_n \wedge S^m$ for some continuous based map $f\colon S^{k+n} \longrightarrow X_n$. So

$$(X \wedge h)_*[f \wedge S^m] \ =\ [(X_n \wedge h) \circ (f \wedge S^m)] \ =\ [(f \wedge S^m) \circ (S^{n+k} \wedge h)] \ =\ [f \wedge S^m] \cdot \deg(h)\ . \qquad \square$$

Corollary 1.14. *For every morphism* $f\colon X \longrightarrow Y$ *of sequential spectra, the following conditions are equivalent.*
(a) *The morphism* $f\colon X \longrightarrow Y$ *is a stable equivalence.*
(b) *The morphism* $\Omega f\colon \Omega X \longrightarrow \Omega Y$ *is a stable equivalence.*
(c) *The morphism* $f\colon X \wedge S^1 \longrightarrow Y \wedge S^1$ *is a stable equivalence.*

A morphism $g\colon A \wedge S^1 \longrightarrow X$ *of sequential spectra is a stable equivalence if and only if its adjoint* $\hat{g}\colon A \longrightarrow \Omega X$ *is a stable equivalence.*

Proof. We only need to prove the last statement. The morphism g and its adjoint are related by $g = \epsilon \circ (\hat{g} \wedge S^1)$ where $\epsilon\colon (\Omega X) \wedge S^1 \longrightarrow X$ is the counit of the adjunction. The counit is a stable equivalence by Proposition 1.13. We conclude that the morphism g is a stable equivalence if and only if the morphism

$\hat{g} \wedge S^1 \colon A \wedge S^1 \longrightarrow (\Omega X) \wedge S^1$ is. Since suspension shifts homotopy groups, this happens if and only if g is a stable equivalence. □

As far as I can see, the suspension functor $- \wedge S^1$ does not in general preserve weak equivalences of spaces if these are not well-pointed. Hence $- \wedge S^1$ does not in general preserve level equivalences of sequential spectra. However, $- \wedge S^1$ preserves stable equivalences, ultimately because of the suspension isomorphism.

1.1. **Mapping cone and homotopy fiber.** Now we review the mapping cone and the homotopy fiber of a map of based spaces in some detail, along with their relationships to one another and to suspension and loop space. The *(reduced) mapping cone* Cf of a morphism of based spaces $f \colon A \longrightarrow B$ is defined by

$$Cf \;=\; (A \wedge [0,1]) \cup_f B \; .$$

Here the unit interval $[0,1]$ is pointed by $0 \in [0,1]$, so that $A \wedge [0,1]$ is the reduced cone of A. The mapping cone comes with an inclusion $i \colon B \longrightarrow Cf$ and a projection $p \colon Cf \longrightarrow A \wedge S^1$; the projection sends B to the basepoint and is given on $A \wedge [0,1]$ by $p(a \wedge x) = a \wedge \mathbf{t}(x)$ where

$$\mathbf{t} \colon [0,1] \longrightarrow S^1 \qquad \text{is defined as} \qquad \mathbf{t}(x) = \frac{2x-1}{x(1-x)} \; .$$

What is relevant about the map $\mathbf{t}$ is not the precise formula, but that it passes to a homeomorphism between the quotient space $[0,1]/\{0,1\}$ and the circle S^1, and that it satisfies $\mathbf{t}(1-x) = -\mathbf{t}(x)$.

We observe that an iteration of the mapping cone construction yields the suspension of A, up to homotopy.

Lemma 1.15. *Let $f \colon A \longrightarrow B$ be any continuous based map.*

(i) *The collapse map*

$$* \cup p \;:\; Ci = (B \wedge [0,1]) \cup_i Cf \;\longrightarrow\; A \wedge S^1$$

is a based homotopy equivalence.

(ii) *The square*

$$\begin{tikzcd} Ci \arrow[r, "p\cup *"] \arrow[d, "*\cup p"'] & B \wedge S^1 \arrow[d, "B\wedge\tau"] \\ A \wedge S^1 \arrow[r, "f\wedge S^1"'] & B \wedge S^1 \end{tikzcd}$$

commutes up to natural, based homotopy, where τ is the involution of S^1 given by $\tau(x) = -x$.

(iii) *Let $\beta \colon Z \longrightarrow B$ be a continuous based map such that the composite $i\beta \colon Z \longrightarrow Cf$ is null-homotopic. Then there exists a based map $h \colon Z \wedge S^1 \longrightarrow A \wedge S^1$ such that $(f \wedge S^1) \circ h \colon Z \wedge S^1 \longrightarrow B \wedge S^1$ is homotopic to $\beta \wedge S^1$.*

Proof. (i) A homotopy inverse $r \colon A \wedge S^1 \longrightarrow (B \wedge [0,1]) \cup_i Cf$ of $* \cup p$ is defined by the formula

$$r(a \wedge x) \;=\; \begin{cases} a \wedge 2x & \text{in } Cf \text{ for } 0 \le x \le 1/2, \text{ and} \\ f(a) \wedge (2-2x) & \text{in } B \wedge [0,1] \text{ for } 1/2 \le x \le 1. \end{cases}$$

We give explicit based homotopies between the two composites r and $*\cup p$ and the respective identity maps. The space $Ci = (B \wedge [0,1]) \cup_i Cf$ is homeomorphic to the quotient of the disjoint union of $B \wedge [0,1]$ and $A \wedge [0,1]$ by the equivalence relation that identifies $f(a) \wedge 1$ in $B \wedge [0,1]$ with $a \wedge 1$ in $A \wedge [0,1]$ for all $a \in A$. So we can define a homotopy on the space Ci by gluing two compatible homotopies. The homotopy

$$[0,1] \times (B \wedge [0,1]) \;\longrightarrow\; Ci \;, \quad (t, b \wedge x) \longmapsto b \wedge (1-t)x \quad \text{in } B \wedge [0,1] \; .$$

and the homotopy

$$[0,1] \times (A \wedge [0,1]) \;\longrightarrow\; Ci \;,\; (t, a \wedge x) \mapsto \begin{cases} a \wedge (1+t)x & \text{in } Cf \text{ for } 0 \le x \le 1/(1+t), \text{ and} \\ f(a) \wedge (2 - x(1+t)) & \text{in } B \wedge [0,1] \text{ for } 1/(1+t) \le x \le 1, \end{cases}$$

are compatible, and the combined homotopy starts at $t = 0$ with the identity and ends at $t = 1$ with the map $r \circ (* \cup p)$.

A homotopy from the identity of $A \wedge S^1$ to $(* \cup p) \circ r$ is given by

$$[0,1] \times (A \wedge S^1) \longrightarrow A \wedge S^1 , \quad (t,\ a \wedge x) \longmapsto a \wedge (1+t)$$

which is to be interpreted as the basepoint if $(1+t)x \geq 1$.

(ii) Again we glue the desired homotopy from two pieces, namely

$$[0,1] \times (B \wedge [0,1]) \longrightarrow B \wedge S^1 , \quad (t,\ b \wedge x) \longmapsto b \wedge (1+t-x) ,$$

which has to be interpreted as the basepoint if $x \leq t$ and

$$[0,1] \times (A \wedge [0,1]) \longrightarrow B \wedge S^1 , \quad (t,\ a \wedge x) \longmapsto f(a) \wedge (t+x-1)$$

which has to be interpreted as the basepoint if $t + x \leq 1$. The two homotopies are compatible and the combined homotopy starts with the map $(B \wedge \tau) \circ (p \cup *)$ for $t = 0$ and it ends with the map $(f \wedge S^1) \circ (* \cup p)$ for $t = 1$.

(iii) Let $H \colon Z \wedge [0,1] \longrightarrow Cf$ be a based null-homotopy of the composite $i\beta \colon Z \longrightarrow Cf$, i.e., $H(z \wedge 1) = i(\beta(z))$ for all $z \in Z$. The composite $p_A H \colon Z \wedge [0,1] \longrightarrow A \wedge S^1$ then factors as $p_A H = hp_Z$ for a unique map $h \colon Z \wedge S^1 \longrightarrow A \wedge S^1$.

To analyze $(f \wedge S^1) \circ h$ we compose it with the map $* \cup p_Z \colon (Z \wedge [0,1]) \cup_{Z \times 1} (Z \wedge [0,1]) \longrightarrow Z \wedge S^1$ which collapses the second cone and which is a homotopy equivalence by (i). We obtain a sequence of equalities and homotopies

$$\begin{aligned} (f \wedge S^1) \circ h \circ (* \cup p_Z) &= (f \wedge S^1) \circ (* \cup p_A) \circ ((\beta \wedge [0,1]) \cup H) \\ &\simeq (B \wedge \tau) \circ (p_B \cup *) \circ ((\beta \wedge [0,1]) \cup H) \\ &= (B \wedge \tau) \circ (\beta \wedge S^1) \circ (p_Z \cup *) \\ &= (\beta \wedge S^1) \circ (Z \wedge \tau) \circ (p_Z \cup *) \ \simeq\ (\beta \wedge S^1) \circ (* \cup p_Z) \end{aligned}$$

Here $(\beta \wedge [0,1]) \cup H \colon CZ \cup_{Z \times 1} CZ \longrightarrow CB \cup_i Cf = C(i)$. The two homotopies result from part (ii) applied to f respectively the identity of Z. Since the map $* \cup p_Z$ is a homotopy equivalence, this proves that $(f \wedge S^1) \circ h$ is homotopic to $\beta \wedge S^1$. □

Now we can introduce mapping cones for sequential spectra. The *mapping cone* Cf of a morphism of sequential spectra $f \colon X \longrightarrow Y$ is define levelwise:

$$(Cf)_n \ =\ C(f_n) \ =\ (X_n \wedge [0,1]) \cup_{f_n} Y_n , \tag{1.16}$$

the reduced mapping cone of $f_n \colon X_n \longrightarrow Y_n$. The structure maps are induced by the structure maps of X and Y, and they do not interact with the cone coordinate. The inclusions $i_n \colon Y_n \longrightarrow C(f_n)$ and projections $p_n \colon C(f_n) \longrightarrow X_n \wedge S^1$ assemble into morphisms of sequential spectra $i \colon Y \longrightarrow Cf$ and $p \colon Cf \longrightarrow X \wedge S^1$.

We define a *connecting homomorphism* $\delta \colon \pi_{k+1}(Cf) \longrightarrow \pi_k(X)$ as the composite

$$\pi_{k+1}(Cf) \xrightarrow{p_*} \pi_{k+1}(X \wedge S^1) \xrightarrow{-\wedge S^{-1}} \pi_k(X) , \tag{1.17}$$

where the second map is the inverse of the suspension isomorphism $- \wedge S^1 \colon \pi_k(X) \longrightarrow \pi_{k+1}(X \wedge S^1)$. If we unravel all the definitions, we see that δ sends the class represented by a based map $g \colon S^{n+k+1} \longrightarrow Cf_n$ to $(-1)^{n+k}$ times the class of the composite

$$S^{n+k+1} \xrightarrow{g} Cf_n \xrightarrow{p_n} X_n \wedge S^1 \xrightarrow{\text{twist}} S^1 \wedge X_n \xrightarrow{\sigma_n} X_{1+n} .$$

Proposition 1.18. *For every morphism $f \colon X \longrightarrow Y$ of sequential spectra the long sequence of abelian groups*

$$\cdots \longrightarrow \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{i_*} \pi_k(Cf) \xrightarrow{\delta} \pi_{k-1}(X) \longrightarrow \cdots$$

is exact.

Proof. We start with exactness at $\pi_k(Y)$. The composite of $f\colon X \longrightarrow Y$ and the inclusion $Y \longrightarrow Cf$ is levelwise the constant map at the basepoint, so it induces the trivial map on π_k. It remains to show that every element in the kernel of i_* is in the image of f_*. Let $\beta\colon S^{n+k} \longrightarrow Y_n$ represent an element in the kernel. By increasing n, if necessary, we can assume that $i\beta\colon S^{n+k} \longrightarrow C(f_n)$ is null-homotopic. By Lemma 1.15 (iii) there is a based map $h\colon S^{n+k+1} \longrightarrow X_n \wedge S^1$ such that $(f_n \wedge S^1) \circ h$ is homotopic to $\beta \wedge S^1$. The composite

$$\tilde{h} \ : \ S^{1+n+k} \xrightarrow{\chi_{1,n+k}} S^{n+k+1} \xrightarrow{\ h\ } X_n \wedge S^1 \xrightarrow{\tau_{X_n,S^1}} S^1 \wedge X_n$$

then has the property that $(S^1 \wedge f_n) \circ \tilde{h}$ is homotopic to $S^1 \wedge \beta$. The map $\sigma_n \circ \tilde{h}\colon S^{1+n+k} \longrightarrow X_{1+n}$ represents a homotopy class in $\pi_k(X)$ and we have

$$f_*[\sigma_n \circ \tilde{h}] \ = \ [f_{1+n} \circ \sigma_n \circ \tilde{h}] \ = \ [\sigma_n \circ (S^1 \wedge f_n) \circ \tilde{h}] \ = \ [\sigma_n \circ (S^1 \wedge \beta)] \ = \ [\beta] \ .$$

So the class represented by β is in the image of $f_*\colon \pi_k(X) \longrightarrow \pi_k(Y)$.

We now deduce the exactness at $\pi_k(Cf)$ and $\pi_{k-1}(X)$ by comparing the mapping cone sequence for $f\colon X \longrightarrow Y$ to the mapping cone sequence for the morphism $i\colon Y \longrightarrow Cf$ (shifted to the left). The collapse map

$$* \cup p \ : \ Ci \ = \ CY \cup_i Cf \ \longrightarrow \ X \wedge S^1$$

is levelwise a homotopy equivalence by Lemma 1.15 (i), and thus induces an isomorphism of homotopy groups. Now we consider the diagram

$$\begin{tikzcd}
Cf \arrow[r, "i_i"] \arrow[dr, "p"'] & Ci \arrow[r, "p\cup *"] \arrow[d, "*\cup p"] & Y \wedge S^1 \arrow[d, "Y\wedge\tau"] \\
 & X \wedge S^1 \arrow[r, "f\wedge S^1"'] & Y \wedge S^1
\end{tikzcd}$$

whose upper row is part of the mapping cone sequence for the morphism $i\colon Y \longrightarrow Cf$. The left triangle commutes on the nose and the right triangle commutes up to based homotopy by Lemma 1.15 (ii). The involution $\tau\colon S^1 \longrightarrow S^1$ has degree -1, so the automorphism $Y \wedge \tau$ of $Y \wedge S^1$ induces multiplication by -1 on homotopy groups. We get a commutative diagram

$$\begin{tikzcd}
\pi_k(Y) \arrow[r, "i_*"] \arrow[d, equal] & \pi_k(Cf) \arrow[r, "(i_i)_*"] \arrow[d, equal] & \pi_k(Ci) \arrow[r, "\delta"] \arrow[d, "(-\wedge S^{-1})\circ(*\cup p)_*"', "\cong"] & \pi_{k-1}(Y) \arrow[d, "(-1)\cdot"] \\
\pi_k(Y) \arrow[r, "i_*"'] & \pi_k(Cf) \arrow[r, "\delta"'] & \pi_{k-1}(X) \arrow[r, "f_*"'] & \pi_{k-1}(Y)
\end{tikzcd}$$

(using for the right square the naturality of the suspension isomorphism). By the previous paragraph, applied to $i\colon Y \longrightarrow Cf$ instead of f, the upper row is exact at $\pi_k(Cf)$. Since all vertical maps are isomorphisms, the original lower row is exact at $\pi_k(Cf)$. But the morphism f was arbitrary, so when applied to $i\colon Y \longrightarrow Cf$ instead of f, we obtain that the upper row is exact at $\pi_k(Ci)$. Since all vertical maps are isomorphisms, the original lower row is exact at $\pi_{k-1}(X)$. This finishes the proof. □

A continuous map $f\colon A \longrightarrow B$ of spaces is an *h-cofibration* if it has the homotopy extension property, i.e., given a continuous map $\varphi\colon B \longrightarrow X$ and a homotopy $H\colon A \times [0,1] \longrightarrow X$ such that $H(-,0) = \varphi f$, there is a homotopy $\bar{H}\colon B \times [0,1] \longrightarrow X$ such that $\bar{H} \circ (f \times [0,1]) = H$ and $\bar{H}(-,0) = \varphi$. An equivalent condition is that the map $A \times [0,1] \cup_{f\times 0} B \longrightarrow B \times [0,1]$ has a retraction. For every h-cofibration the map $Cf \longrightarrow B/A$ which collapses the cone of A to a point is a based homotopy equivalence, see for example [19, Proposition 0.17] or [53, Proposition 5.1.10] with $B = *$.

Let $f\colon X \longrightarrow Y$ be a morphism of sequential spectra that is levelwise an h-cofibration. Then by the above, the morphism $c\colon Cf \longrightarrow Y/X$ that collapses the cone of X is a level equivalence, and so it induces an isomorphism of homotopy groups. We can thus define another connecting homomorphism

$$\delta \ : \ \pi_k(Y/X) \ \longrightarrow \ \pi_{k-1}(X) \tag{1.19}$$

as the composite of the inverse of the isomorphism $c_*\colon \pi_k(Cf) \longrightarrow \pi_k(Y/X)$ and the connecting homomorphism $\pi_k(Cf) \longrightarrow \pi_{k-1}(X)$ defined in (1.17).

Corollary 1.20. *Let $f\colon X \longrightarrow Y$ be a morphism of sequential spectra that is levelwise an h-cofibration and denote by $q\colon Y \longrightarrow Y/X$ the quotient map. Then the long sequence of homotopy groups*

$$\cdots \longrightarrow \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{q_*} \pi_k(Y/X) \xrightarrow{\delta} \pi_{k-1}(X) \longrightarrow \cdots$$

is exact.

Now we discuss the *homotopy fiber*, a construction 'dual' to the mapping cone. The homotopy fiber of a morphism $f\colon A \longrightarrow B$ of based spaces is the fiber product

$$Ff \;=\; * \times_B B^{[0,1]} \times_B A \;=\; \{(\lambda, a) \in B^{[0,1]} \times A \mid \lambda(0) = *,\, \lambda(1) = f(a)\} ,$$

i.e., the space of paths in B starting at the basepoint and equipped with a lift of the endpoint to A. As basepoint of the homotopy fiber we take the pair consisting of the constant path at the basepoint of B and the basepoint of A. The homotopy fiber comes with maps

$$\Omega B \xrightarrow{i} Ff \xrightarrow{q} A ;$$

the map q is the projection to the second factor and the value of the map i on a loop $\omega\colon S^1 \longrightarrow B$ is $i(\omega) = (\omega \circ \mathbf{t}, *)$.

We can apply the homotopy fiber levelwise to a morphism $f\colon X \longrightarrow Y$ of sequential spectra. The homotopy fiber Ff is defined by

$$(Ff)_n \;=\; F(f_n) ,$$

the homotopy fiber of $f_n\colon X_n \longrightarrow Y_n$. The inclusions $i_n\colon \Omega(Y_n) \longrightarrow (Ff)_n$ and projections $q_n\colon (Ff)_n \longrightarrow X_n$ assemble into morphisms of sequential spectra $i\colon \Omega Y \longrightarrow Ff$ and $p\colon Ff \longrightarrow X$.

We define a *connecting homomorphism* $\delta\colon \pi_{k+1}(Y) \longrightarrow \pi_k(Ff)$ as the composite

$$\pi_{k+1}(Y) \xrightarrow{\alpha^{-1}} \pi_k(\Omega Y) \xrightarrow{i_*} \pi_k(Ff) , \tag{1.21}$$

where $\alpha\colon \pi_k(\Omega Y) \longrightarrow \pi_{1+k}(Y)$ is the loop isomorphism.

We can compare the mapping cone and homotopy fiber as follows. For a map $f\colon A \longrightarrow B$ of based spaces we define a map $\bar{h}\colon F(f) \times [0,1] \longrightarrow (A \wedge [0,1]) \cup_f B = Cf$ by

$$(\lambda, a, t) \;\longmapsto\; \begin{cases} a \wedge 2t & \text{for } 0 \le t \le 1/2, \text{ and} \\ \lambda(2-2t) & \text{for } 1/2 \le t \le 1. \end{cases}$$

We note that the two formulas match at $t = 1/2$ because $\lambda(1) = f(a) = a \wedge 1$ in Cf. Since $\bar{h}(\lambda, a, 0)$ and $\bar{h}(\lambda, a, 1)$ are the basepoint of the mapping cone for all (λ, a) in Ff, the map $\bar{h}$ factors over a based map

$$h \;:\; (Ff) \wedge S^1 \;\longrightarrow\; Cf ,$$

which satisfies $h \circ q = \bar{h}$ and is natural in f. So for a morphism $f\colon X \longrightarrow Y$ of sequential spectra, the maps h for the various levels together form a natural morphism

$$h \;:\; (Ff) \wedge S^1 \;\longrightarrow\; Cf .$$

Proposition 1.22. *For every morphism $f\colon X \longrightarrow Y$ of sequential spectra the long sequence of abelian groups*

$$\cdots \longrightarrow \pi_k(Ff) \xrightarrow{q_*} \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{\delta} \pi_{k-1}(Ff) \longrightarrow \cdots$$

is exact and the morphism $h\colon (Ff) \wedge S^1 \longrightarrow Cf$ is a stable equivalence.

Proof. The long sequence is exact because it is obtained from the unstable long exact sequences for the homotopy fiber sequences $(Ff)_n \longrightarrow X_n \longrightarrow Y_n$ by passage to the colimit (which is exact).

For showing that h is a stable equivalence it suffices to show that the composite $h_* \circ (- \wedge S^1) \colon \pi_k(Ff) \longrightarrow \pi_{k+1}(Cf)$ is an isomorphism. We claim that the diagram

$$\begin{tikzcd}
\pi_{k+1}(Y) \arrow[r, "\delta"] \arrow[d, "(-1)\cdot"'] & \pi_k(Ff) \arrow[r, "q_*"] \arrow[d, "h_* \circ (- \wedge S^1)"'] & \pi_k(X) \arrow[d, equal] \\
\pi_{k+1}(Y) \arrow[r, "i_*"'] & \pi_{k+1}(Cf) \arrow[r, "\delta"'] & \pi_k(X)
\end{tikzcd}$$

commutes. The morphism $h_* \circ (- \wedge S^1) \colon \pi_k(Ff) \longrightarrow \pi_{k+1}(Cf)$ and the identity maps of the homotopy groups of X and Y thus give a natural map from the long exact sequence of the homotopy fiber to the long exact sequence of the mapping cone, with an extra sign. A sign does not affect exactness of a sequence, and so the five lemma shows that $h_* \circ (- \wedge S^1)$ is an isomorphism. Hence h is a stable equivalence.

We still have to justify the commutativity of the previous diagram. For the right square this is the definition of the connecting homomorphism, naturality of the suspension isomorphism and the fact that the composite

$$(Ff) \wedge S^1 \xrightarrow{\ h\ } Cf \xrightarrow{\ p\ } X \wedge S^1$$

is homotopic to $q \wedge S^1$ via the homotopy

$$[0,1] \times ((Ff) \wedge S^1) \longrightarrow X \wedge S^1 , \quad (t,\ (\lambda, a) \wedge s) \longmapsto \begin{cases} a \wedge 2s/(2-t) & \text{for } 0 \leq s \leq 1 - t/2, \text{ and} \\ * & \text{for } 1 - t/2 \leq s \leq 1, \end{cases}$$

(to be interpreted levelwise). Indeed, these facts together supply the relation

$$\begin{aligned} \delta \circ h_* \circ (- \wedge S^1) &= (- \wedge S^{-1}) \circ p_* \circ h_* \circ (- \wedge S^1) \\ &= (- \wedge S^{-1}) \circ (q \wedge S^1)_* \circ (- \wedge S^1) \\ &= (- \wedge S^{-1}) \circ (- \wedge S^1) \circ q_* \ = \ q_* \ . \end{aligned}$$

For the left square we need that the diagram

$$\begin{tikzcd}
(\Omega Y) \wedge S^1 \arrow[r, "i \wedge \tau"] \arrow[d, "\epsilon"'] & (Ff) \wedge S^1 \arrow[d, "h"] \\
Y \arrow[r, "i"'] & Cf
\end{tikzcd}$$

commutes up to based homotopy, where ϵ is the adjunction counit. One possible such homotopy is

$$\begin{aligned} [0,1] \times ((\Omega Y) \wedge S^1) &\longrightarrow \quad Cf \\ (t,\ \omega \wedge x) \quad &\longmapsto \begin{cases} * & \text{for } 0 \leq x \leq t/2, \text{ and} \\ \omega(2(1-t)/(2-x)) & \text{for } t/2 \leq x \leq 1. \end{cases} \end{aligned}$$

Given this, we have

$$\begin{aligned} h_*(\delta(y) \wedge S^1) &= h_*(i_*(\alpha^{-1}(y)) \wedge S^1) \ = \ (h \circ (i \wedge S^1))_*(\alpha^{-1}(y) \wedge S^1) \\ &= -(i \circ \epsilon)_*(\alpha^{-1}(y) \wedge S^1) \ =_{(1.12)} \ -i_*(y) \end{aligned}$$

and this finishes the proof. □

For every Serre fibration $\varphi \colon E \longrightarrow B$ of topological spaces the map $c \colon F \longrightarrow F(\varphi)$ from the strict fiber to the homotopy fiber that sends $e \in F$ to (const_*, e). is a weak equivalence. We let $f \colon X \longrightarrow Y$ be a morphism of sequential spectra that is levelwise a Serre fibration; then by the above the morphism $c \colon F \longrightarrow F(f)$ from the strict fiber to the homotopy fiber of f is a level equivalence. So we can define another connecting morphism

$$\delta \ : \ \pi_k(Y) \ \longrightarrow \ \pi_{k-1}(F)$$

as the composite of the connecting homomorphism $\pi_k(Y) \longrightarrow \pi_{k-1}(Ff)$ defined in (1.21) and the inverse of the isomorphism $c_* \colon \pi_{k-1}(Ff) \longrightarrow \pi_{k-1}(F)$.

Corollary 1.23. *Let $f \colon X \longrightarrow Y$ be a morphism of sequential spectra that is levelwise a Serre fibration; let $\iota \colon F \longrightarrow X$ denote the inclusion of the fiber over the basepoint. Then the long sequence of homotopy groups*

$$\cdots \longrightarrow \pi_k(F) \xrightarrow{\iota_*} \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{\delta} \pi_{k-1}(F) \longrightarrow \cdots$$

is exact.

We draw some consequences of our previous results.

Proposition 1.24. (i) *For every family of sequential spectra $\{X^i\}_{i\in I}$ and every integer k the canonical map*

$$\bigoplus_{i\in I} \pi_k(X^i) \longrightarrow \pi_k\left(\bigvee_{i\in I} X^i\right)$$

is an isomorphism of abelian groups.

(ii) *For every finite indexing set I, every family $\{X^i\}_{i\in I}$ of sequential spectra and every integer k the canonical map*

$$\pi_k\left(\prod_{i\in I} X^i\right) \longrightarrow \prod_{i\in I} \pi_k(X^i)$$

is an isomorphism of abelian groups.

(iii) *For every finite family of sequential spectra the canonical morphism from the wedge to the product is a stable equivalence.*

Proof. (i) We first show the special case of two summands. If A and B are two sequential spectra, then the wedge inclusion $i_A \colon A \longrightarrow A \vee B$ has a retraction. So the associated long exact homotopy group sequence of Proposition 1.18 splits into short exact sequences

$$0 \longrightarrow \pi_k(A) \xrightarrow{(i_A)_*} \pi_k(A \vee B) \xrightarrow{i_*} \pi_k(C(i_A)) \longrightarrow 0 .$$

The mapping cone $C(i_A)$ is isomorphic to $(CA) \vee B$ and thus homotopy equivalent to B. So we can replace $\pi_k(C(i_A))$ by $\pi_k(B)$ and conclude that $\pi_k(A \vee B)$ splits as the sum of $\pi_k(A)$ and $\pi_k(B)$, via the canonical map. The case of a finite indexing set I now follows by induction.

In the general case we consider the composite

$$\bigoplus\nolimits_{i\in I} \pi_k(X^i) \longrightarrow \pi_k\left(\bigvee\nolimits_{i\in I} X^i\right) \longrightarrow \prod\nolimits_{i\in I} \pi_k(X^i) ,$$

where the second map is induced by the projections to the wedge summands. This composite is the canonical map from a direct sum to a product of abelian groups, hence injective. So the first map is injective as well. For surjectivity we consider a continuous based map $f \colon S^{n+k} \longrightarrow \bigvee_{i\in I} X^i_n$ that represents an element in the k-th homotopy group of $\bigvee_{i\in I} X^i$. Since the source of f is compact, there is a finite subset J of I such that f has image in $\bigvee_{j\in J} X^j_n$, see for example [43, Proposition A.18]. Then the given class is in the image of $\pi_k\left(\bigvee_{j\in J} X^j\right)$; since J is finite, the class is in the image of the canonical map, by the previous paragraph.

(ii) Unstable homotopy groups commute with products, which for finite indexing sets are also sums, which commute with filtered colimits.

(iii) This is a direct consequence of (i) and (ii). More precisely, for finite indexing set I and every integer k the composite map

$$\bigoplus_{i\in I} \pi_k(X^i) \longrightarrow \pi_k(\bigvee_{i\in I} X^i) \longrightarrow \pi_k(\prod_{i\in I} X^i) \longrightarrow \prod_{i\in I} \pi_k(X^i)$$

is an isomorphism, where the first and last maps are the canonical ones. These canonical maps are isomorphisms by parts (i) respectively (ii), hence so is the middle map. □

Remark 1.25. The restriction to *finite* indexing sets in parts (ii) of the previous corollary is essential, and it ultimately comes from the fact that infinite products do not in general commute with sequential colimits. Here is an explicit example: we consider the spectra $\mathbb{S}^{\leq i}$ obtained by truncating the sphere spectrum above level i, i.e.,

$$(\mathbb{S}^{\leq i})_n \;=\; \begin{cases} S^n & \text{for } n \leq i, \\ * & \text{for } n \geq i+1 \end{cases}$$

with structure maps as a quotient spectrum of $\mathbb{S}$. Then $\mathbb{S}^{\leq i}$ has trivial homotopy groups for all i. The 0th homotopy group of the product $\prod_{i\geq 1}\mathbb{S}^{\leq i}$ is the colimit of the sequence of maps

$$\prod_{i\geq n}\pi_n(S^n) \;\longrightarrow\; \prod_{i\geq n+1}\pi_{n+1}(S^{n+1})$$

which first projects away from the factor indexed by $i = n$ and then takes a product of the suspensions homomorphisms $-\wedge S^1\colon \pi_n(S^n) \longrightarrow \pi_{n+1}(S^{n+1})$. The colimit is thus isomorphic to the quotient of an infinite product of copies of the group $\mathbb{Z}$ by the direct sum of the same number of copies of $\mathbb{Z}$. Hence the right hand side of the canonical map

$$\pi_0\left(\prod\nolimits_{i\geq 1}\mathbb{S}^{\leq i}\right) \;\longrightarrow\; \prod\nolimits_{i\geq 1}\pi_0(\mathbb{S}^{\leq i})$$

is trivial, while the left hand side is not.

Proposition 1.26. (i) *Let $e^m\colon X^m \longrightarrow X^{m+1}$ be morphisms of sequential spectra that are levelwise closed embeddings, for $m \geq 0$. Let X^∞ be a colimit of the sequence $\{e^m\}_{m\geq 0}$. Then for every integer k the canonical map*

$$\operatorname{colim}_{m\geq 0}\, \pi_k(X^m) \;\longrightarrow\; \pi_k(X^\infty)$$

is an isomorphism.

(ii) *Let $e^m\colon X^m \longrightarrow X^{m+1}$ and $f^m\colon Y^m \longrightarrow Y^{m+1}$ be morphisms of sequential spectra that are levelwise closed embeddings, for $m \geq 0$. Let $\psi^m\colon X^m \longrightarrow Y^m$ be stable equivalences that satisfy $\psi^{m+1}\circ e^m = f^m\circ\psi^m$ for all $m \geq 0$. Then the induced morphism $\psi^\infty\colon X^\infty \longrightarrow Y^\infty$ between the colimits of the sequences is a stable equivalence.*

(iii) *Let $f^m\colon Y^m \longrightarrow Y^{m+1}$ be stable equivalences of sequential spectra that are levelwise closed embeddings, for $m \geq 0$. Then the canonical morphism $f^\infty\colon Y^0 \longrightarrow Y^\infty$ to a colimit of the sequence $\{f^m\}_{m\geq 0}$ is a stable equivalence.*

Proof. (i) We let $f\colon S^{n+k} \longrightarrow X_n^\infty$ be a based continuous map that represents a class in $\pi_k(X^\infty)$. Since the sphere S^{n+k} is compact and X_n^∞ is a colimit of the sequence of closed embeddings $X_n^m \longrightarrow X_n^{m+1}$, the map f factors through a continuous map

$$\bar f \;:\; S^{n+k} \;\longrightarrow\; X_n^m$$

for some $m \geq 0$, see for example [21, Proposition 2.4.2] or [43, Proposition A.15]. The same reasoning applies to homotopies, so the canonical map

$$\operatorname{colim}_{m\geq 0}\, \pi_{n+k}(X_n^m) \;\longrightarrow\; \pi_{n+k}(X_n^\infty)$$

is bijective. Passing to colimits over n proves the claim.

Parts (ii) and (iii) are direct consequences of (i). □

Proposition 1.27.

(i) *A wedge of stable equivalences is a stable equivalence.*
(ii) *A finite product of stable equivalences is a stable equivalence.*

(iii) *Consider a commutative square of orthogonal spectra*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "j"'] & D \end{tikzcd} \tag{1.28}$$

and let $h = (Cf) \cup g\colon Ci \longrightarrow Cj$ *be the map induced by* f *and* g *on mapping cones. Then if two of the three morphisms* f, g *and* h *are stable equivalences, so is the third.*

(iv) *Consider a commutative square* (1.28) *of sequential spectra. Let* $e\colon Fi \longrightarrow Fj$ *be the map induced by* f *and* g *on homotopy fibers. Then if two of the three morphisms* e, f *and* g *are stable equivalences, so is the third.*

(v) *Consider a commutative square* (1.28) *of sequential spectra for which one of the following conditions holds:*
(a) *the square is a pushout and* i *or* f *is levelwise an h-cofibration.*
(b) *the square is a pullback and* j *or* g *is a levelwise a Serre fibration.*
Then f *is a stable equivalence if and only if* g *is.*

(vi) *Let* K *be a based space that admits a CW-structure. Then the functor* $- \wedge K$ *preserves stable equivalences of sequential spectra.*

(vii) *Let* K *be a based space that admits a finite CW-structure. Then the functor* $\mathrm{map}_*(K, -)$ *preserves stable equivalences of sequential spectra.*

Proof. Part (i) holds because the homotopy groups of a wedge are the direct sum of the homotopy groups of the wedge summands (Proposition 1.24 (i)). Part (ii) holds because the homotopy groups of a finite product are the product of the homotopy groups of the factors (Proposition 1.24 (ii)). Part (iii) follows by applying the 5-lemma to the long exact sequences of the mapping cones of i and j (Proposition 1.18). Part (iv) follows by applying the 5-lemma to the long exact sequences of the homotopy fibers of i and j (Proposition 1.22).

(v) We start with case (a) of a pushout square. Since the square is a pushout, the morphism $j\colon C \longrightarrow D$ descends to an isomorphism $j/i\colon C/A \cong D/B$ between the two vertical quotient spectra; and the morphism $g\colon B \longrightarrow D$ descends to an isomorphism $g/f\colon B/A \cong D/C$ between the two horizontal quotient spectra.

If f is levelwise an h-cofibration, then the long exact homotopy group sequence for the quotient spectrum C/A (Corollary 1.20) shows that f is a stable equivalence if and only of C/A has trivial homotopy groups. Since h-cofibrations are stable under cobase change, the same is true for g: the morphism g is a stable equivalence if and only of D/B has trivial homotopy groups. So f is a stable equivalence if and only if g is.

If i is an h-cofibration, the argument is similar, but slightly different. In this case we compare the two long exact homotopy group sequences for the horizontal quotient spectra (Corollary 1.20). Since $g/f\colon B/A \cong D/C$ is an isomorphism, the 5-lemma shows that f is stable equivalence if and only if g is. The comparison map for the homotopy groups of the

The case (b) of a pullback square is strictly dual, using strict fibers instead of quotient spectra, and Corollary 1.23 in place of Corollary 1.20.

(vi) The functor $- \wedge K$ preserves mapping cones, so by the long exact homotopy group sequence of Proposition 1.18 it suffices to show the following special case: let X be a sequential spectrum all of whose homotopy groups vanish; then all homotopy groups of the spectrum $X \wedge K$ vanish, too.

We let K_n denote the n-skeleton in a CW-structure on K. We show first, by induction on n, that the spectrum $X \wedge K_n$ has trivial homotopy groups. The induction starts with $n = -1$, where there is nothing to show. For $n \geq 0$ the quotient K_n/K_{n-1} is homeomorphic to a wedge of n-spheres. Since homotopy groups take wedges to sums, the suspension isomorphism allows us to rewrite the homotopy groups of $X \wedge (K_n/K_{n-1})$ as

$$\pi_k(X \wedge (K_n/K_{n-1})) \;\cong\; \pi_k(X \wedge (\bigvee\nolimits_I S^n)) \;\cong\; \pi_k(\bigvee\nolimits_I (X \wedge S^n)) \;\cong\; \bigoplus\nolimits_I \pi_{k-n}(X) \ .$$

This group is trivial by the hypothesis on X.

The inclusion $K_{n-1} \longrightarrow K_n$ is an h-cofibration of based spaces, so the induced morphism $X \wedge K_{n-1} \longrightarrow X \wedge K_n$ is an h-cofibration of sequential spectra; these morphisms are then in particular levelwise closed embeddings, giving rise to a long exact sequence of homotopy groups (Corollary 1.20). By the previous paragraph and the inductive hypothesis, the spectrum $X \wedge K_n$ has vanishing homotopy groups. This completes the inductive step.

Since K is the sequential colimit, along h-cofibrations of based spaces, of the skeleta K_n, the spectrum $X \wedge K$ is the sequential colimit, along h-cofibrations sequential spectra, of the sequence with terms $X \wedge K_n$. These h-cofibrations are in particular levelwise closed embeddings, see for example [43, Proposition A.31]. So homotopy groups commute with such sequential colimits (Proposition 1.26 (i)), so also $X \wedge K$ has vanishing homotopy groups.

(vii) We start with a special case and let X be a sequential spectrum whose homotopy groups vanish. We show first that then the homotopy groups of the spectrum $\mathrm{map}_*(K, X)$ vanish, too. We argue by induction over the number of cells in a CW-structure of K. The induction starts when K consists only of the basepoint, in which case $\mathrm{map}_*(K, X)$ is a trivial spectrum and there is nothing to show. For the inductive step we assume that the homotopy groups of $\mathrm{map}_*(K, X)$ vanish and L is obtained from K by attaching an n-cell. Then the restriction map $\mathrm{map}_*(L, X) \longrightarrow \mathrm{map}_*(K, X)$ is levelwise a Serre fibration whose fiber is isomorphic to

$$\mathrm{map}_*(L/K, X) \;\cong\; \mathrm{map}_*(S^n, X) \;\cong\; \Omega^n X \ .$$

The homotopy groups of this spectrum are isomorphic to the shifted homotopy groups of X, and these vanish by assumption. The long exact sequence of Corollary 1.23 and the inductive hypothesis then show that the homotopy groups of $\mathrm{map}_*(K, X)$ vanish.

The functor $\mathrm{map}_*(K, -)$ commutes with homotopy fibers; so two applications of the long exact homotopy group sequence of a homotopy fiber (Proposition 1.22) reduce the general case to the special case. □

2. Ω-spectra

Definition 2.1. A sequential spectrum is an *Ω-spectrum* if for all $n \geq 0$ the map $\tilde{\sigma}_n \colon X_n \longrightarrow \Omega X_{1+n}$ which is adjoint to the structure map $\sigma_n \colon S^1 \wedge X_n \longrightarrow X_{1+n}$ is a weak homotopy equivalence.

Example 2.2. Eilenberg–MacLane spectra as discussed in Example 1.8 are Ω-spectra. If X is an Ω-spectrum, then so is $\mathrm{map}_*(K, X)$, provided we also assume that K is cofibrant (for example a CW-complex). Indeed, under this hypothesis, the mapping space functor $\mathrm{map}_*(K, -)$ takes the weak equivalence $\tilde{\sigma}_n \colon X_n \longrightarrow \Omega X_{n+1}$ to a weak equivalence

$$\mathrm{map}_*(K, X_n) \xrightarrow{\mathrm{map}_*(K, \tilde{\sigma}_n)} \mathrm{map}_*(K, \Omega X_{n+1}) \cong \Omega(\mathrm{map}_*(K, X)_{n+1}) \;=\; (\Omega\, \mathrm{map}_*(K, X))_{1+n} \ .$$

We will see below that every sequential spectrum admits a natural stable equivalence to a functorially assigned Ω-spectrum.

Proposition 2.3. (i) *Let X be an Ω-spectrum. Then for all integers k and natural numbers n with $k + n \geq 0$, the canonical map*

$$\pi_{k+n}(X_n) \;\longrightarrow\; \pi_k(X)$$

is bijective.

(ii) *Every stable equivalence between Ω-spectra is a level equivalence.*

Proof. (i) The homotopy groups of ΩX_{n+1} are isomorphic to the homotopy groups of X_{n+1}, shifted by one dimension. So the colimit system that defines $\pi_k(X)$ is isomorphic to the colimit system

$$\pi_{n+k}(X_n) \;\longrightarrow\; \pi_{n+k}(\Omega X_{1+n}) \;\longrightarrow\; \pi_{n+k}(\Omega^2 X_{2+n}) \;\longrightarrow\; \cdots \ , \tag{2.4}$$

where the maps in the system are induced by the maps $\tilde{\sigma}_n$ adjoint to the structure maps. In an Ω-spectrum, the maps $\tilde{\sigma}_n$ are weak equivalences, so all maps in the sequence (2.4) are bijective, hence so is the map from each term to the colimit $\pi_{k-n}(X)$.

(ii) We let $f\colon X \longrightarrow Y$ be a stable equivalence between Ω-spectra. Then for all $l, n \geq 0$, the lower horizontal map in the commutative diagram

$$\begin{tikzcd} \pi_l(X_n) \arrow[r, "\pi_l(f_n)"] \arrow[d] & \pi_l(Y_n) \arrow[d] \\ \pi_{l-n}(X) \arrow[r, "\pi_{l-n}(f)"'] & \pi_{l-n}(Y) \end{tikzcd}$$

is bijective. The two vertical maps are bijective by part (i). So the map $f_n\colon X_n \longrightarrow Y_n$ induces a bijection of path components and isomorphisms of all homotopy groups *based at the distinguished basepoint* of X_n. A popular lapse is to hasten to conclude from this that f_n is a weak equivalence. However, X_n need not be path connected, and we do not yet have any hold on the homotopy groups in the path components other than the distinguished one. So an additional argument is necessary at this point.

Since $f_{1+n}\colon X_{1+n} \longrightarrow Y_{1+n}$ induces isomorphisms of homotopy groups at the distinguished basepoints, it restricts to a weak equivalence $f^\circ_{1+n}\colon X^\circ_{1+n} \longrightarrow Y^\circ_{1+n}$ between the path components of the distinguished base points in X_{1+n} and Y_{1+n}. Since looping shifts homotopy groups, also the map $\Omega(f^\circ_{1+n})\colon \Omega(X^\circ_{1+n}) \longrightarrow \Omega(Y^\circ_{1+n})$ is a weak equivalence. The loop space construction only sees the path component of the base point, i.e., the inclusion $X^\circ_{1+n} \longrightarrow X_{1+n}$ induces a homeomorphism $\Omega(X^\circ_{1+n}) \cong \Omega(X_{1+n})$. Hence also the map $\Omega(f_{1+n})\colon \Omega(X_{1+n}) \longrightarrow \Omega(Y_{1+n})$ is a weak equivalence. The vertical maps in the following commutative diagram are weak equivalence by hypothesis:

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "\tilde\sigma_n"', "\sim"] & Y_n \arrow[d, "\sim"', "\tilde\sigma_n"] \\ \Omega(X_{1+n}) \arrow[r, "\sim", "\Omega(f_{1+n})"'] & \Omega(Y_{1+n}) \end{tikzcd}$$

Since the other three maps in the diagram are weak equivalences, so is the map $f_n\colon X_n \longrightarrow Y_n$. Since this holds for all $n \geq 0$, the morphism f is a level equivalence. $\square$

Example 2.5 (Mapping telescope)**.** We recall the *reduced mapping telescope* of a sequence of based continuous maps

$$A^0 \xrightarrow{f^0} A^1 \xrightarrow{f^1} A^2 \xrightarrow{f^2} \cdots , \tag{2.6}$$

a homotopy invariant version of the sequential colimit. Its key property is that the homotopy groups of the telescope are the colimits of the homotopy groups.

The *mapping telescope* $\mathrm{tel}_i\, A^i$ of the sequence (2.6) is defined as the coequalizer of two continuous maps

$$\bigvee_{i\geq 0} A^i \quad \rightrightarrows \quad \bigvee_{i\geq 0} [i, i+1]_+ \wedge A^i$$

Here $[i, i+1]$ denotes a copy of the unit interval, to which we then add a disjoint basepoint. So $[i, i+1]_+ \wedge A^i$ is a reduced version of the product of an interval with A^i, i.e., the copy of $[i, i+1] \times \{*\}$ is collapsed. One of the morphisms takes A^i to $\{i+1\}_+ \wedge A^i$ by the identity, the other one takes A^i to $\{i+1\}_+ \wedge A^{i+1}$ by the morphism f^i.

We write

$$\kappa_i \ : \ A^i \ \longrightarrow \ \mathrm{tel}_i\, A^i$$

for the continuous map that first embeds A^i as $\{i\}_+ \wedge A^i$ into the $\bigvee_{i\geq 0} [i, i+1]_+ \wedge A^i$, and then maps to the coequalizer $\mathrm{tel}_i\, A^i$. The composite

$$A^i \ \xrightarrow{f^i} \ A^{i+1} \ \xrightarrow{\kappa_{i+1}} \ \mathrm{tel}_i\, A^i$$

is based homotopic to κ_i, a homotopy being the inclusion of the 'segment' $[i,i+1]_+ \wedge A^i$ into the telescope. So the following diagram of homotopy groups commutes for all $k \geq 0$:

$$\begin{tikzcd} \pi_k(A^i) \arrow[r, "\pi_k(f^i)"] \arrow[dr, "\kappa_i"'] & \pi_k(A^{i+1}) \arrow[d, "\kappa_{i+1}"] \\ & \pi_k(\mathrm{tel}_i\, A^i) \end{tikzcd}$$

The maps κ_i thus assemble into a map

$$\kappa_\infty \ : \ \mathrm{colim}_i\ \pi_k(A^i) \ \longrightarrow \ \pi_k(\mathrm{tel}_i\, A^i)\ .$$

Like other colimit-type constructions, the mapping telescope construction readily extends to the category of sequential spectra by doing everything levelwise. So the *mapping telescope* of a sequence of morphisms of sequential spectra

$$X^0 \ \xrightarrow{f^0}\ X^1 \ \xrightarrow{f^1}\ X^2 \ \xrightarrow{f^2}\ \cdots \tag{2.7}$$

has levels

$$(\mathrm{tel}_i\, X^i)_n \ = \ \mathrm{tel}_i(X^i_n)\ ,$$

the mapping telescope of the sequence of based continuous maps $f^i_n \colon X^i_n \longrightarrow X^{i+1}_n$. Much like for colimits and smash products with based spaces, the structure maps go along for the ride. For varying $n \geq 0$, the maps $\kappa_i \colon X^i_n \longrightarrow \mathrm{tel}_i\, X^i_n$ then form a morphism of sequential spectra $\kappa_i \colon X^i \longrightarrow \mathrm{tel}_i\, X^i$, and for every integer $k \in \mathbb{Z}$, the induced homomorphisms of homotopy groups $\pi_k(\kappa_i) \colon \pi_k(X^i) \longrightarrow \pi_k(\mathrm{tel}_i\, X^i)$ are compatible as i increases, and assemble into a homomorphism of abelian groups

$$\kappa_\infty \ : \ \mathrm{colim}_i\ \pi_k(X^i) \ \longrightarrow \ \pi_k(\mathrm{tel}_i\, X^i)\ .$$

Construction 2.8 (Assembly maps for mapping telescopes)**.**

$$\mathrm{tel}_i\, \Omega(A^i) \ \longrightarrow \ \Omega(\mathrm{tel}_i\, A^i) \tag{2.9}$$

The assembly map is a weak homotopy equivalence, and it makes the following diagram commute:

$$\begin{tikzcd} \Omega(A^0) \arrow[d, "0\wedge -"'] \arrow[dr, bend left, "\Omega(0\wedge -)"] & \\ \mathrm{tel}_i\, \Omega(A^i) \arrow[r, "\sim"'] & \Omega(\mathrm{tel}_i\, A^i) \end{tikzcd} \tag{2.10}$$

Proposition 2.11.

(i) *For every sequence* (2.6) *of based continuous maps and every* $k \geq 0$, *the map* $\kappa_\infty \colon \mathrm{colim}_i\, \pi_k(A^i) \longrightarrow \pi_k(\mathrm{tel}_i\, A^i)$ *is bijective.*

(ii) *The assembly map* $\mathrm{tel}_i(\Omega(A^i)) \longrightarrow \Omega(\mathrm{tel}_i\, A^i)$ *is a weak equivalence.*

(iii) *For every sequence* (2.7) *of morphisms of sequential spectra, and every* $k \in \mathbb{Z}$, *the homomorphism* $\kappa_\infty \colon \mathrm{colim}_i\, \pi_k(X^i) \longrightarrow \pi_k(\mathrm{tel}_i\, X^i)$ *is an isomorphism of abelian groups.*

Proof. (i) We use the 'partial telescopes' $\mathrm{tel}_{[0,n]}\, A^i$, the coequalizer of two continuous maps

$$\bigvee_{i=0}^{n-1} A^i \ \ \rightrightarrows \ \ \bigvee_{i=0}^{n} [i,i+1]^+ \wedge A^i$$

defined as before. The space $\mathrm{tel}_{[0,n]}\, A^i$ includes into the next space $\mathrm{tel}_{[0,n+1]}\, A^i$ by a closed embedding, with (categorical) colimit the mapping telescope. We let $c_n : \mathrm{tel}_{[0,n]}\, A^i \longrightarrow A^n$ denote the map which projects each wedge summand $[i, i+1]^+ \wedge A^i$ onto A^i and then applies the morphism $f^{n-1} \cdots f^i : A^i \longrightarrow A^n$; this maps c_n is a homotopy equivalence. The commutative diagram

$$\begin{tikzcd} \mathrm{tel}_{[0,0]} A^i \arrow[r] \arrow[d, "c_0"', "\simeq"] & \mathrm{tel}_{[0,1]} A^i \arrow[r] \arrow[d, "c_1"', "\simeq"] & \mathrm{tel}_{[0,2]} A^i \arrow[r] \arrow[d, "\simeq"', "c_2"] & \cdots \\ A^0 \arrow[r, "f^0"'] & A^1 \arrow[r, "f^1"'] & A^2 \arrow[r, "f^2"'] & \cdots \end{tikzcd}$$

induces a morphism of horizontal sequential colimits of homotopy groups. For every $n \geq 0$, the composite morphism

$$\mathrm{tel}_{[0,n]} A^i \xrightarrow{c_n} A^n \xrightarrow{\kappa_n} \mathrm{tel}_i A^i$$

is homotopic to the inclusion. So the composite

$$\mathrm{colim}_n \, \pi_k(\mathrm{tel}_{[0,n]} A^i) \xrightarrow{\cong} \mathrm{colim}_n \, \pi_k(A^n) \xrightarrow{\kappa_\infty} \pi_k(\mathrm{tel}_i A^i)$$

agrees with the canonical morphism coming from the fact that $\mathrm{tel}_i A^i$ is the sequential colimit of the partial telescopes. Each partial telescope $\mathrm{tel}_{[0,n]} A^i$ maps to the subsequent one $\mathrm{tel}_{[0,n+1]} A^i$ by a levelwise closed embedding, so the latter composite homomorphism is bijective, by the unstable precursor of Proposition 1.26 (i). Hence also κ_∞ is bijective.

(ii) Since sequential colimits commute with each other, the colimit $\mathrm{colim}_i \, \pi_k(X^i)$ can also be calculated by first fixing an $n \geq 0$ with $n + k \geq 0$, then forming the colimit $\mathrm{colim}_i \, \pi_{n+k}(X^i_n)$ of unstable homotopy groups, and then passing to colimit in n over the stabilization maps. Part (i) identifies $\mathrm{colim}_i \, \pi_{n+k}(X^i_n)$ with $\pi_{n+k}((\mathrm{tel}_i X^i)_n)$, and the colimit over n of these groups defined $\pi_k(\mathrm{tel}_i X^i)$. Altogether, this shows the claim. □

Given a sequence (2.6) of based continuous maps, we obtain a 'shifted sequence' by omitting the starting term and reindexing the other terms. Moreover, the given maps form a morphism from the original sequence to the shifted sequence:

(2.12)
$$\begin{tikzcd} A^0 \arrow[r, "f^0"] \arrow[d, "f^0"'] & A^1 \arrow[r, "f^1"] \arrow[d, "f^1"'] & A^2 \arrow[r, "f^2"] \arrow[d, "f^2"'] & \cdots \\ A^1 \arrow[r, "f^1"'] & A^2 \arrow[r, "f^2"'] & A^3 \arrow[r, "f^3"'] & \cdots \end{tikzcd}$$

Proposition 2.13. *For every sequence* (2.6) *of based continuous maps, the map*

$$\mathrm{tel}_i f^i \ : \ \mathrm{tel}_i A^i \longrightarrow \mathrm{tel}_i A^{i+1}$$

induced by the diagram (2.12) *on horizontal mapping telescopes is a based homotopy equivalence.*

Proof. The mapping telescope of the shifted sequence in the lower row of (2.12) embeds into the mapping telescope of the original sequence (2.6) by gluing the maps

$$[i, i+1]_+ \wedge A^{i+1} \longrightarrow [i+1, i+2]_+ \wedge A^{i+1} \ , \quad x \wedge a \mapsto (x+1) \wedge a$$

for $i \geq 0$. This image of this embedding is the complement of the initial half-open segment $[0,1)_+ \wedge A^0$. The two composite, in either order, of this embedding $\mathrm{tel}_i A^{i+1} \longrightarrow \mathrm{tel}_i A^i$ and the map $\mathrm{tel}_i f^i$ moves the respective mapping telescope 'one unit to the right', while applying the appropriate instance of f^i. Each of these composites are based homotopic to the respective identity maps, proving the claim. □

Construction 2.14 (The sequential spectrum RX)**.** We define a functor $R \colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{S}p^{\mathbb{N}}$ and a stable equivalence $\tilde{\sigma} \colon X \longrightarrow RX$. For a sequential spectrum X, we set

$$(RX)_n \ = \ \Omega(X_{1+n}) \ .$$

We define the structure map

$$\sigma_n^{RX} \ : \ S^1 \wedge (RX)_n \longrightarrow (RX)_{1+n}$$

as the composite

$$S^1 \wedge \Omega(X_{1+n}) \xrightarrow[s\wedge f \mapsto f(s)]{\text{ev}} X_{1+n} \xrightarrow{\tilde\sigma^X_{1+n}} \Omega(X_{2+n}) \ .$$

Explicitly,

$$\sigma^{RX}_n(s \wedge f)(t) \ = \ \sigma^X_{1+n}(t \wedge f(s)) \ ,$$

for $s,t \in S^1$, and $f\colon S^1 \longrightarrow X_{1+n}$. The evaluation map $\text{ev}\colon S^1 \wedge \Omega(X_{1+n}) \longrightarrow X_{1+n}$ is the counit of the adjunction $(S^1 \wedge -, \Omega)$, so $\sigma^X_n = \text{ev} \circ (S^1 \wedge \tilde\sigma^X_n)$. Hence the two triangles in the following diagram commute:

$$\begin{tikzcd}
S^1 \wedge X_n \arrow[r, "\sigma^X_n"] \arrow[d, "S^1\wedge\tilde\sigma^X_n"'] & X_{1+n} \arrow[d, "\tilde\sigma^X_{1+n}"] \\
S^1 \wedge (RX)_n \arrow[r, "\sigma^{RX}_n"'] \arrow[ur, "\text{ev}"] & (RX)_{1+n}
\end{tikzcd} \tag{2.15}$$

The commutativity of the composite outer square (2.15) precisely means that the collection of adjoint structure maps $\tilde\sigma^X_n : X_n \longrightarrow \Omega(X_{1+n}) = (RX)_n$ form a morphism of sequential spectra $\tilde\sigma^X : X \longrightarrow RX$.

One should beware of the order in which s and t appear in the formula for $\sigma^{RX}_n(s \wedge f)(t)$. Had we instead defined $\sigma^{RX}_n(s \wedge f)(t)$ as $\sigma^X_{1+n}(s \wedge f(t))$, we would get a sequential spectrum that it not isomorphic to RX, and for which the adjoint structure maps $\tilde\sigma^X_n : X_n \longrightarrow \Omega(X_{1+n}) = (RX)_n$ for *not* form a morphism of sequential spectra.

Proposition 2.16.

(i) *For every sequential spectrum X, the morphism $\tilde\sigma^X : X \longrightarrow RX$ is a stable equivalence.*
(ii) *The functor $R\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{S}p^{\mathbb{N}}$ preserves stable equivalences.*

Proof. (i) The following diagram commutes by naturality of the suspension homomorphism and by the commutativity of (2.15):

$$\begin{tikzcd}
\pi_{n+k}(X_n) \arrow[r, "S^1\wedge-"] \arrow[d, "\pi_{n+k}(\tilde\sigma^X_n)"'] & \pi_{1+n+k}(S^1 \wedge X_n) \arrow[r, "\pi_{1+n+k}(\sigma^X_n)"] \arrow[d, "\pi_{1+n+k}(S^1\wedge\tilde\sigma^X_n)"'] & \pi_{1+n+k}(X_{1+n}) \arrow[d, "\pi_{1+n+k}(\sigma^X_{1+n})"] \\
\pi_{n+k}(\Omega(X_{1+n})) \arrow[r, "S^1\wedge-"'] & \pi_{1+n+k}(S^1 \wedge \Omega(X_{1+n})) \arrow[r, "\pi_{1+n+k}(\sigma^{RX}_n)"'] \arrow[ur, "\pi_{1+n+k}(\text{ev})"] & \pi_{1+n+k}(\Omega(X_{2+n}))
\end{tikzcd}$$

So upon passing to sequential colimits over n in the horizontal direction, the maps

$$\pi_{1+n+k}(\text{ev}) \circ (S^1 \wedge -) \ : \ \pi_{n+k}(\Omega(X_{1+n})) \ \longrightarrow \ \pi_{1+n+k}(X_{1+n}) \ ,$$

which are in fact instances of adjunction isomorphisms, induce an inverse to the map $\pi_k(\tilde\sigma)\colon \pi_k(X) \longrightarrow \pi_k(RX)$. So $\pi_k(\tilde\sigma)$ is bijective for all $k \in \mathbb{Z}$, proving the claim.

(ii) Let $f\colon X \longrightarrow Y$ be a stable equivalence of sequential spectra. Naturality of $\tilde\sigma$ provides the relation $R(f) \circ \tilde\sigma^X = \tilde\sigma^Y \circ f$. Since $\tilde\sigma^X$ and $\tilde\sigma^Y$ are stable equivalences by (i), and f is a stable equivalence by hypothesis, also $R(f)$ is a stable equivalence. □

Construction 2.17 (The Ω-spectrum QX)**.** Now we define the sequential spectrum QX as the mapping telescope of the sequence of stable equivalences of sequential spectra

$$X \xrightarrow{\tilde\sigma^X} RX \xrightarrow{R(\tilde\sigma^X)} \cdots \longrightarrow R^m(X) \xrightarrow{R^m(\tilde\sigma^X)} R^{m+1}(X) \longrightarrow \cdots \tag{2.18}$$

We write $i\colon X \longrightarrow QX$ for the morphism (earlier called κ_0) that embeds X as the 'beginning' of the mapping telescope.

Theorem 2.19. *For every sequential spectrum X, the morphism $i\colon X \longrightarrow QX$ is a stable equivalence, and the sequential spectrum QX is an Ω-spectrum.*

Proof. Each of the morphisms $R^m(\tilde{\sigma}^X)\colon R^m(X) \longrightarrow R^{m+1}(X)$ is a stable equivalence by Proposition 2.16. Hence so is the morphism $i\colon X \longrightarrow \mathrm{tel}_m R^m(X) = QX$, by Proposition 2.11 (iii).

Unraveling of definitions shows that the diagram

$$\begin{tikzcd}
X_n \arrow[r, "\tilde{\sigma}_n^X"] \arrow[d, "\tilde{\sigma}_n^X"'] & (RX)_n \arrow[r, "R(\tilde{\sigma}^X)_n"] \arrow[d, "\tilde{\sigma}_n^{RX}"'] & (R^2(X))_n \arrow[r, "(R^2(\tilde{\sigma}^X))_n"] \arrow[d, "\tilde{\sigma}_n^{R^2(X)}"'] & \dots \\
\Omega(X_{1+n}) \arrow[r, "\Omega((\tilde{\sigma}^X)_{1+n})"'] & \Omega((RX)_{1+n}) \arrow[r, "\Omega((R(\tilde{\sigma}^X))_{1+n})"'] & \Omega((R^2(X))_{1+n}) \arrow[r, "\Omega((R^2(\tilde{\sigma}^X))_{1+n})"'] & \dots
\end{tikzcd}$$

equals the following diagram:

$$\begin{tikzcd}
X_n \arrow[r, "\tilde{\sigma}_n^X"] \arrow[d, "\tilde{\sigma}_n^X"'] & \Omega(X_{1+n}) \arrow[r, "\Omega(\tilde{\sigma}_{1+n}^X)"] \arrow[d, "\Omega(\tilde{\sigma}_{1+n}^X)"'] & \Omega(\Omega(X_{2+n})) \arrow[r, "\Omega(\Omega(\tilde{\sigma}_{2+n}^X))"] \arrow[d, "\Omega(\Omega(\tilde{\sigma}_{2+n}^X))"'] & \dots \\
\Omega(X_{1+n}) \arrow[r, "\Omega(\tilde{\sigma}_{1+n}^X)"'] & \Omega(\Omega(X_{2+n})) \arrow[r, "\Omega(\Omega(\tilde{\sigma}_{2+n}^X))"'] & \Omega(\Omega(\Omega(X_{3+n}))) \arrow[r, "\Omega(\Omega(\Omega(\tilde{\sigma}_{3+n}^X)))"'] & \dots
\end{tikzcd} \tag{2.20}$$

The essential point is that the lower sequence is the shift, in the sense of (2.12), of the upper sequence, and that the vertical maps are the same maps as in the upper sequence. The adjoint structure map $(QX)_n \longrightarrow \Omega((QX)_{1+n})$ of the spectrum QX is the composite

$$(QX)_n = \mathrm{tel}_m(R^m(X))_n \;\longrightarrow\; \mathrm{tel}_m\,\Omega((R^m(X))_{1+n}) \;\xrightarrow{\text{assembly}}\; \Omega(\mathrm{tel}_m(R^m(X))_{1+n}) = \Omega((QX)_{1+n})\ ,$$

where the first map is the effect on horizontal mapping telescopes of the diagram (2.20). The first map is a weak equivalence by Proposition 2.13, and the assembly map is a weak equivalence by Proposition 2.11 (ii). Thus the adjoint structure maps of the spectrum QX are weak equivalence, proving that QX is an Ω-spectrum. □

We can now how that the stable homotopy category is equivalent to the localization of the category of Ω-spectra at the class of level equivalences. This implements the perspective on spectra as *infinite loop spaces*. Indeed, an Ω-spectrum X is precisely the data that witnesses the initial space X_0 as the loop space of X_1, via the adjoint structure map $\tilde{\sigma}_0\colon X_0 \xrightarrow{\sim} \Omega(X_1)$; and X_1 is similarly witnessed as the loop space of X_2, and so on ad infinitum. In this sense, Ω-spectra 'are' infinite loop spaces.

Definition 2.21. The *stable homotopy category* $\mathcal{SH}$ is the localization of the category of sequential spectra at the class of stable equivalences.

We write $\mathcal{S}p^{\mathbb{N}}_{\Omega}$ for the full subcategory of $\mathcal{S}p^{\mathbb{N}}$ spanned by the Ω-spectra.

Corollary 2.22. *The inclusion $\mathcal{S}p^{\mathbb{N}}_{\Omega} \longrightarrow \mathcal{S}p^{\mathbb{N}}$ of the full subcategory of Ω-spectra into sequential spectra descends to an equivalence of localizations*

$$\mathcal{S}p^{\mathbb{N}}_{\Omega}[\text{level eq}^{-1}] \;\longrightarrow\; \mathcal{S}p^{\mathbb{N}}[\text{stable eq}^{-1}] = \mathcal{SH}\ .$$

Proof. We write

$$\gamma_{\mathrm{st}}\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{S}p^{\mathbb{N}}[\text{stable eq}^{-1}] = \mathcal{SH} \qquad \text{and} \qquad \gamma_{\mathrm{lv}}\colon \mathcal{S}p^{\mathbb{N}}_{\Omega} \longrightarrow \mathcal{S}p^{\mathbb{N}}[\text{level eq}^{-1}]$$

for the respective localization functors. Level equivalences are in particular stable equivalences, so the composite

$$\mathcal{S}p^{\mathbb{N}}_{\Omega} \xrightarrow{\text{incl}} \mathcal{S}p^{\mathbb{N}} \xrightarrow{\gamma_{\mathrm{st}}} \mathcal{SH}$$

sends all level equivalences to isomorphisms. It thus factors uniquely through a functor $F\colon \mathcal{S}p^{\mathbb{N}}_{\Omega}[\text{level eq}^{-1}] \longrightarrow \mathcal{SH}$ such that $F \circ \gamma_{\mathrm{lv}}$ equals the restriction of γ_{st} to the subcategory $\mathcal{S}p^{\mathbb{N}}_{\Omega}$.

By Theorem 2.19, the functor $Q\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{S}p^{\mathbb{N}}$ takes values in the full subcategory of Ω-spectra. Because $i\colon X \longrightarrow QX$ is a natural stable equivalence, the functor Q takes stable equivalences to stable equivalences between Ω-spectra. The latter are level equivalences by Proposition 2.3 (ii). So the composite

$$\mathcal{S}p^{\mathbb{N}} \xrightarrow{\ Q\ } \mathcal{S}p^{\mathbb{N}}_{\Omega} \xrightarrow{\ \gamma_{\mathrm{lv}}\ } \mathcal{S}p^{\mathbb{N}}_{\Omega}[\text{level eq}^{-1}]$$

sends all stable equivalences to isomorphisms. It thus factors uniquely through a functor $G\colon \mathcal{SH} \longrightarrow \mathcal{S}p^{\mathbb{N}}_{\Omega}[\text{level eq}^{-1}]$ such that $G \circ \gamma_{\mathrm{st}} = \gamma_{\mathrm{lv}} \circ Q$.

By the universal property of localizations applied to natural transformations, the natural stable equivalence $i\colon X \longrightarrow QX$ descends to a natural isomorphism

$$\mathrm{Id}_{\mathcal{SH}} \ \cong\ F \circ G\ .$$

When X is an Ω-spectrum, then $i\colon X \longrightarrow QX$ is a stable equivalence between Ω-spectra, and hence a level equivalence, again by Proposition 2.3 (ii). So the restriction of i to $\mathcal{S}p^{\mathbb{N}}_{\Omega}$ descends to a natural isomorphism

$$\mathrm{Id}_{\mathcal{S}p^{\mathbb{N}}_{\Omega}[\text{level eq}^{-1}]} \ \cong\ G \circ F\ .$$

This concludes the proof. □

Remark 2.23 (Relation to the ∞-category of spectra)**.** We can now sketch how sequential spectra form a model for the ∞-category of spectra. One possible way of defining the latter is as the ∞-categorical version of infinite loop spaces. More precisely, we let $\mathcal{S}_*$ denote the ∞-category of pointed spaces, defined, for example, in one of the following equivalence ways:

- as the ∞-categorical localization of the category of pointed compactly generated spaces at the class of weak equivalences;
- as the coherent nerve of the topological category of the category of based spaces that admit a CW-structure;
- as the ∞-categorical localization of the category of pointed simplicial sets at the class of weak equivalences;
- as the coherent nerve of the simplicial category of based Kan complexes.

If we use first of these three possible definitions, the loop functor $\Omega\colon \mathbf{T}_* \longrightarrow \mathbf{T}_*$ on based compactly generated spaces descends to a functor $\Omega\colon \mathcal{S}_* \longrightarrow \mathcal{S}_*$ of ∞-categorical localizations, simply because it preserves weak equivalences. The descended functor is equivalent to the 'abstract' loop functor that exists on any pointed ∞-category with finite limits. The *∞-category of spectra* may then be defined as a limit, in the ∞-category $\mathcal{C}at_\infty$ of large ∞-categories, of the tower

$$\cdots \longrightarrow \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*\ . \tag{2.24}$$

Objects of such a limit in $\mathcal{C}at_\infty$ can be thought of as families $(X_n, \tau_n)_{n\geq 0}$ consisting of object X_n of $\mathcal{S}_*$, and equivalences $\tau_n\colon X_n \sim \Omega(X_{1+n})$, i.e., infinite loop spaces.

Here is how this description of spectra can be related to sequential spectra. We write $\gamma^{\infty}_{\mathrm{lv}}\colon N(\mathcal{S}p^{\mathbb{N}}) \longrightarrow \mathcal{S}p^{\mathbb{N}}_{\Omega}[\![\text{level eq}^{-1}]\!]$ for an ∞-categorical localization of the category of Ω-spectra at the class of level equivalences. So $\gamma^{\infty}_{\mathrm{lv}}$ is an initial (in the ∞-categorical sense) example of an ∞-functor from the nerve of the category $\mathcal{S}p^{\mathbb{N}}_{\Omega}$ that takes all level equivalences to equivalences. The evaluation functors $\mathrm{ev}_n\colon \mathcal{S}p^{\mathbb{N}}_{\Omega} \longrightarrow \mathbf{T}_*$ participate in a diagram of categories and functors:

$$\begin{tikzcd}
 & & \mathcal{S}p^{\mathbb{N}}_{\Omega} \arrow[dl, "\mathrm{ev}_3"'] \arrow[d, "\mathrm{ev}_2"'] \arrow[dr, "\mathrm{ev}_1"] \arrow[drr, "\mathrm{ev}_0"] & & \\
\cdots \arrow[r] & \mathbf{T}_* \arrow[r, "\Omega"'] & \mathbf{T}_* \arrow[r, "\Omega"'] & \mathbf{T}_* \arrow[r, "\Omega"'] & \mathbf{T}_*
\end{tikzcd}$$

This diagram does *not* commute; however, the the collection of adjoint structure maps $\tilde{\sigma}_n\colon X_n \longrightarrow \Omega(X_{1+n})$ form a natural weak equivalence from ev_n to the composite $\Omega \circ \mathrm{ev}_{1+n}$.

We write $\gamma^{\infty}_{we}\colon N(\mathbf{T}_*) \longrightarrow \mathcal{S}_*$ for a localization functor that witnesses the ∞-category of based spaces as the localization of $\mathbf{T}_*$ at the class of weak equivalences. As natural weak equivalences, the adjoint structure maps $\tilde{\sigma}_n\colon \mathrm{ev}_n \xrightarrow{\sim} \Omega \circ \mathrm{ev}_{1+n}$ descend to natural equivalences that witness that all the triangles in the following diagram of ∞-categories commute:

$$\begin{tikzcd}
 & & N(\mathcal{S}p^{\mathbb{N}}_{\Omega}) \arrow[dl, "\gamma^{\infty}_{we}\circ\mathrm{ev}_3"'] \arrow[d, "\gamma^{\infty}_{we}\circ\mathrm{ev}_2"'] \arrow[dr, "\gamma^{\infty}_{we}\circ\mathrm{ev}_1"] \arrow[drr, "\gamma^{\infty}_{we}\circ\mathrm{ev}_0"] & & \\
\cdots \arrow[r] & \mathcal{S}_* \arrow[r, "\Omega"'] & \mathcal{S}_* \arrow[r, "\Omega"'] & \mathcal{S}_* \arrow[r, "\Omega"'] & \mathcal{S}_*
\end{tikzcd}$$

The universal property of a limit in $\mathcal{C}at_\infty$ turns this whole collection of data into an essentially unique functor

$$N(\mathcal{S}p^{\mathbb{N}}_{\Omega}) \longrightarrow \lim(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*)\ .$$

This functor sends level equivalence of Ω-spectra to equivalence in the limit ∞-category, so it descends to an essentially unique functor

$$N(\mathcal{S}p^{\mathbb{N}}_{\Omega})[\![\text{level eq}^{-1}]\!] \longrightarrow \lim(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*)\ . \tag{2.25}$$

To show that the functor (2.25) is an equivalence of ∞-categories, we can apply one of many useful model-to-infinity comparison theorems. Here is a sketch of how one can proceed. We must first switch from spaces to simplicial sets; this step is necessary to then apply results of Harpaz [18] that require model categories enjoying the technical condition 'combinatorial', a condition that spaces fail to satisfy. To this end we literally copy the Definition 1.1 of sequential spectra while replacing 'space' by 'simplicial set' and the sphere S^1 by its simplicial avatar $\Delta[1]/(\partial\Delta[1])$. The resulting category $\mathcal{S}p^{\mathbb{N}}_{\mathrm{sset}}$ of *simplicial sequential spectra* was first considered by Bousfield and Friedlander [7, Definition 2.1]. Morphisms in $\mathcal{S}p^{\mathbb{N}}_{\mathrm{sset}}$ are called *strict weak equivalences* or *stable weak equivalences* if they become level equivalences or stable equivalences, respectively, of sequential spectra (of topological space) after taken geometric realization levelwise. Bousfield and Friedlander show in Proposition 2.2 and Theorem 2.3 of [7] that both notions of equivalences participate in proper simplicial model category structures. The adjoint functor pair $(|-|, \mathcal{S}ing)$ of geometric realization and singular complex extends, by levelwise application, to an adjoint functor pair between topological and simplicial sequential spectra, and this adjoint pair descends to equivalences of localizations

$$|-|\ :\ \mathcal{S}p^{\mathbb{N}}_{\mathrm{sset}}[\![\text{stable weak eq}^{-1}]\!] \underset{\sim}{\overset{\sim}{\rightleftarrows}} \mathcal{S}p^{\mathbb{N}}[\![\text{stable eq}^{-1}]\!]\ :\ \mathcal{S}ing$$

Slightly more is true: these adjoint functors form a 'Quillen equivalence' for the stable model structures. A result of Harpaz [18, Theorem 1.3], explained in more detail in [18, Example 1.5], shows that the localization of the category $\mathcal{S}p^{\mathbb{N}}_{\mathrm{sset}}$ of simplicial sequential spectra at the stable equivalences presents the limit, in the ∞-category of ∞-categories, of the tower of localizations

$$\cdots \longrightarrow \mathrm{sset}_*[\![\text{weak eq}^{-1}]\!] \xrightarrow{\Omega} \mathrm{sset}_*[\![\text{weak eq}^{-1}]\!] \xrightarrow{\Omega} \mathrm{sset}_*[\![\text{weak eq}^{-1}]\!] \xrightarrow{\Omega} \mathrm{sset}_*[\![\text{weak eq}^{-1}]\!]\ .$$

This sequence is one possibility to model the tower of ∞-categories (2.24).

To relate this to the full category of all sequential spectra (not necessarily Ω-spectra), we use the natural stable equivalence $i\colon X \longrightarrow QX$ from Theorem 2.19. The arguments to prove Corollary 2.22 are formal and robust enough to apply in much the same way to the ∞-categorical localizations $\mathcal{S}p^{\mathbb{N}}_{\Omega}[\![\text{level eq}^{-1}]\!]$ and $\mathcal{S}p^{\mathbb{N}}[\![\text{stable eq}^{-1}]\!]$ as opposed to the 1-categorical localizations $\mathcal{S}p^{\mathbb{N}}_{\Omega}[\text{level eq}^{-1}]$ and $\mathcal{S}p^{\mathbb{N}}[\text{stable eq}^{-1}] = \mathcal{SH}$. Appropriately adapted, they prove that the inclusion $\mathcal{S}p^{\mathbb{N}}_{\Omega} \longrightarrow \mathcal{S}p^{\mathbb{N}}$ of the full subcategory of Ω-spectra into sequential spectra descends to an equivalence of ∞-categorical localizations

$$\mathcal{S}p^{\mathbb{N}}_{\Omega}[\![\text{level eq}^{-1}]\!] \xrightarrow{\sim} \mathcal{S}p^{\mathbb{N}}[\![\text{stable eq}^{-1}]\!]\ .$$

When combined with the equivalence (2.25), the upshot of all this discussion is that the localization of sequential spectra at the class of stable equivalences 'is' the ∞-category of spectra.

3. Additivity of the stable homotopy category

In this section we show that the stable homotopy category is additive. The main work was essentially already done in Proposition 1.24, where we showed that wedges and finite products of stable equivalences of sequential spectra are stable equivalences, and that the canonical map from a finite wedge to a finite product is a stable equivalence. The rest is pure category theory; more precisely, as we shall explain in this section, additivity of the localization essentially follows by certain general properties of localizations.

Definition 3.1. A *relative category* is a pair $(\mathcal{D},\mathcal{W})$ consisting of a category $\mathcal{C}$ and a class $\mathcal{W}$ of morphisms of $\mathcal{C}$.

Given a relative category $(\mathcal{D},\mathcal{W})$, we will often want to consider the localization of $\mathcal{D}$ by $\mathcal{W}$. We shall write $\gamma\colon \mathcal{D}\longrightarrow\mathcal{D}[\mathcal{W}^{-1}]$ for such a localization. We had previously discusses localizations in some detail, but here were the main points.

- A localization is define by the following universal property: for every $\mathcal{W}$-inverting functor $F\colon\mathcal{D}\longrightarrow\mathcal{E}$ (i.e., one that sends all morphisms in $\mathcal{W}$ to isomorphism), there is a unique functor $G\colon\mathcal{D}[\mathcal{W}^{-1}]\longrightarrow\mathcal{E}$ such that $G\circ\gamma=F$.
- Localizations need not exist in general due to size issue that we shall consistently ignore. If a localization exists, it is unique up to unique isomorphism under $\mathcal{D}$.
- If a localization exists, it is bijective on objects. It it sometimes convenient to assume that $\mathcal{D}[\mathcal{W}^{-1}]$ has the same objects as $\mathcal{D}$, and that the localization functor γ is the identity on objects.
- Every localization $\gamma\colon\mathcal{D}\longrightarrow\mathcal{D}[\mathcal{W}^{-1}]$ has a refined universal property: for every category $\mathcal{E}$, the restriction (precomposition) functor

$$\mathrm{Fun}(\gamma,\mathcal{E})\ :\ \mathrm{Fun}(\mathcal{D}[\mathcal{W}^{-1}],\mathcal{E})\ \longrightarrow\ \mathrm{Fun}(\mathcal{D},\mathcal{E}) \tag{3.2}$$

  is an isomorphism of functor categories onto the full subcategory of $\mathrm{Fun}^{\mathcal{W}}(\mathcal{D},\mathcal{E})$ spanned by the $\mathcal{W}$-inverting functors. The additional information encoded in this refined formulation is that nor only $\mathcal{W}$-inverting functors, but also natural transformations between $\mathcal{W}$-inverting functors, descend uniquely to the localization.
- Some prefer to define localizations by a 2-categorical universal property, instead of the 1-categorical universal property. Namely by requiring that for every category $\mathcal{E}$, the restriction functor (3.2) is an equivalence (as opposed to an isomorphism) from $\mathrm{Fun}(\mathcal{D}[\mathcal{W}^{-1}],\mathcal{E})$ to the full subcategory of $\mathrm{Fun}^{\mathcal{W}}(\mathcal{D},\mathcal{E})$ spanned by the $\mathcal{W}$-inverting functors. If this is taken as the definition, localizations are no longer unique up to isomorphism, and not generally bijective on objects. However, they are unique up to equivalence of categories under $\mathcal{D}$, and the equivalence itself is unique up to natural isomorphism.

Localizations are also compatible with passage to opposites. More precisely, if $\gamma\colon\mathcal{D}\longrightarrow\mathcal{D}[\mathcal{W}^{-1}]$ is a localization of the relative category $(\mathcal{D},\mathcal{W})$, then the functor $\gamma^{\mathrm{op}}\colon\mathcal{D}^{\mathrm{op}}\longrightarrow(\mathcal{D}[\mathcal{W}^{-1}])^{\mathrm{op}}$ is a localization of the relative category $(\mathcal{D}^{\mathrm{op}},\mathcal{W}^{\mathrm{op}})$. Informally:

$$(\mathcal{D}[\mathcal{W}^{-1}])^{\mathrm{op}}\ =\ (\mathcal{D}^{\mathrm{op}})[(\mathcal{W}^{\mathrm{op}})^{-1}]$$

As we make precise in Proposition 3.7, localization also commutes with finite products. Informally:

$$(\mathcal{C}\times\mathcal{D})[\mathcal{V}^{-1}\times\mathcal{W}^{-1}]\ =\ \mathcal{C}[\mathcal{V}^{-1}]\times\mathcal{D}[\mathcal{W}^{-1}]\ .$$

Remark 3.3 (Localizations of ∞-categories)**.** Instead of localizing a relative category $(\mathcal{D},\mathcal{W})$ in the world of 1-categories, one can also localize $\mathcal{D}$ (or rather its nerve) in the world of ∞-categories. The definition and key properties are then analogous, mutatis mutandis. An ∞-categorical localization is defined as a functor $\gamma\colon N(\mathcal{D})\longrightarrow\mathcal{D}[\![\mathcal{W}^{-1}]\!]$ of ∞-categories such that for every ∞-category $\mathcal{E}$, the restriction functor

$$\mathrm{Fun}(\gamma,\mathcal{E})\ :\ \mathrm{Fun}(\mathcal{D}[\mathcal{W}^{-1}],\mathcal{E})\ \longrightarrow\ \mathrm{Fun}(\mathcal{D},\mathcal{E})$$

is an equivalence functor ∞-categories onto the full ∞-subcategory of $\mathrm{Fun}^{\mathcal{W}}(\mathcal{D},\mathcal{E})$ spanned by the $\mathcal{W}$-inverting functors, see for example [11, Definition 7.1.2]. Modulo the same size issues as in the 1-categorical context, localizations of ∞-categories exist, are unique up to equivalence of ∞-categories under $\mathcal{D}$, and the equivalences themselves naturally form a contractible ∞-groupoid (Kan complex), see [11, Proposition 7.1.3]. In this context, localization is again compatible with passage to opposite categories, see [11, Proposition 7.1.7]. Finally, the two notions of localizations are intimately related. If $\gamma\colon N(\mathcal{D}) \longrightarrow \mathcal{D}[\![\mathcal{W}^{-1}]\!]$ is an ∞-categorical localization by $\mathcal{W}$, then the functor of 1-categories

$$h(\gamma) \ : \ \mathcal{D} \cong h(N(\mathcal{D})) \ \longrightarrow \ h(\mathcal{D}[\![\mathcal{W}^{-1}]\!])$$

induced on homotopy categories is a 1-categorical localization. Informally speaking: $\mathcal{D}[\mathcal{W}^{-1}] = h(\mathcal{D}[\![\mathcal{W}^{-1}]\!])$.

Our first result in this section is that initial and terminal objects descend to localizations.

Proposition 3.4. *Let $(\mathcal{D}, \mathcal{W})$ be a relative category, and let $\gamma\colon \mathcal{D} \longrightarrow \mathcal{D}[\mathcal{W}^{-1}]$ be a localizations by $\mathcal{W}$. Let x be an object of $\mathcal{D}$.*

(i) *If x is terminal in $\mathcal{D}$, then $\gamma(x)$ is terminal in $\mathcal{D}[\mathcal{W}^{-1}]$.*
(ii) *If x is initial in $\mathcal{D}$, then $\gamma(\emptyset)$ is initial in $\mathcal{D}[\mathcal{W}^{-1}]$.*
(iii) *If x is a zero object of $\mathcal{D}$, then $\gamma(x)$ is a zero object of $\mathcal{D}[\mathcal{W}^{-1}]$.*

Proof. (i) For a $\mathcal{D}$-object A, we let $t_A\colon A \longrightarrow x$ denote the unique morphism. As A varies these morphisms form a natural transformation $t\colon \mathrm{Id} \longrightarrow c[x]$, where $c[x]\colon \mathcal{D} \longrightarrow \mathcal{D}$ is the constant functor with value x. Composing with the localization functor yields a natural transformation

$$\mathrm{Id}_{\mathcal{D}[\mathcal{W}^{-1}]} \circ \gamma = \gamma \ \xrightarrow{\gamma \diamond t} \ \gamma \diamond c[x] = c[\gamma(x)] \circ \gamma$$

of $\mathcal{W}$-inverting functors $\mathcal{D} \longrightarrow \mathcal{D}[\mathcal{W}^{-1}]$. The universal property of the localization yields a unique natural transformation

$$s \ : \ \mathrm{Id}_{\mathcal{D}[\mathcal{W}^{-1}]} \ \longrightarrow \ c[\gamma(x)]$$

such that

$$s \diamond \gamma \ = \ \gamma \diamond t \ : \ \gamma \longrightarrow \gamma \diamond c[x] \ .$$

Evaluating this equality of transformation at the terminal object x yields

$$s_{\gamma(x)} \ = \ \gamma(t_x) \ = \ \gamma(\mathrm{Id}_x) \ = \ \mathrm{Id}_{\gamma(x)} \ .$$

Now we let $f\colon A \longrightarrow \gamma(x)$ be any morphism in $\mathcal{D}[\mathcal{W}^{-1}]$. Then the following diagram commutes by naturality of the transformation s:

$$\begin{tikzcd} A \arrow[r, "s_A"] \arrow[d, "f"'] & \gamma(x) \arrow[d, equal] \\ \gamma(x) \arrow[r, "s_{\gamma(x)}"'] & \gamma(x) \end{tikzcd}$$

Hence

$$f \ = \ s_{\gamma(x)} \circ f \ = \ s_A \ .$$

So the morphism set $\mathcal{D}[\mathcal{W}^{-1}](A, \gamma(x))$ has only one element, the morphism s_A.

(ii) Since $\gamma\colon \mathcal{D} \longrightarrow \mathcal{D}[\mathcal{W}^{-1}]$ is a localizations by $\mathcal{W}$, the functor $\gamma^{\mathrm{op}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow (\mathcal{D}[\mathcal{W}^{-1}])^{\mathrm{op}}$ is a localizations by $\mathcal{W}^{\mathrm{op}}$. The initial object x of $\mathcal{D}$ is a terminal object of $\mathcal{D}^{\mathrm{op}}$. So $\gamma^{\mathrm{op}}(x)$ is terminal in $(\mathcal{D}[\mathcal{W}^{-1}])^{\mathrm{op}}$ by (i). So $\gamma(x)$ is initial in $\mathcal{D}[\mathcal{W}^{-1}]$.

Part (iii) is the combination of (i) and (ii). $\square$

Construction 3.5 (Homotopical adjunctions descend to localizations)**.** We let $(\mathcal{C}, \mathcal{V})$ and $(\mathcal{D}, \mathcal{W})$ be relative categories, and we let

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] & \mathcal{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

be an adjunction with unit $\eta\colon 1_{\mathcal{C}} \longrightarrow RL$ and counit $\epsilon\colon LR \longrightarrow 1_{\mathcal{D}}$. As we shall explain, the entire adjunction data descends to the localizations *if L and R are homotopical*, i.e., if $L(\mathcal{V}) \subset \mathcal{W}$ and $R(\mathcal{W}) \subset \mathcal{V}$. The entire argument is completely formal, using only universal properties.

We write $\kappa\colon \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{V}^{-1}]$ and $\gamma\colon \mathcal{D} \longrightarrow \mathcal{D}[\mathcal{W}^{-1}]$ for localizations by $\mathcal{V}$ and $\mathcal{W}$, respectively. Indeed, if L and R are homotopical then the functor $\gamma\circ L\colon \mathcal{C} \longrightarrow \mathcal{D}[\mathcal{W}^{-1}]$ inverts $\mathcal{V}$, and the functor $\kappa\circ R\colon \mathcal{D} \longrightarrow \mathcal{C}[\mathcal{V}^{-1}]$ inverts $\mathcal{W}$. So there are unique functors $\bar{L}\colon \mathcal{C}[\mathcal{V}^{-1}] \longrightarrow \mathcal{D}[\mathcal{W}^{-1}]$ and $\bar{R}\colon \mathcal{D}[\mathcal{W}^{-1}] \longrightarrow \mathcal{C}[\mathcal{V}^{-1}]$ such that

$$\bar{L}\circ\kappa \ = \ \gamma\circ L \qquad \text{and} \qquad \bar{R}\circ\gamma \ = \ \kappa\circ R\ .$$

Composing the localization κ and the adjunction unit yields a natural transformation

$$\kappa\diamond\eta \ : \ \kappa \ \longrightarrow \ \kappa\circ R\circ L \ = \ \bar{R}\circ\bar{L}\circ\kappa$$

of functors $\mathcal{C} \longrightarrow \mathcal{C}[\mathcal{V}^{-1}]$. The refined universal property of the localization κ provides a unique natural transformation

$$\bar{\eta} \ : \ 1_{\mathcal{C}[\mathcal{V}^{-1}]} \ \longrightarrow \ \bar{R}\circ\bar{L}$$

of endofunctors of $\mathcal{C}[\mathcal{V}^{-1}]$ such that

$$\bar{\eta}\diamond\kappa \ = \ \kappa\diamond\eta\ .$$

Analogously, the adjunction counit descends uniquely to a natural transformation $\bar{\epsilon}\colon \bar{L}\circ\bar{R} \longrightarrow 1_{\mathcal{D}[\mathcal{W}^{-1}]}$ such that

$$\bar{\epsilon}\diamond\gamma \ = \ \gamma\diamond\epsilon\ .$$

We claim that the data $(\bar{L},\bar{R},\bar{\eta},\bar{\epsilon})$ is an adjunction, i.e., it satisfies the triangle identities. We verify this for one of the triangle identities, the other argument being analogous. The triangle identity for the original adjunction reads

$$(R\diamond\epsilon)\circ(\eta\diamond R) \ = \ 1_R \quad ,$$

an equality of natural endotransformations of the functor $R\colon \mathcal{D} \longrightarrow \mathcal{C}$. Composing with the localization functor κ yields:

$$\begin{aligned}
1_{\bar{R}}\diamond\gamma \ &= \ 1_{\bar{R}\circ\gamma} \ = \ 1_{\kappa\circ R} \ = \ \kappa\diamond 1_R \\
&= \ \kappa\diamond((R\diamond\epsilon)\circ(\eta\diamond R)) \\
&= \ (\kappa\diamond(R\diamond\epsilon))\circ(\kappa\diamond(\eta\diamond R)) \\
&= \ ((\kappa\circ R)\diamond\epsilon)\circ((\kappa\diamond\eta)\diamond R) \\
&= \ ((\bar{R}\circ\gamma)\diamond\epsilon)\circ((\bar{\eta}\diamond\kappa)\diamond R) \\
&= \ (\bar{R}\diamond(\gamma\diamond\epsilon))\circ(\bar{\eta}\diamond(\kappa\circ R)) \\
&= \ (\bar{R}\diamond(\bar{\epsilon}\diamond\gamma))\circ(\bar{\eta}\diamond(\bar{R}\circ\gamma)) \\
&= \ ((\bar{R}\diamond\bar{\epsilon})\diamond\gamma)\circ((\bar{\eta}\diamond\bar{R})\diamond\gamma) \\
&= \ ((\bar{R}\diamond\bar{\epsilon})\circ(\bar{\eta}\diamond\bar{R}))\diamond\gamma
\end{aligned}$$

The uniqueness part of the refined universal property of the localization γ thus shows the equality

$$1_{\bar{R}} \ = \ (\bar{R}\diamond\bar{\epsilon})\circ(\bar{\eta}\diamond\bar{R})$$

of natural endotransformations of the functor $\bar{R}\colon \mathcal{D}[\mathcal{W}^{-1}] \longrightarrow \mathcal{C}[\mathcal{V}^{-1}]$. This is one of the triangle equalities for the localized adjunction; the other one is proved analogously.

Remark 3.6. There is a more general context in which adjunctions descend to localizations, namely when they are *derivable*. In this case the functors L and R need not be fully homotopical, and thus do not literally descend to the localizations. Hence in this more general context, 'descend' takes on a more general meaning.

Now we make precise in which sense localization of categories commutes with finite products. It can informally be stated as a equivalence (or even an isomorphism) of categories

$$(\mathcal{C}\times\mathcal{D})[\mathcal{V}^{-1}\times\mathcal{W}^{-1}] \ \simeq \ \mathcal{C}[\mathcal{V}^{-1}]\times\mathcal{D}[\mathcal{W}^{-1}]\ .$$

We alert the reader that localization of categories does not in general commute with infinite products, see the mathoverflow post by Karol Szumiło:

`https://mathoverflow.net/questions/139020/is-the-localisation-of-a-product-of-categories-the-product-o`

Proposition 3.7. *Let $(\mathcal{C},\mathcal{V})$ and $(\mathcal{D},\mathcal{W})$ be relative categories, and let $\kappa\colon \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{V}^{-1}]$ and $\gamma\colon \mathcal{D} \longrightarrow \mathcal{D}[\mathcal{W}^{-1}]$ be localizations by $\mathcal{V}$ and $\mathcal{W}$, respectively. Then the product functor*

$$\kappa \times \gamma \ : \ \mathcal{C} \times \mathcal{D} \ \longrightarrow \ \mathcal{C}[\mathcal{V}^{-1}] \times \mathcal{D}[\mathcal{W}^{-1}]$$

is a localization by the class $\mathcal{V} \times \mathcal{W}$.

Proof. We show that the functor $\kappa \times \gamma$ has the universal property of a localization by the class $\mathcal{V} \times \mathcal{W}$. We may assume without loss of generality that the classes $\mathcal{V}$ and $\mathcal{W}$ contain all identities, since including these does not change the localization. We write $\mathcal{V} \times \mathcal{D}$ for the class of morphisms of $\mathcal{C} \times \mathcal{D}$ of the form $(v, 1_D)$ for some $v \in \mathcal{V}$ and some object d of $\mathcal{D}$. And similarly for the class $\mathcal{C} \times \mathcal{W}$. Then both $\mathcal{V} \times \mathcal{D}$ and $\mathcal{C} \times \mathcal{W}$ are contained in $\mathcal{V} \times \mathcal{W}$; and every morphisms in $\mathcal{V} \times \mathcal{W}$ is a composite of one morphism from $\mathcal{V} \times \mathcal{D}$ and one from $\mathcal{C} \times \mathcal{W}$. Hence a functor $F\colon \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$ inverts $\mathcal{V} \times \mathcal{W}$ if and only if it inverts both $\mathcal{V} \times \mathcal{D}$ and $\mathcal{C} \times \mathcal{W}$.

Hence the currying isomorphism of functor categories

$$\mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \ \cong \ \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\mathcal{D}, \mathcal{E}))$$

takes the full subcategory $\mathrm{Fun}^{\mathcal{V}\times\mathcal{W}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ to the full subcategory $\mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathrm{Fun}^{\mathcal{W}}(\mathcal{D}, \mathcal{E}))$. In the commutative diagram of categories and functors

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{C}[\mathcal{V}^{-1}] \times \mathcal{D}[\mathcal{W}^{-1}], \mathcal{E}) \arrow[r, "\mathrm{curry}", "\cong"'] \arrow[d, "\mathrm{Fun}(\kappa\times\gamma,\mathcal{E})"'] & \mathrm{Fun}(\mathcal{C}[\mathcal{V}^{-1}], \mathrm{Fun}(\mathcal{D}[\mathcal{W}^{-1}], \mathcal{E})) \arrow[d, "\mathrm{Fun}(\kappa,\mathrm{Fun}(\gamma,\mathcal{E}))"] \\
\mathrm{Fun}^{\mathcal{V}\times\mathcal{W}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \arrow[r, "\cong", "\mathrm{curry}"'] & \mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathrm{Fun}^{\mathcal{W}}(\mathcal{D}, \mathcal{E}))
\end{tikzcd}$$

the right vertical functor is an isomorphism of categories by the universal properties of κ and γ. Hence also the left vertical functor is an isomorphism of categories. This shows that $\kappa \times \gamma$ indeed has the required universal property of a localization by $\mathcal{V} \times \mathcal{W}$. □

Now we shall prove that finite products descend to localizations, provided the class of morphisms to be inverted is closed under finite products. And dually for coproducts. For this we recall that a category $\mathcal{D}$ has binary products if and only if the diagonal functor

$$\mathrm{diag} \ : \ \mathcal{D} \ \longrightarrow \ \mathcal{D} \times \mathcal{D}$$

that sends an object A to $\mathrm{diag}(A) = (A, A)$ and a morphism f to $\mathrm{diag}(f) = (f, f)$ has a right adjoint. Indeed, if $\mathcal{D}$ has binary products, we can choose for all pairs of $\mathcal{D}$-objects X and Y a product $P(X, Y)$, as well as morphisms $p_{X,Y}\colon P(X, Y) \longrightarrow X$ and $q_{X,Y}\colon P(X, Y) \longrightarrow Y$ that witness the universal property, i.e., such that for all $\mathcal{D}$-objects A, the map

$$\mathcal{D}(A, P(X, Y)) \ \longrightarrow \ \mathcal{D}(A, X) \times \mathcal{D}(A, Y) \ , \quad f \longmapsto (p_{X,Y} \circ f, q_{X,Y} \circ f)$$

is bijective. These data extend to a functor

$$P \ : \ \mathcal{D} \times \mathcal{D} \ \longrightarrow \ \mathcal{D}$$

by letting

$$P(\alpha\colon X \longrightarrow X', \beta\colon Y \longrightarrow Y') \ : \ P(X, Y) \ \longrightarrow P(X', Y')$$

be the unique morphism such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[d, "\alpha"'] & P(X,Y) \arrow[l, "p_{X,Y}"'] \arrow[r, "q_{X,Y}"] \arrow[d, "P(\alpha,\beta)"'] & Y \arrow[d, "\beta"] \\
X' & P(X',Y') \arrow[l, "p_{X',Y'}"] \arrow[r, "q_{X',Y'}"'] & Y'
\end{tikzcd}$$

By design, the product projections form natural transformation

$$p \ : \ P \ \longrightarrow \ \pi_1 \qquad \text{and} \qquad q \ : \ P \ \longrightarrow \ \pi_2$$

of functors $\mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}$, where π_1 and π_2 are the two projections. Equivalently, the two projections together form a natural transformation

$$(p, q) \; : \; \mathrm{diag} \circ P \; \longrightarrow \; \mathrm{Id}_{\mathcal{D}\times\mathcal{D}}$$

of endofunctors of $\mathcal{D} \times \mathcal{D}$. And the universal property of products is equivalent to the requirement that (p, q) is the counit of an adjunction for (diag, P).

Proposition 3.8. *Let $(\mathcal{D}, \mathcal{W})$ be a relative category, and let $\gamma\colon \mathcal{D} \longrightarrow \mathcal{D}[\mathcal{W}^{-1}]$ be a localization by $\mathcal{W}$.*

(i) *Suppose that $\mathcal{D}$ has finite products, and that every product of two morphisms in $\mathcal{W}$ belongs to $\mathcal{W}$. Then the category $\mathcal{D}[\mathcal{W}^{-1}]$ has finite products and the localization functor γ preserves finite products.*

(ii) *Suppose that $\mathcal{D}$ has finite coproducts, and that every coproduct of two morphisms in $\mathcal{W}$ belongs to $\mathcal{W}$. Then the category $\mathcal{D}[\mathcal{W}^{-1}]$ has finite coproducts and the localization functor γ preserves finite coproducts.*

Proof. (i) Since the localization functor preserves the terminal object by Proposition 3.4, and since every object of $\mathcal{D}[\mathcal{W}^{-1}]$ is in the image of γ, it remains to show that γ preserves binary products.

By Proposition 3.7, the functor $\gamma \times \gamma\colon \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}[\mathcal{W}^{-1}] \times \mathcal{D}[\mathcal{W}^{-1}]$ is a localization by $\mathcal{W} \times \mathcal{W}$. The following diagram of categories and functors commutes:

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\mathrm{diag}"] \arrow[d, "\gamma"'] & \mathcal{D} \times \mathcal{D} \arrow[d, "\gamma\times\gamma"] \\
\mathcal{D}[\mathcal{W}^{-1}] \arrow[r, "\mathrm{diag}"'] & \mathcal{D}[\mathcal{W}^{-1}] \times \mathcal{D}[\mathcal{W}^{-1}]
\end{tikzcd}$$

In this sense, the diagonal functor of $\mathcal{D}$ descends to the diagonal functor of $\mathcal{D}[\mathcal{W}^{-1}]$.

Since $\mathcal{D}$ has finite products, the diagonal functor $\mathrm{diag}\colon \mathcal{D} \longrightarrow \mathcal{D}\times\mathcal{D}$ has a right adjoint $P\colon \mathcal{D}\times\mathcal{D} \longrightarrow \mathcal{D}$. The diagonal functor sends $\mathcal{W}$ to $\mathcal{W} \times \mathcal{W}$. And by the hypothesis, the product functor P sends $\mathcal{W} \times \mathcal{W}$ to $\mathcal{W}$. As we explained in Construction 3.5, the adjunction descends to the localizations. In particular, the descended left adjoint $\mathrm{diag}\colon \mathcal{D}[\mathcal{W}^{-1}] \longrightarrow \mathcal{D}[\mathcal{W}^{-1}] \times \mathcal{D}[\mathcal{W}^{-1}]$ again has a right adjoint. But this precisely means that the category $\mathcal{D}[\mathcal{W}^{-1}]$ has binary products.

(ii) We can either dualize the argument of part (i), or simply apply part (i) to the opposite category $\mathcal{D}^{\mathrm{op}}$, exploiting that $\gamma^{\mathrm{op}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow (\mathcal{D}[\mathcal{W}^{-1}])^{\mathrm{op}}$ is a localization of the relative category by $(\mathcal{D}^{\mathrm{op}}, \mathcal{W}^{\mathrm{op}})$. □

Remark 3.9. The various properties of 1-categorical localizations that we established above have direct analogues for ∞-categorical localizations. Moreover, the proof by universal properties that we describe can be adapted to also prove the ∞-categorical analogs. For example, the image in the ∞-categorical localization $\mathcal{D}[\![\mathcal{W}^{-1}]\!]$ of a terminal object in $\mathcal{D}$ is terminal, in the ∞-categorical sense, and similarly for initial objects and zero objects [ref?]. And ∞-categorical localizations are compatible with products. Indeed, if $\kappa\colon \mathcal{C} \longrightarrow \mathcal{C}[\![\mathcal{V}^{-1}]\!]$ and $\gamma\colon \mathcal{D} \longrightarrow \mathcal{D}[\![\mathcal{W}^{-1}]\!]$ are localizations of ∞-categories by classes of morphisms $\mathcal{V}$ and $\mathcal{W}$, then $\kappa \times \gamma\colon \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{C}[\![\mathcal{V}^{-1}]\!] \times \mathcal{D}[\![\mathcal{W}^{-1}]\!]$ is a localization by $\mathcal{V} \times \mathcal{W}$, compare [11, Proposition 7.1.13].

There is also a good theory of adjunctions of ∞-categories, enjoying the following properties:

- any adjunction of ∞-categories in which both functors are homotopical for given classes of weak equivalence descends canonically to an adjunction between the localizations, see [11, Proposition 7.1.14].
- Given an ∞-category $\mathcal{C}$ with finite products and a class of morphisms $\mathcal{W}$ that is stable under finite products, then the localization $\mathcal{C}[\![\mathcal{W}^{-1}]\!]$ admits finite products, and the localization functor preserves finite products, see [11, Corollary 7.1.16]. The analogous property holds for finite coproducts, by passage to opposite ∞-categories.

Now we have all necessary tools to show that the stable homotopy category is additive, i.e., there is a natural abelian group structure on the homomorphism sets such that composition is biadditive. We first review the latter concept in some detail. It may seem as if 'additive category' is extra structure on a category (namely the addition on morphism sets), but in fact, 'additive category' is really a property of a category (namely having finite sums which are isomorphic to products). So we present the construction of the addition on hom-sets in this generality.

Definition 3.10. A category $\mathcal{C}$ is *preadditive* if it has a zero object, finite coproducts, and for every pair of objects X and Y the morphisms $p_1 = \mathrm{Id} + 0 : X \amalg Y \longrightarrow X$ and $p_2 = 0 + \mathrm{Id} : X \amalg Y \longrightarrow Y$ make $X \amalg Y$ into a product of X and Y, where '0' is the unique morphism which factors through a zero object.

In other words, we demand that for every object A the map

$$\mathcal{C}(A, X \amalg Y) \longrightarrow \mathcal{C}(A, X) \times \mathcal{C}(A, Y) , \quad f \mapsto (p_1 f, p_2 f)$$

is bijective.

Example 3.11. The category of abelian groups is preadditive, and so is the category of left modules over any ring. The category of abelian monoids is preadditive.

Construction 3.12. Let $\mathcal{C}$ be a preadditive category. We can define a binary operation on the morphism set $\mathcal{C}(A, X)$ for every pair of objects A and X. Given morphisms $a, b : A \longrightarrow X$ we let $a \perp b : A \longrightarrow X \amalg X$ be the unique morphism such that $p_1(a \perp b) = a$ and $p_2(a \perp b) = b$. Then we define $a + b : A \longrightarrow X$ as $\nabla(a \perp b)$ where $\nabla = \mathrm{Id} + \mathrm{Id} : X \amalg X \longrightarrow X$ is the fold morphism.

Proposition 3.13. *Let $\mathcal{C}$ be a preadditive category.*

(i) *For every pair of objects A and X of $\mathcal{C}$ the binary operation $+$ makes the set $\mathcal{C}(A, X)$ of morphisms into an abelian monoid with the zero morphism as neutral element. Moreover, the monoid structure is natural for all morphisms in both variables, or, equivalently, composition is biadditive.*

(ii) *If, moreover, the shearing morphism $p_1 \perp \nabla : X \amalg X \longrightarrow X \amalg X$ is an isomorphism, then the abelian monoid $\mathcal{C}(A, X)$ has additive inverse, i.e., is an abelian group, for every object A.*

(iii) *Let $F, G : \mathcal{C} \longrightarrow \mathcal{A}b\mathcal{M}on$ be two functors that preserve zero objects. Suppose moreover that the functor G preserves finite coproducts. Then every natural transformation between the underlying set-valued functors of F and G is automatically additive.*

(iv) *Let $G : \mathcal{C} \longrightarrow \mathcal{A}b\mathcal{M}on$ be a functor that preserves zero objects and finite coproducts. Then for all objects A and X of $\mathcal{C}$ and every element $a \in G(A)$ the evaluation map*

$$\mathcal{C}(A, X) \longrightarrow G(X) , \quad f \longmapsto G(f)(a)$$

is a monoid homomorphism.

Proof. (i) The proof is lengthy, but completely formal. For the associativity of '+' we consider three morphisms $a, b, c : A \longrightarrow X$. Then $a + (b + c)$ respectively $(a + b) + c$ are the two upper and lower composites around the commutative diagram:

$$\begin{tikzcd}
 & & X \amalg X \arrow[dr, "\nabla"] & \\
A \arrow[urr, bend left, "a\perp(b+c)"] \arrow[r, "a\perp b\perp c"] \arrow[drr, bend right, "(a+b)\perp c"'] & X \amalg X \amalg X \arrow[ur, "\mathrm{Id}\amalg\nabla"] \arrow[dr, "\nabla\amalg\mathrm{Id}"'] & & X \\
 & & X \amalg X \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

The commutativity is a consequence of two elementary facts: first, $b \perp a = (a \perp b)\tau$ where $\tau : X \amalg X \longrightarrow X \amalg X$ is the automorphism which interchanges the two factors; this follows from $p_1 \tau = p_2$ and $p_2 \tau = p_1$. Second, the fold morphism is commutative, i.e., $\nabla\tau = \nabla : X \amalg X \longrightarrow X$. Altogether we get

$$a + b \; = \; \nabla(a \perp b) \; = \; \nabla\tau(a \perp b) \; = \; \nabla(b \perp a) \; = \; b + a \; .$$

As before we denote by $0 \in \mathcal{C}(A, X)$ the unique morphism which factors through a zero object. Then we have $a \perp 0 = i_1 a$ in $\mathcal{C}(A, X \amalg X)$ where $i_1 : X \longrightarrow X \amalg X$ is the embedding as the first factor. Hence $a + 0 = \nabla(a \perp 0) = \nabla i_1 a = a$; by commutativity we also have $0 + a = a$.

Now we verify naturality of the addition on $\mathcal{C}(A,X)$ in A and X. To check $c(a+b) = ca+cb$ for $a,b : A \longrightarrow X$ and $c : X \longrightarrow Y$ we consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "a\perp b"'] \arrow[rr, bend left, "ca\perp cb"] \arrow[d, "a\perp b"'] & X \amalg X \arrow[r, "c\amalg c"'] \arrow[d, "\nabla"] & Y \amalg Y \arrow[d, "\nabla"] \\
X \amalg X \arrow[r, "\nabla"'] & X \arrow[r, "c"'] & Y
\end{tikzcd}$$

in which the composite through the lower left corner is $c(a+b)$. We have

$$\begin{aligned} p_1(c \amalg c)(a\perp b) &= (\mathrm{Id}_Y + 0)(c\amalg c)(a\perp b) \ = \ (c+0)(a\perp b) \\ &= c(\mathrm{Id}_X + 0)(a\perp b) \ = \ cp_1(a\perp b) \ = \ ca \ = \ p_1(ca\perp cb) \end{aligned}$$

and similarly for p_2 instead p_1. So $(c\amalg c)(a\perp b) = ca\perp cb$ since both sides have the same 'projections' to the two summands of $Y\amalg Y$. Since the composite through the upper right corner is $ca+cb$, we have shown $c(a+b) = ca+cb$.

Naturality in A is even easier. For a morphism $d : E \longrightarrow A$ we have $(a\perp b)d = ad\perp bd : E \longrightarrow X\amalg X$ since both sides have the same 'projections' ad and bd, respectively, to the two summands of $X\amalg X$. Thus $(a+b)d = ad+bd$ by the definition of '+'.

(ii) An arbitrary abelian monoid M has additive inverses if and only if the map

$$M^2 \ \longrightarrow \ M^2 \ , \quad (x,y) \longmapsto (x, x+y)$$

is bijective. Indeed, the inverse of $x\in M$ is the second component of the preimage of $(x,0)$.

Every morphism $f : A \longrightarrow X\amalg X$ satisfies $f = (p_1 f)\perp(p_2 f)$, and hence

$$(p_1 f) + (p_2 f) \ = \ \nabla((p_1 f)\perp(p_2 f)) \ = \ \nabla f \ .$$

So for the abelian monoid $\mathcal{C}(A,X)$ the square

$$\begin{tikzcd}
\mathcal{C}(A, X\amalg X) \arrow[r, "(p_1\perp\nabla)\circ -"] \arrow[d, "(p_1\circ -, p_2\circ -)"', "\cong"] & \mathcal{C}(A, X\amalg X) \arrow[d, "\cong"', "(p_1\circ -, p_2\circ -)"] \\
\mathcal{C}(A,X)^2 \arrow[r, "(a,b)\mapsto(a,a+b)"'] & \mathcal{C}(A,X)^2
\end{tikzcd}$$

commutes. Moreover, both vertical maps are bijective. Since $\nabla\perp p_1$ is an isomorphism, the upper map is bijective, hence so is the lower map, and so the monoid $\mathcal{C}(A,X)$ has inverses.

(iii) Let $\tau : F \Longrightarrow G$ be any natural transformation between the underlying set-valued functors, i.e., the maps $\tau_X : F(X) \longrightarrow G(X)$ are not assumed to be additive. We write $i_1, i_2 : X \longrightarrow X\amalg X$ for the two morphisms that exhibit $X\amalg X$ as a coproduct of two instances of X. Because F and G preserve zero objects, they also preserve zero morphisms. We consider two classes x and y in $F(X)$; we claim that

$$\tau_{X\amalg X}(F(i_1)(x) + F(i_2)(y)) \ = \ G(i_1)(\tau_X(x)) + G(i_2)(\tau_X(y)) \tag{3.14}$$

in the abelian monoid $G(X\amalg X)$. To show this we observe that

$$\begin{aligned} G(p_1)(\tau_{X\amalg X}(F(i_1)(x)+F(i_2)(y))) &= \tau_X(F(p_1)(F(i_1)(x)+F(i_2)(y))) \\ &= \tau_X(F(p_1 i_1)(x) + F(p_1 i_2)(y))) \\ &= \tau_X(F(\mathrm{Id}_X)(x) + F(0)(y)) \ = \ \tau_X(x) \\ &= G(\mathrm{Id}_X)(\tau_X(x)) + G(0)(\tau_X(x)) \\ &= G(p_1 i_1)(\tau_X(x)) + G(p_1 i_2)(\tau_X(y)) \\ &= G(p_1)(G(i_1)(\tau_X(x)) + G(i_2)(\tau_X(y))) \end{aligned}$$

in $G(X)$. Similarly,

$$G(p_2)(\tau_{X\amalg X}(F(i_1)(x)+F(i_2)(y))) \;=\; G(p_2)(G(i_1)(\tau_X(x))+G(i_2)(\tau_X(y)))\ .$$

Since the functor G preserves coproducts, and because abelian monoids form a preadditive category, the morphism

$$(G(p_1),G(p_2)) \;:\; G(X\amalg X) \;\longrightarrow\; G(X)\times G(X)$$

is bijective; so this shows the relation (3.14). The fold morphism $\nabla : X\amalg X \longrightarrow X$ satisfies $\nabla\circ i_1 = \nabla\circ i_2 = \mathrm{Id}_X$. So

$$F(\nabla)(F(i_1)(x)+F(i_2)(y)) \;=\; F(\nabla\circ i_1)(x)+F(\nabla\circ i_2)(y) \;=\; x+y\ .$$

So we can finally conclude with the desired relation:

$$\begin{aligned}
\tau_X(x+y) \;&=\; \tau_X(F(\nabla)(F(i_1)(x)+F(i_2)(y)))\\
&=\; G(\nabla)(\tau_{X\amalg X}(F(i_1)(x)+F(i_2)(y)))\\
\overset{(3.14)}{}&=\; G(\nabla)(G(i_1)(\tau_X(x))+G(i_2)(\tau_X(y)))\\
&=\; G(\nabla\circ i_1)(\tau_X(x))+G(\nabla\circ i_2)(\tau_X(y)) \;=\; \tau_X(x)+\tau_X(y)\ .
\end{aligned}$$

Part (iv) is a special case of (iii), with $F=\mathcal{C}(A,-)$ a represented functor. □

Remark 3.15. The hypothesis in part (iii) of Proposition 3.13 that the functor G preserves finite coproducts cannot be dropped. To see this, we consider the endofunctor $\tilde{\mathbb{Z}}[-]\colon \mathcal{A}b \longrightarrow \mathcal{A}b$ on the category of abelian groups that takes an abelian group A the $\tilde{\mathbb{Z}}[A]=\mathbb{Z}[A]/\mathbb{Z}[\{0\}]$, the reduced free abelian group generated by the underlying based set $(A,0)$ of A. The functor $\tilde{\mathbb{Z}}[-]$ preserves zero objects, but not binary coproducts. And the set-valued natural transformation

$$\tau \;:\; \mathrm{Id}_{\mathcal{A}b} \longrightarrow \tilde{\mathbb{Z}}[-]\ ,\quad \tau_A \;:\; A\longrightarrow \tilde{\mathbb{Z}}[A]\ ,\quad \tau_A(a) \;=\; 1\cdot[a]$$

is not additive.

Definition 3.16. A category $\mathcal{C}$ is *additive* if it is preadditive and for every object X of $\mathcal{C}$, the shearing morphism $\nabla\perp p_1 : X\amalg X \longrightarrow X\amalg X$ is an isomorphism.

In an additive category, all the abelian monoids $\mathcal{C}(A,X)$ of morphisms have additive inverses, i.e., are abelian groups, by Proposition 3.13 (ii).

Theorem 3.17. *Let $(\mathcal{D},\mathcal{W})$ be a relative category, and let $\gamma\colon \mathcal{D} \longrightarrow \mathcal{D}[\mathcal{W}^{-1}]$ be a localization by $\mathcal{W}$. Suppose that*

(a) *the category $\mathcal{D}$ has a zero object, finite products and finite coproducts;*
(b) *the coproduct and the product of two morphisms in $\mathcal{W}$ is again in $\mathcal{W}$;*
(c) *for all objects X and Y of $\mathcal{D}$ the canonical morphism $X\vee Y \longrightarrow X\times Y$ is in $\mathcal{W}$.*

Then the category $\mathcal{D}[\mathcal{W}^{-1}]$ is preadditive.

If moreover for every object X of $\mathcal{D}$ the shearing morphism $(\nabla,p_2)\colon X\vee X \longrightarrow X\times X$ is in $\mathcal{W}$, then the category $\mathcal{D}[\mathcal{W}^{-1}]$ is additive.

Proposition 1.24 verifies the essential hypotheses of Theorem 3.17 for the category of sequential spectra relative to the class of stable equivalences. So the fact that the stable homotopy category is additive is essentially a corollary of Proposition 1.24 and Theorem 3.17. We spell this out in more detail now.

Theorem 3.18.

(i) *The localization functor $\gamma\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{SH}$ preserves finite coproducts and finite products.*
(ii) *The stable homotopy category $\mathcal{SH}$ is additive.*
(iii) *A morphism $f\colon X \longrightarrow Y$ of sequential spectra is a stable equivalence if and only if $\gamma(f)$ is an isomorphism in $\mathcal{SH}$.*

Proof. The category $\mathcal{S}p^{\mathbb{N}}$ of sequential spectra has all limits and colimits, and these are calculated levelwise. In particular, $\mathcal{S}p^{\mathbb{N}}$ has finite coproducts and finite products. The class of stable equivalences is closed under coproducts and finite products by Proposition 1.27. So applying Proposition 3.8 to the relative category $(\mathcal{S}p^{\mathbb{N}}, \text{stable equivalences})$ yields part (i), and also shows that the localization $\mathcal{SH}$ has finite coproducts and finite products. The trivial spectrum, consisting of a one-point space in every level, is a zero object in $\mathcal{S}p^{\mathbb{N}}$. Hence its image under the localization functor $\gamma\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{SH}$ is a zero object of the stable homotopy category, by Proposition 3.4 (iii).

For all sequential spectra X and Y, the canonical morphism $k\colon X \vee Y \longrightarrow X \times Y$ is a stable equivalence by Proposition 1.24 (iii). So all hypotheses of Theorem 3.17 hold for the relative category $(\mathcal{S}p^{\mathbb{N}}, \text{stable equivalences})$, and we can conclude that $\mathcal{SH}$ is preadditive. To prove that $\mathcal{SH}$ is additive, it remains to show that for every sequential spectrum X, the shearing morphism $(\nabla, p_2)\colon X \vee X \longrightarrow X \times X$ is a stable equivalence. For every integer k the composite map

$$\pi_k(X) \oplus \pi_k(X) \xrightarrow[\cong]{(i_1)_* + (i_2)_*} \pi_k(X \vee X) \xrightarrow{\pi_k(\nabla, p_2)} \pi_k(X \times X) \xrightarrow[\cong]{((p_1)_*, (p_2)_*)} \pi_k(X) \times \pi_k(X)$$

sends (x, y) to $(x + y, y)$ where the first and last maps are the canonical ones. The composite map is an isomorphism since $\pi_k(X)$ is a group, i.e., has additive inverses. Since the two canonical maps are isomorphisms by Proposition 1.24, so is the middle map. So $(\nabla, p_1)\colon X \vee X \longrightarrow X \times X$ induces isomorphisms of homotopy groups, and is thus a stable equivalence. This completes the proof of (ii).

(iii) The localization functor sends stable equivalences to isomorphisms by design. So it remains to show the converse, i.e., that whenever $\gamma(f)$ is an isomorphism, then f is a stable equivalence. For $k \in \mathbb{Z}$, the functor $\pi_k\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{A}b$ takes stable equivalences to isomorphisms. We write $\bar{\pi}_k\colon \mathcal{SH} \longrightarrow \mathcal{A}b$ for the unique factorization of π_k through the localization functor $\gamma\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{SH}$, i.e., such that $\bar{\pi}_k \circ \gamma = \pi_k$. If $\gamma(f)$ is an isomorphism in $\mathcal{SH}$, then $\pi_k(f) = \bar{\pi}_k(\gamma(f))$ is an isomorphism for all $k \in \mathbb{Z}$, so f is a stable equivalence. $\square$

Remark 3.19. We shall show later that the stable homotopy category has arbitrary set coproducts and products, not only finite ones. And the localization functor $\gamma\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{SH}$ preserves arbitrary coproducts, but *not* arbitrary products.

Remark 3.20 (Additivity of the ∞-category of spectra)**.** The essentially formal arguments involved in the proof of Theorem 3.18 all carry over to the ∞-categorical context, and show that the ∞-category of spectra is additive. For this purpose, it is convenient to define the ∞-category of spectra as the localization of the category of sequential spectra at the class of stable equivalences, i.e., $\mathcal{S}p_\infty = \mathcal{S}p^{\mathbb{N}}[\![\text{stable eq}^{-1}]\!]$; for the relation to other definitions, see Remark 2.23.

An ∞-category $\mathcal{C}$ is called *preadditive* if it admits a zero object, finite product and finite coproducts, and if for all objects x and y of $\mathcal{C}$, the canonical morphism $x \vee y \longrightarrow x \times y$ is an equivalence. A preadditive ∞-category $\mathcal{C}$ is *additive* if for all objects x of $\mathcal{C}$, the shearing morphism $(\nabla, p_2)\colon x \vee x \longrightarrow x \times x$ is an equivalence. If $\mathcal{C}$ is preadditive, then its homotopy category $h(\mathcal{C})$ is additive in the sense of Definition 3.10. Moreover, $\mathcal{C}$ is then additive if and only if $h(\mathcal{C})$ is additive in the sense of Definition 3.16. All of these claim holds because (co-)products in ∞-categories, if they exists, are also (co-)products in the homotopy category [ref].

The category $\mathcal{S}p^{\mathbb{N}}$ of sequential spectra has finite products, and stable equivalences are preserved by finite products, by Proposition 1.27 (ii). So the ∞-categorical localization $\mathcal{S}p_\infty$ has finite products, and the localization functor preserves them, by [11, Corollary 7.1.6]. Dually, has $\mathcal{S}p^{\mathbb{N}}$ finite coproducts, and stable equivalences are preserved by finite coproducts, by Proposition 1.27 (i). So applying [11, Corollary 7.1.6] to the opposite category of $\mathcal{S}p^{\mathbb{N}}$ shows that $\mathcal{S}p_\infty$ has finite coproducts. Since the localization functor $\gamma\colon N(\mathcal{S}p^{\mathbb{N}}) \longrightarrow \mathcal{S}p_\infty$ moreover preserves finite product and coproducts, also by [11, Corollary 7.1.6], the fact that the canonical morphisms $X \vee Y \longrightarrow X \times Y$ and the shearing morphism $(\nabla, p_2)\colon X \vee X \longrightarrow X \times X$ in the 1-category of sequential spectra are stable equivalences become the statements that the canonical and shearing morphisms are equivalence in $\mathcal{S}p_\infty$. So the ∞-category of spectra is additive. Hence also the homotopy category $h(\mathcal{S}p_\infty) = \mathcal{SH}$ is additive, recovering Theorem 3.18 (ii).

4. Cofibration categories

We have introduced the stable homotopy category $\mathcal{SH}$ as the localization of the category of sequential spectra at the class of stable equivalences. We were able to show with this definition that the stable homotopy category is additive.

Another important structure on $\mathcal{SH}$ is that of a *triangulated category*. To establish this structure and to get a better hold on the morphisms groups in $\mathcal{SH}$, we will use additional structure on the model $\mathcal{S}p^{\mathbb{N}}$. We will use *cofibration categories*. This notion was first introduced and studied (in the dual formulation) by Brown [8] under the name 'categories of fibrant objects'. Closely related sets of axioms have been explored by various authors, compare Remark 4.2.

Definition 4.1. A *cofibration category* is a category $\mathcal{C}$ equipped with two classes of morphisms, called *cofibrations* and *weak equivalences*, respectively, that satisfy the following axioms (C1)–(C4).

(C1) All isomorphisms are cofibrations and weak equivalences. Cofibrations are stable under composition. The category $\mathcal{C}$ has an initial object and every morphism from an initial object is a cofibration.

(C2) Given two composable morphisms f and g in $\mathcal{C}$, if two of the three morphisms f, g and gf are weak equivalences, then so is the third.

(C3) Given a cofibration $i\colon A \longrightarrow B$ and any morphism $f\colon A \longrightarrow C$, there exists a pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i"'] & C \arrow[d, tail, "j"] \\ B \arrow[r] & D \end{tikzcd}$$

in $\mathcal{C}$ and the morphism j is a cofibration. If additionally i is a weak equivalence, then so is j.

(C4) Every morphism in $\mathcal{C}$ can be factored as the composite of a cofibration followed by a weak equivalence.

An *acyclic cofibration* is a morphism that is both a cofibration and a weak equivalence.

We will often decorate cofibrations by a tail at the arrow, as in $\rightarrowtail$; will often denote weak equivalences by a tilde over the arrow, as in $\xrightarrow{\sim}$; hence acyclic cofibrations come with a tail and a tilde, as in $\overset{\sim}{\rightarrowtail}$.

We record some elementary consequences of the axioms:

- In a cofibration category a coproduct $B \vee C$ of any two objects in $\mathcal{C}$ exists by (C3) with A an initial object, and the canonical morphisms from B and C to $B \vee C$ are cofibrations.
- If $i\colon A \longrightarrow B$ and $i'\colon A' \longrightarrow B'$ are cofibrations, so is their coproduct $i \amalg i'\colon A \amalg A' \longrightarrow B \amalg B'$. Indeed, applying (C3) to the two pushout squares

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, tail] & B \arrow[d, tail] \\ A \amalg A' \arrow[r, tail, "i \amalg A'"'] & B \amalg A' \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, tail, "i'"] \arrow[d, tail] & B' \arrow[d, tail] \\ B \amalg A' \arrow[r, tail, "B \amalg i'"'] & B \amalg B' \end{tikzcd}$$

shows that $i \amalg A'$ and $B \amalg i'$ are cofibrations, hence so is their composite, by (C1). The same argument shows that whenever i and i' are acyclic cofibrations, so is $i \amalg i'$.

The *homotopy category* of a cofibration category is a localization at the class of weak equivalences, i.e., a functor $\gamma\colon \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ that takes all weak equivalences to isomorphisms and is initial among such functors. The homotopy category always exists if one is willing to pass to a larger universe. To get a locally small homotopy category (i.e., have 'small hom-sets'), additional assumptions are necessary; one possibility is to assume that $\mathcal{C}$ has 'enough fibrant objects', compare Remark 4.14. We recall some basic facts about the homotopy category of a cofibration category in Theorem 4.13.

Remark 4.2. The above notion of cofibration category is due to K. S. Brown [8]. More precisely, Brown introduced 'categories of fibrant objects', and the axioms (C1)–(C4) are equivalent to the duals of the axioms (A)–(E) of Part I.1 in [8]. The concept of a cofibration category is a substantial generalization of Quillen's notion of a 'closed model category' [39]: from a Quillen model category one obtains a cofibration category by restricting to the full subcategory of cofibrant objects and forgetting the class of fibrations.

Cofibration categories are closely related to 'categories with cofibrations and weak equivalences' in the sense of Waldhausen [56]. In fact, a category with cofibrations and weak equivalences that also satisfies the *saturation axiom* [56, 1.2] and the *cylinder axiom* [56, 1.6] is in particular a cofibration category as in Definition 4.1. Further relevant references on closely related axiomatic frameworks are Baues' monograph [3] and Cisinski's article [10]. Radulescu-Banu's extensive paper [40] is the most comprehensive source for basic results on cofibration categories and, among other things, contains a survey of the different kinds of cofibration categories and their relationships.

A property that we will frequently use is the following *gluing lemma*. A proof of the gluing lemma can be found in Lemma 1.4.1 (1) of [40].

Proposition 4.3 (Gluing lemma)**.** *Let $\mathcal{C}$ be a cofibration category. Consider a commutative $\mathcal{C}$-diagram*

$$\begin{tikzcd} B \arrow[d, "\sim"'] & A \arrow[l, tail, "i"'] \arrow[r] \arrow[d, "\sim"'] & C \arrow[d, "\sim"] \\ B' & A' \arrow[l, tail, "i'"] \arrow[r] & C' \end{tikzcd}$$

such that i and i' are cofibrations and all three vertical morphisms are weak equivalences. The the induced morphism between the horizontal pushouts $B \cup_A C \longrightarrow B' \cup_{A'} C'$ is a weak equivalence.

A special case of the gluing lemma is particularly important. Let $i\colon A \longrightarrow B$ be a cofibration, and let $f\colon A \longrightarrow C$ be a weak equivalence. Applying the gluing lemma to the commutative diagram

$$\begin{tikzcd} B \arrow[d, equal] & A \arrow[l, tail, "i"'] \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "\sim"', "f"] \\ B & A \arrow[l, tail, "i"] \arrow[r, "f"'] & C \end{tikzcd}$$

yields that the morphism g in the pushout square

$$\begin{tikzcd} A \arrow[r, "f", "\sim"'] \arrow[d, tail, "i"'] & C \arrow[d, tail] \\ B \arrow[r, "\sim", "g"'] & B \cup_A C \end{tikzcd}$$

is a weak equivalence, too.

4.1. **The homotopy relation.** One key feature that makes the homotopy category of a cofibration category more manageable that an arbitrary localization is a 'calculus of fractions' for morphisms in the homotopy category. We will develop this part of the theory now.

Definition 4.4. Let A be an object of a cofibration category $\mathcal{C}$. A *cylinder object* for A is a quadruple (I, i_0, i_1, p) consisting of an object I, morphisms $i_0, i_1\colon A \longrightarrow I$ and a weak equivalence $p\colon I \longrightarrow A$ satisfying $pi_0 = pi_1 = \mathrm{Id}_A$ and such that $i_0 + i_1\colon A \amalg A \longrightarrow I$ is a cofibration.

Two morphisms $f, g\colon A \longrightarrow Z$ in a cofibration category are *homotopic* if there exists a cylinder object (I, i_0, i_1, p) for A and a morphism $H\colon I \longrightarrow Z$ (the *homotopy*) such that $f = Hi_0$ and $g = Hi_1$.

Every object has a cylinder object: axiom (C4) lets us factor the fold map $\nabla = \mathrm{Id} + \mathrm{Id}\colon A \amalg A \longrightarrow A$ as a cofibration $i_0 + i_1\colon A \amalg A \longrightarrow I$ followed by a weak equivalence $p\colon I \longrightarrow A$.

Since the morphism p in a cylinder object is a weak equivalence, $\gamma(p)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$ and so $\gamma(i_0) = \gamma(i_1)$ since they share $\gamma(p)$ as common left inverse. So if f and g are homotopic via H, then

$$\gamma(f) \ = \ \gamma(H)\gamma(i_0) \ = \ \gamma(H)\gamma(i_1) \ = \ \gamma(g) \ .$$

In other words: homotopic morphisms become equal in the homotopy category. The converse is not true in general, but part (ii) of the 'calculus of fractions' (Theorem 4.13) says that the converse is true up to post-composition with a weak equivalence.

Proposition 4.5. *Let A and Z be objects in a cofibration category $\mathcal{C}$.*

(i) *'Homotopy' is an equivalence relation on the set of morphisms $\mathcal{C}(A, Z)$.*
(ii) *Postcomposition with any $\mathcal{C}$-morphism $Z \longrightarrow \bar{Z}$ preserves the homotopy relation.*
(iii) *Let $f, g\colon A \longrightarrow Z$ be homotopic, and let $\varphi\colon \bar{A} \longrightarrow A$ be any $\mathcal{C}$-morphism. Then there is an acyclic cofibration $s\colon Z \longrightarrow Z'$ such that the two morphisms $sf\varphi, sg\varphi\colon A \longrightarrow Z'$ are homotopic; moreover, the homotopy can be witnessed by any cylinder object for $\bar{A}$.*
(iv) *Let $f, g\colon A \longrightarrow Z$ and $\tau\colon \bar{A} \longrightarrow A$ be $\mathcal{C}$-morphism such that τ is a weak equivalence. If $f\tau, g\tau\colon \bar{A} \longrightarrow Z$ are homotopic, then f and g are homotopic.*

Proof. (i) For reflexivity we let (I, i_0, i_1, p) be any cylinder object for A. Then for any morphism $f\colon A \longrightarrow Z$, the morphism $fp\colon I \longrightarrow Z$ is a homotopy from f to itself.

For symmetry we let $H\colon I \longrightarrow Z$ be a homotopy from $f\colon A \longrightarrow Z$ to $g\colon A \longrightarrow Z$, based on some cylinder object (I, i_0, i_1, p). Interchanging the roles of i_0 and i_1 yields another cylinder object (I, i_1, i_0, p) for A. The same morphism $H\colon I \longrightarrow Z$ is now a homotopy from g to f based on this new cylinder object.

As a preparation for the transitivity relation we explain how two cylinder objects (I, i_0, i_1, p) and (J, j_0, j_1, q) for A can be glued into a third cylinder object $(I \cup_A J, l_0, l_1, r)$. We define $I \cup_A J$ by a choice of pushout

$$\begin{tikzcd} A \arrow[r, tail, "j_0"] \arrow[d, tail, "i_1"'] & J \arrow[d, tail, "b"] \\ I \arrow[r, tail, "a"'] & I \cup_A J \end{tikzcd}$$

Such pushout exists because i_1 and j_0 are cofibrations. Since i_1 and j_0 are acyclic cofibrations, so are a and b, by (C3). We define

$$l_0 \ = \ ai_0 \qquad \text{and} \qquad l_1 \ = \ bj_1 \ .$$

The universal property of the pushout provides a unique morphism $r\colon I \cup_A J \longrightarrow Z$ such that

$$ra \ = \ p \qquad \text{and} \qquad rb \ = \ q \ .$$

Then

$$rl_0 \ = \ rai_0 \ = \ pi_1 \ = \ \mathrm{Id}_A \ ,$$

and similarly $rl_1 = \mathrm{Id}_A$. Since $a\colon I \longrightarrow I \cup_A J$ and $p\colon Z \longrightarrow A$ are weak equivalences, the 2-out-of-3 axiom (C2) and the relation $ra = p$ show that r is a weak equivalence.

Since the coproduct of two cofibrations is a cofibration, the pushout square

$$\begin{tikzcd} A \amalg A \amalg A \amalg A \arrow[r, tail, "i_0+i_1+j_0+j_1"] \arrow[d, "A\amalg\nabla\amalg A"'] & I \amalg J \arrow[d] \\ A \amalg A \amalg A \arrow[r, tail, "l_0+l_{1/2}+l_1"'] & I \cup_A J \end{tikzcd}$$

shows that the lower horizontal morphism is a cofibration. Since embedding $A \amalg A \longrightarrow A \amalg A \amalg A$ as the first and last summand is a cofibration, too, we conclude that $l_0 + l_1\colon A \amalg A \longrightarrow I \cup_A J$ is a cofibration. This concludes the proof that the quadruple $(I \cup_A J, l_0, l_1, r)$ is indeed a cylinder object for A.

For transitivity we consider three morphisms $f, g, h\colon A \longrightarrow Z$, a homotopy $H\colon I \longrightarrow Z$ from f to g based on a cylinder object (I, i_0, i_1, p), and a homotopy $K\colon J \longrightarrow Z$ from g to h based on the cylinder object (J, j_0, j_1, q). Because

$$Hi_1 \ = \ g \ = Kj_0 \ : \ A \ \longrightarrow \ Z \ ,$$

the universal property of the pushout provides a unique morphism $H \cup K\colon I \cup_A J \longrightarrow Z$ such that

$$(H \cup K)a \ = \ H \qquad \text{and} \qquad (H \cup K)b \ = \ K \ .$$

Then

$$(H \cup K)l_0 \ = \ (H \cup K)ai_0 \ = \ Hi_0 \ = \ f \ ,$$

and similarly $(H \cup K)l_1 = h$. In other words: $H \cup K$ is a homotopy from f to h.

(ii) Given a homotopy $H\colon I \longrightarrow Z$ between two morphisms $f\colon A \longrightarrow Z$ to $g\colon A \longrightarrow Z$ and any morphism $\psi\colon Z \longrightarrow \bar{Z}$, then ψH is a homotopy, based on the same cylinder object, from ψf to ψg. So 'homotopy' is stable under postcomposition.

(iii) We let (I, i_0, i_1, p) and (J, j_0, j_1, q) be cylinder objects for A and $\bar{A}$, respectively. We start with a preliminary construction that fixes the defect that cylinder objects we not assumed to be functorial. The left vertical morphism in the commutative square

$$\begin{tikzcd}
\bar{A} \amalg \bar{A} \arrow[r, tail, "i_0\varphi + i_1\varphi"] \arrow[d, tail, "j_0 + j_1"'] & I \arrow[d, "p"] \\
J \arrow[r, "\varphi q"'] & A
\end{tikzcd}$$

is a cofibration; so a pushout of the initial part of the diagram exists. We apply the factorization axiom (C4) to the morphism

$$(\varphi q) \cup p \ : \ J \cup_{\bar{A} \amalg \bar{A}} I \ \longrightarrow \ A \ ;$$

we obtain a cofibration and a weak equivalence

$$\bar{\varphi} \cup t \ : \ J \cup_{\bar{A} \amalg \bar{A}} I \ \longrightarrow \ I' \qquad \text{and} \qquad p' \ : \ I' \ \longrightarrow \ A$$

whose composite is $(\varphi q) \cup p$. We define

$$i'_0 \ = \ ti_0 \ : A \ \longrightarrow \ I' \qquad \text{and} \qquad i'_1 \ = \ ti_1 \ : A \ \longrightarrow \ I' \ ;$$

then the following diagram commutes:

$$\begin{tikzcd}
\bar{A} \amalg \bar{A} \arrow[rr, "\varphi \amalg \varphi"] \arrow[d, tail, "j_0 + j_1"'] & & A \amalg A \arrow[dl, tail, "i'_0 + i'_1"'] \arrow[dr, tail, "i_0 + i_1"] & \\
J \arrow[r, "\bar{\varphi}"] \arrow[d, "q"', "\sim"] & I' \arrow[dr, "p'"', "\sim"] & & I \arrow[ll, tail, "t", "\sim"'] \arrow[dl, "p", "\sim"'] \\
\bar{A} \arrow[rr, "\varphi"'] & & A &
\end{tikzcd} \tag{4.6}$$

We claim that the quadruple $(I', i'_0.i'_1, p')$ is a new cylinder object for A. Because the morphisms p and p' are weak equivalences, so is $t\colon I \longrightarrow I'$ by 2-out-of-3. Because $j_0 + j_1\colon \bar{A} \amalg \bar{A} \longrightarrow J$ is a cofibration, so is the canonical morphism $I \longrightarrow J \cup_{\bar{A} \amalg \bar{A}} I$. Since the morphism $\bar{\varphi} \cup t\colon J \cup_{\bar{A} \amalg \bar{A}} I \longrightarrow I'$ is a cofibration by design, we conclude that $t\colon I \longrightarrow I'$ is a cofibration. Because $i_0 + i_1\colon A \amalg A \longrightarrow I$ is a cofibration, so is

$$i'_0 + i'_1 \ = \ t \circ (i_0 + i_1) \ : \ A \amalg A \longrightarrow I' \ .$$

The upshot of this discussion is that we have constructed a new cylinder object $(I', i'_0.i'_1, p')$ for A, an acyclic cofibration $t\colon I \longrightarrow I'$, and a morphism $\bar{\varphi}\colon J \longrightarrow I'$ such that the diagram (4.6) commutes.

Now we prove part (iii). We let $H\colon I \longrightarrow Z$ be a homotopy from $f\colon A \longrightarrow Z$ to $g\colon A \longrightarrow Z$, based on the cylinder object (I, i_0, i_1, p). We define the acyclic cofibration $s\colon Z \longrightarrow Z'$ by a choice of pushout:

$$\begin{tikzcd} I \arrow[r, tail, "t", "\sim"'] \arrow[d, "H"'] & I' \arrow[d, "\kappa"] \\ Z \arrow[r, tail, "\sim", "s"'] & Z' \end{tikzcd}$$

Then

$$\kappa\bar{\varphi}j_0 \;=\; \kappa i_0'\varphi \;=\; \kappa t i_0\varphi \;=\; sHi_0\varphi \;=\; sf\varphi \;,$$

and similarly $\kappa\bar{\varphi}j_1 = sg\varphi$. In other words, the morphism $\kappa\bar{\varphi}\colon J \longrightarrow Z'$ is a homotopy from $sf\varphi$ to $sg\varphi$.

(iv) Since $f\tau$ is homotopic to $g\tau$, there is a cylinder object (J, j_0, j_1, q) for $\bar{A}$ and a homotopy $H\colon J \longrightarrow Z$ from $f\tau$ to $g\tau$. We form a pushout

$$\begin{tikzcd} \bar{A} \amalg \bar{A} \arrow[r, tail, "j_0+j_1"] \arrow[d, "\tau\amalg\tau"', "\sim"] & J \arrow[d, "\sim"', "\bar{\tau}"] \\ A \amalg A \arrow[r, tail, "i_0+i_1"'] & I \end{tikzcd}$$

The morphism $i_0 + i_1\colon A \amalg A \longrightarrow I$ is then a cofibration because $j_0 + j_1$ is; and the morphism $\bar{\tau}$ is a weak equivalence because $\tau \amalg \tau$ is, by the gluing lemma. The universal property of the pushout provides a unique morphism $p\colon I \longrightarrow A$ such that

$$pi_0 \;=\; pi_1 \;=\; \mathrm{Id}_A \qquad \text{and} \qquad p\bar{\tau} \;=\; \tau q \;.$$

Because τ, $\bar{\tau}$ and q are weak equivalences, so is p, by 2-out-of-3. We conclude that (I, i_0, i_1, p) is a cylinder object for A.

The relations

$$f\tau \;=\; Hj_0 \qquad \text{and} \qquad g\tau \;=\; Hj_1$$

and the universal property of the pushout provide a unique morphism $K\colon I \longrightarrow Z$ such that

$$Ki_0 \;=\; f \;, \quad Ki_1 \;=\; g \qquad \text{and} \qquad K\bar{\tau} \;=\; H \;.$$

In particular, $\bar{K}$ is a homotopy from f to g. $\square$

4.2. **Localization by fractions.** We can now exhibit a localization of a cofibration category at the class of weak equivalences, by a construction that resembles the definition of fractions.

Construction 4.7. We let $\mathcal{C}$ be a cofibration category. We define a category $\mathrm{Ho}(\mathcal{C})$ with the same objects as $\mathcal{C}$. Morphisms in $\mathrm{Ho}(\mathcal{C})$ from A to B are equivalence classes of pairs (f, τ) consisting of $\mathcal{C}$-morphisms $f\colon A \longrightarrow Z$ and $\tau\colon B \longrightarrow Z$ with the same target, and such that τ is an acyclic cofibration. Two such pairs (f, τ) and (f', τ') are *equivalent* if there are acyclic cofibrations $a\colon Z \longrightarrow \bar{Z}$ and $b\colon Z' \longrightarrow \bar{Z}$ such that the following diagram commutes up to homotopy:

$$\begin{tikzcd} & Z \arrow[d, tail, "a"', "\sim"] & \\ A \arrow[ur, "f"] \arrow[dr, "f'"'] & \bar{Z} & B \arrow[ul, tail, "\tau"', "\sim"] \arrow[dl, tail, "\sim", "\tau'"'] \\ & Z' \arrow[u, tail, "b", "\sim"'] & \end{tikzcd}$$

We write $(f, \tau) \approx (f', \tau')$ for this relation.

Proposition 4.8. *The relation $\approx$ is an equivalence relation.*

Proof. The relation $\approx$ is clearly reflexive and symmetric, because the homotopy relation is reflexive and symmetric. But we need to argue that it is also transitive.

We suppose that $(f,\tau)\approx(f',\tau')$ via two acyclic cofibrations $a\colon Z\longrightarrow\bar Z$ and $b\colon Z'\longrightarrow\bar Z$, i.e., such that af is homotopic to $bf'\colon A\longrightarrow\bar Z'$, and $a\tau$ is homotopic to $b\tau'$. And we suppose that also $(f',\tau')\approx(f'',\tau'')$ via two acyclic cofibrations $a'\colon Z'\longrightarrow\bar Z'$ and $b'\colon Z''\longrightarrow\bar Z'$, i.e., such that $a'f'$ is homotopic to $b'f''\colon A\longrightarrow\bar Z'$, and $a'\tau'$ is homotopic to $b'\tau''$.

Because b and a' are acyclic cofibrations, we can choose a pushout:

$$\begin{tikzcd}
Z' \arrow[r, tail, "a'", "\sim"'] \arrow[d, tail, "b"', "\sim"] & \bar Z' \arrow[d, tail, "\beta", "\sim"'] \\
\bar Z \arrow[r, tail, "\sim", "\alpha"'] & E
\end{tikzcd}$$

Moreover, the morphisms α and β are also acyclic cofibrations. The homotopy relation is compatible with postcomposition, so

$$\alpha af\sim\alpha bf'=\beta a'f'\sim\beta b'f''\qquad\text{and}\qquad\alpha a\tau\sim\alpha b\tau'=\beta a'\tau'\sim\beta b'\tau''\ .$$

Since the homotopy relation is transitive, the acyclic cofibrations $\alpha a\colon Z\longrightarrow E$ and $\beta b'\colon Z''\longrightarrow E$ witness that $(f,\tau)\approx(f'',\tau'')$. $\square$

Construction 4.9. We continue with the definition of the category $\mathrm{Ho}(\mathcal C)$. Morphisms in $\mathrm{Ho}(\mathcal C)$ from A to B are equivalence classes under the relation $\approx$ of pairs $(f\colon A\longrightarrow Z,\tau\colon B\longrightarrow Z)$ such that τ is an acyclic cofibration. We write

$$\tau\backslash f\ :\ A\ \longrightarrow\ B$$

for the equivalence class of the pair (f,τ).

Now we define composition in the category $\mathrm{Ho}(\mathcal C)$. We consider two pairs of morphisms (f,τ) and (g,σ) that represent morphisms $\tau\backslash f\colon A\longrightarrow B$ and $\sigma\backslash g\colon B\longrightarrow C$ in $\mathrm{Ho}(\mathcal C)$. Because $\tau\colon B\longrightarrow Z$ is an acyclic cofibration, there is a pushout in $\mathcal C$:

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, tail, "\tau"', "\sim"] & Y \arrow[d, tail, "\psi", "\sim"'] \\
Z \arrow[r, "\varphi"'] & W
\end{tikzcd}\tag{4.10}$$

Moreover, the morphism ψ is an acyclic cofibration, too. We then define the composite by

$$(\sigma\backslash g)\circ(\tau\backslash f)\ =\ (\psi\sigma)\backslash(\varphi f)\ .\tag{4.11}$$

Theorem 4.12. *Let $\mathcal C$ be a cofibration category.*

(i) *The composition* (4.11) *is well-defined and makes* $\mathrm{Ho}(\mathcal C)$ *into a category.*
(ii) *The assignments $\gamma(A)=A$ and $\gamma(f)=\mathrm{Id}\backslash f$ define a functor $\gamma\colon\mathcal C\longrightarrow\mathrm{Ho}(\mathcal C)$.*
(iii) *For every acyclic cofibration $\tau\colon B\longrightarrow Z$ in $\mathcal C$, the morphism $\gamma(\tau)$ is an isomorphism with inverse $\tau\backslash\mathrm{Id}_Z$, and the relation*
$$\tau\backslash f\ =\ \gamma(\tau)^{-1}\circ\gamma(f)$$
holds for all $\mathcal C$-morphisms $f\colon A\longrightarrow Z$.
(iv) *The functor $\gamma\colon\mathcal C\longrightarrow\mathrm{Ho}(\mathcal C)$ takes weak equivalences to isomorphisms.*
(v) *The functor $\gamma\colon\mathcal C\longrightarrow\mathrm{Ho}(\mathcal C)$ is a localization of $\mathcal C$ at the class of weak equivalences.*

Proof. (i) We start by observing that the composition (4.11) does not depend on the choice of pushout (4.10). Indeed, any two choices of pushout are canonically isomorphic, and the resulting pairs $(\varphi f,\psi\sigma)$ are then $\approx$-equivalent via isomorphisms.

The equivalence relation $\approx$ is generated by three 'elementary' instances:

(1) for every acyclic cofibration $a\colon Z\longrightarrow\bar Z$, the pair (f,τ) is equivalent to $(af,a\tau)$;
(2) for all pairs of homotopic morphisms $f,f'\colon A\longrightarrow Z$, the pair (f,τ) is equivalent to (f',τ);

(3) for all pairs of homotopic acyclic cofibrations $\tau, \tau' \colon B \longrightarrow Z$, the pair (f, τ) is equivalent to (f, τ').

So it suffices to show that pre- and postcomposition is compatible with each of these elementary relations.

We start with postcomposition by the morphism represented by a pair (g, σ), for a $\mathcal{C}$-morphisms $g \colon B \longrightarrow Y$ and an acyclic cofibration $\sigma \colon C \longrightarrow Y$.

Relation (1): We choose two pushout squares in $\mathcal{C}$:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, tail, "\tau"', "\sim"] & Y \arrow[d, tail, "\psi", "\sim"'] \\ Z \arrow[r, "\varphi"'] \arrow[d, tail, "a"', "\sim"] & W \arrow[d, tail, "\lambda", "\sim"'] \\ \bar{Z} \arrow[r, "\kappa"'] & V \end{tikzcd}$$

The composite is then a pushout, too. So

$$(\sigma\backslash g) \circ ((a\tau)\backslash(af)) \ = \ (\lambda\psi\sigma)\backslash(\kappa a f) \ = \ (\lambda\psi\sigma)\backslash(\lambda\varphi f) \ = \ (\psi\sigma)\backslash(\varphi f) \ = \ (\sigma\backslash g) \circ (\tau\backslash f) \ .$$

Relation (2): We choose a pushout (4.10). Postcomposition preserves the homotopy relation; because $f, f' \colon A \longrightarrow Z$ are homotopic, so are $\varphi f, \varphi f' \colon A \longrightarrow W$. Hence $(\varphi f, \psi\sigma) \approx (\varphi f', \psi\sigma)$.

Relation (3): we choose three pushouts

$$\begin{tikzcd} & B \arrow[r, tail, "\tau", "\sim"'] \arrow[d, "g"'] & Z \arrow[d, "\varphi"] \\ B \arrow[r, "g"] \arrow[d, tail, "\tau'"', "\sim"] & Y \arrow[r, tail, "\sim", "\psi"'] \arrow[d, tail, "\psi'", "\sim"'] & W \arrow[d, tail, "\alpha", "\sim"'] \\ Z \arrow[r, "\varphi'"'] & W' \arrow[r, tail, "\sim", "\beta"'] & V \end{tikzcd}$$

Because τ and τ' are acyclic cofibrations, so are the morphisms ψ, ψ', α and β. Postcomposition preserves the homotopy relation; since τ and τ' are homotopic, we conclude that

$$\alpha\varphi\tau \ = \ \alpha\psi g \ = \ \beta\psi' g \ = \ \beta\varphi'\tau' \ \sim \ \beta\varphi'\tau \ .$$

Because τ is a weak equivalence, Proposition 4.5 (iv) shows that $\alpha\varphi \colon Z \longrightarrow V$ is homotopic to $\beta\varphi'$.

Unfortunately, precomposition with $f \colon A \longrightarrow Z$ need not preserve the homotopy relation. However, Proposition 4.5 (iii) provides an acyclic cofibration $s \colon V \longrightarrow V'$ such that $s\alpha\varphi f \colon A \longrightarrow V'$ is homotopic to $s\beta\varphi' f$. Moreover,

$$s\alpha\psi\sigma \ = \ s\beta\psi'\sigma \ : \ C \ \longrightarrow \ V' \ .$$

So the acyclic cofibrations $s\alpha \colon W \longrightarrow V'$ and $s\beta \colon W' \longrightarrow V'$ witness that $(\varphi f, \psi\sigma) \approx (\varphi' f, \psi'\sigma)$.

Now we turn to precomposition by the morphism represented by a pair (e, ν), for a $\mathcal{C}$-morphism $e \colon E \longrightarrow X$ and an acyclic cofibration $\nu \colon A \longrightarrow X$.

Relation (1): We choose two pushout squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "\nu"', "\sim"] & Z \arrow[r, "a", "\sim"'] \arrow[d, tail, "\xi", "\sim"'] & \bar{Z} \arrow[d, tail, "\zeta", "\sim"'] \\ X \arrow[r, "\chi"'] & U \arrow[r, "\mu"'] & V \end{tikzcd}$$

The composite is then a pushout, too. So

$$((a\tau)\backslash(af)) \circ (\nu\backslash e) \ = \ (\zeta a\tau)\backslash(\mu\chi e) \ = \ (\mu\xi\tau)\backslash(\mu\chi e) \ = \ (\xi\tau)\backslash(\chi e) \ = \ (\tau\backslash f) \circ (\nu\backslash e) \ .$$

Relation (2): we choose three pushouts

$$\begin{tikzcd}
 & A \arrow[r, tail, "\nu", "\sim"'] \arrow[d, "f"'] & X \arrow[d, "\chi"] \\
A \arrow[r, "f'"] \arrow[d, tail, "\nu"', "\sim"] & Z \arrow[r, tail, "\sim", "\xi"'] \arrow[d, tail, "\sim"', "\xi'"] & U \arrow[d, tail, "\sim"', "\alpha"] \\
X \arrow[r, "\chi'"'] & U' \arrow[r, tail, "\sim", "\beta"'] & V
\end{tikzcd}$$

Because ν is an acyclic cofibration, so are the morphisms ξ, ξ', α and β. Postcomposition preserves the homotopy relation; since f is homotopic to f', we conclude that

$$\alpha\chi\nu \;=\; \alpha\xi f \;=\; \beta\xi' f \;\sim\; \beta\xi' f' \;=\; \beta\chi'\nu \;.$$

Because ν is a weak equivalence, Proposition 4.5 (iv) shows that $\alpha\chi\colon X \longrightarrow V$ is homotopic to $\beta\chi'$. Postcomposition preserves the homotopy relation, so also

$$\alpha\chi e \;\sim\; \beta\chi' e \;.$$

Moreover,

$$\alpha\xi\tau \;=\; \beta\xi'\tau \;:\; B \;\longrightarrow\; V \;.$$

So the acyclic cofibrations $\alpha\colon U \longrightarrow V$ and $\beta\colon U' \longrightarrow V$ witness that $(\chi e, \xi\tau) \approx (\chi' e, \xi'\tau)$.

Relation (3): We choose a pushout

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "\nu"', "\sim"] & Z \arrow[d, tail, "\sim"', "\xi"] \\
X \arrow[r, "\chi"'] & U
\end{tikzcd}$$

Postcomposition preserves the homotopy relation. So if $\tau, \tau'\colon B \longrightarrow Z$ are homotopic, so are $\xi\tau, \xi\tau'\colon B \longrightarrow U$. Hence $(\chi e, \xi\tau) \approx (\chi e, \xi\tau')$.

Now that we know that composition in $\mathrm{Ho}(\mathcal{C})$ is well-defined, we can easily check that it is associative. We consider three pairs (e, ν), (f, τ) and (g, σ) that represent composable fractions. We choose three pushouts

$$\begin{tikzcd}
 & B \arrow[r, "g"] \arrow[d, tail, "\tau"', "\sim"] & Y \arrow[d, tail, "\sim"', "\psi"] \\
A \arrow[r, "f"] \arrow[d, tail, "\nu"', "\sim"] & Z \arrow[r, "\varphi"'] \arrow[d, tail, "\sim"', "\xi"] & W \arrow[d, tail, "\sim"', "\lambda"] \\
X \arrow[r, "\chi"'] & U \arrow[r, "\mu"'] & V
\end{tikzcd}$$

Then the two composite squares are pushouts, too. So

$$\begin{aligned}
(\sigma\backslash g) \circ ((\tau\backslash f) \circ (\nu\backslash e)) \;&=\; (\sigma\backslash g) \circ ((\xi\tau)\backslash(\chi e)) \;=\; (\lambda\psi\sigma)\backslash(\mu\chi e) \\
&=\; ((\psi\sigma)\backslash(\varphi f)) \circ (\nu\backslash e) \;=\; ((\sigma\backslash g) \circ (\tau\backslash f)) \circ (\nu\backslash e) \;.
\end{aligned}$$

For every object A, the fraction $\mathrm{Id}_A\backslash\mathrm{Id}_A$ is clearly a two-sided unit for composition.

(ii) The proof that γ is indeed a functor is straightforward.

(iii) To calculate the composite $(\tau\backslash \mathrm{Id}_Z) \circ \gamma(\tau) = (\tau\backslash \mathrm{Id}_Z) \circ (\mathrm{Id}_Z \backslash\tau)$ we must choose a pushout of two instances of the identity of Z; the square consisting of four instances of Id_Z does the job, and it yields the relation

$$(\tau\backslash \mathrm{Id}_Z) \circ \gamma(\tau) \;=\; (\tau\backslash \mathrm{Id}_Z) \circ (\mathrm{Id}_Z \backslash\tau) \;=\; \tau\backslash\tau \;=\; \mathrm{Id}_B \backslash \mathrm{Id}_B \;.$$

In order to calculate the other composite, we choose a pushout

$$\begin{tikzcd} B \arrow[r, tail, "\tau", "\sim"'] \arrow[d, tail, "\tau"', "\sim"] & Z \arrow[d, tail, "\sim"', "\psi_1"] \\ Z \arrow[r, tail, "\sim", "\psi_0"'] & Z \cup_B Z \end{tikzcd}$$

Because $\psi_0\tau = \psi_1\tau$ and τ is a weak equivalence, Proposition 4.5 (iv) shows that ψ_0 and ψ_1 are homotopic. The acyclic cofibration $\psi_1 \colon Z \longrightarrow Z \cup_B Z$ thus witnesses that $(\mathrm{Id}_Z, \mathrm{Id}_Z) \approx (\psi_0, \psi_1)$. So

$$\gamma(\tau) \circ (\tau \backslash \mathrm{Id}_Z) \; = \; (\mathrm{Id}_Z \backslash \tau) \circ (\tau \backslash \mathrm{Id}_Z) \; = \; \psi_1 \backslash \psi_0 \; = \; \mathrm{Id}_Z \backslash \mathrm{Id}_Z \; .$$

(iv) We **claim** that every $\mathcal{C}$-morphism $f \colon A \longrightarrow B$ admits a factorization $f = qj$ such that $j \colon A \longrightarrow Z$ is a cofibration and the morphism $q \colon Z \longrightarrow B$ is left inverse to an acyclic cofibration $r \colon B \longrightarrow Z$. To prove the claim we use an abstract version of the mapping cylinder factorization. We choose a cylinder object (I, i_0, i_1, p) for A, and form a pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i_1"', "\sim"] & B \arrow[d, tail, "\sim"', "r"] \\ I \arrow[r, "\varphi"'] & I \cup_f B = Z \end{tikzcd}$$

The universal property of the pushout provides a unique morphism $q \colon Z \longrightarrow B$ such that $q\varphi = fp$ and $qr = \mathrm{Id}_B$. In particular, q is left inverse to the acyclic cofibration r. We define $j = \varphi i_0 \colon A \longrightarrow Z$. Then

$$qj \; = \; q\varphi i_0 \; = \; fpi_1 \; = \; f \; .$$

Moreover, the square

$$\begin{tikzcd} A \amalg B \arrow[r, "\mathrm{Id}_A \amalg f"] \arrow[d, tail, "i_0 + i_1"'] & A \amalg B \arrow[d, tail, "j+r"] \\ I \arrow[r, "\varphi"'] & I \cup_f B = Z \end{tikzcd}$$

is a pushout; so $j + r \colon A \amalg B \longrightarrow Z$ is a cofibration. Since the canonical morphism $A \longrightarrow A \amalg B$ is a cofibration, too, so is the morphism j.

Now we can prove part (iv). We factor the given weak equivalence $f \colon A \longrightarrow B$ as provided by the above claim, so that $f = qj$, and $qr = \mathrm{Id}_B$ for an acyclic cofibration $r \colon B \longrightarrow Z$. Because f and q are weak equivalences, so is j; hence j is an acyclic cofibration. Thus $\gamma(r)$ and $\gamma(j)$ are isomorphisms by part (iii). Because

$$\gamma(q) \circ \gamma(r) \; = \; \gamma(qr) \; = \; \mathrm{Id}_B \; ,$$

the morphism $\gamma(q)$ is an isomorphism, and inverse to $\gamma(r)$. So

$$\gamma(f) \; = \; \gamma(q) \circ \gamma(j) \; = \; \gamma(r)^{-1} \circ \gamma(j) \; = \; r \backslash j$$

is an isomorphism.

(v) We let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ be any functor that takes weak equivalences to isomorphisms. We have to show that F factors uniquely through the functor $\gamma \colon \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$.

Claim: let $f, g \colon A \longrightarrow B$ be homotopic $\mathcal{C}$-morphisms. Then $F(f) = F(g)$. Indeed, there is a cylinder object (I, i_0, i_1, p) for A, and a homotopy $H \colon I \longrightarrow B$ such that

$$Hi_0 \; = \; f \qquad \text{and} \qquad Hi_1 \; = \; g \; .$$

Because p is a weak equivalence, the morphism $F(p) \colon FI \longrightarrow FA$ is an isomorphism. Because $F(p)$ is an isomorphism and

$$F(p) \circ F(i_0) \; = \; F(p \circ i_0) \; = \; F(\mathrm{Id}_A) \; = \; F(p \circ i_1) \; = \; F(p) \circ F(i_1) \; ,$$

we conclude that $F(i_0) = F(i_1)$. Hence

$$F(f) \;=\; F(H)\circ F(i_0) \;=\; F(H)\circ F(i_1) \;=\; F(g)\ .$$

Now we can prove the uniqueness property of a localization. Let $G\colon \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathcal{D}$ be any functor such that $G\circ\gamma = F$. Because γ is the identity on objects, G agrees with F on objects. On morphisms we have

$$G(\tau\backslash f) \;=\; G(\gamma(\tau)^{-1}\circ\gamma(f)) \;=\; G(\gamma(\tau))^{-1}\circ G(\gamma(f)) \;=\; F(\tau)^{-1}\circ F(f)\ .$$

So the effect of G on morphisms is determined by the effect of F on morphisms.

For the existence part of a localization, we define $G\colon \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathcal{D}$ on objects by $G(A) = F(A)$. On morphisms, we define

$$G(\tau\backslash f) \;=\; F(\tau)^{-1}\circ F(f)\ .$$

To show that this is well-defined, we suppose that $(f,\tau) \approx (f',\tau')$. Since the equivalence relation $\approx$ is generated by the three elementary instances, it suffices to check these elementary cases.

For the relation (1) we let $a\colon Z \longrightarrow \bar{Z}$ is any acyclic cofibration. Then $F(a)$ is an isomorphism, and hence

$$F(a\tau)^{-1}\circ F(af) \;=\; F(\tau)^{-1}\circ F(a)^{-1}\circ F(a)\circ F(f) \;=\; F(\tau)^{-1}\circ F(f)\ .$$

Relations (2) and (3) are also fine because F takes the same value on homotopic morphisms.

Now we argue that G is indeed a functor. Clearly $G(1_A\backslash 1_A) = \mathrm{Id}_{F(A)}$, so G preserves identities. Now we let (f,τ) and (g,σ) represent two composable morphism $\tau\backslash f\colon A \longrightarrow B$ and $\sigma\backslash g\colon B \longrightarrow C$ in $\mathrm{Ho}(\mathcal{C})$. We choose a pushout square (4.10). Then

$$F(\varphi)\circ F(\tau) \;=\; F(\psi)\circ F(g)\ .$$

Because τ and ψ are weak equivalences, $F(\tau)$ and $F(\psi)$ are isomorphisms, and thus

$$F(\psi)^{-1}\circ F(\varphi) \;=\; F(g)\circ F(\tau)^{-1}\ .$$

Hence

$$\begin{aligned} G((\sigma\backslash g)\circ(\tau\backslash f)) \;&=\; G((\psi\sigma)\backslash(\varphi f)) \;=\; F(\psi\sigma)^{-1}\circ F(\varphi f)\\ &=\; F(\sigma)^{-1}\circ F(\psi)^{-1}\circ F(\varphi)\circ F(f)\\ &=\; F(\sigma)^{-1}\circ F(g)\circ F(\tau)^{-1}\circ F(f) \;=\; G(\sigma\backslash g)\circ G(\tau\backslash f))\ . \end{aligned}$$ □

The specific construction of the localization $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ immediately implies the following corollary, which we will refer to as a 'calculus of fractions' for $\mathrm{Ho}(\mathcal{C})$. The theorem states the main properties of the localization functor without reference to the specific construction.

Theorem 4.13 (Calculus of fractions)**.** *Let $\mathcal{C}$ be a cofibration category and $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ a localization at the class of weak equivalences. Then:*

(i) *Every morphism in $\mathrm{Ho}(\mathcal{C})$ is a 'left fraction', i.e., is of the form $\gamma(\tau)^{-1}\circ\gamma(f)$, where f and τ are $\mathcal{C}$-morphisms with the same target, and τ is an acyclic cofibration.*

(ii) *Given two morphisms $f, g : A \longrightarrow B$ in $\mathcal{C}$, then $\gamma(f) = \gamma(g)$ in $\mathrm{Ho}(\mathcal{C})$ if and only if there is an acyclic cofibration $s : B \longrightarrow \bar{B}$ such that sf and sg are homotopic.*

Proof. Part (i) is a restatement of the relation $\tau\backslash f = \gamma(\tau)^{-1}\circ\gamma(f)$.

(ii) If there is an acyclic cofibration $s : B \longrightarrow \bar{B}$ with $sf \sim sg$, then

$$\gamma(s)\circ\gamma(f) \;=\; \gamma(sf) \;=\; \gamma(sg) \;=\; \gamma(s)\circ\gamma(g)\ .$$

We have exploited that every functor that inverts weak equivalences takes the same value on homotopic morphisms, compare the proof of Theorem 4.12. Because s is a weak equivalence, the morphism $\gamma(s)$ is an isomorphism, and hence $\gamma(f) = \gamma(g)$.

Conversely, the relation $\gamma(f) = \gamma(g)$ means that $(f, \mathrm{Id}_B) \approx (g, \mathrm{Id}_B)$. So there are homotopic acyclic cofibrations $a, b : B \longrightarrow \bar{Z}$ such that af is homotopic to bg. We choose a pushout in $\mathcal{C}$:

$$\begin{tikzcd} B \arrow[r, tail, "b", "\sim"'] \arrow[d, tail, "a"', "\sim"] & \bar{Z} \arrow[d, tail, "\sim"', "\psi"] \\ \bar{Z} \arrow[r, tail, "\varphi"'] & W \end{tikzcd}$$

Postcomposition preserves the homotopy relation, so we conclude that

$$\varphi a \ = \ \psi b \ \sim \ \psi a \ .$$

Because a is a weak equivalence, Proposition 4.5 (iv) shows that φ and ψ are homotopic. Proposition 4.5 (iii) provides an acyclic cofibration $t : W \longrightarrow \bar{B}$ such that

$$t\varphi af \sim t\psi af \ .$$

Because the homotopy relation is transitive, the relations

$$t\varphi af \ \sim \ t\psi af \ \sim \ t\psi bg \ = \ t\varphi ag$$

show that $s = t\varphi a : B \longrightarrow \bar{B}$ can serve as the desired acyclic cofibration. □

Remark 4.14. On the face of it, the homotopy category of a cofibration category raises set-theoretic issues: in general the hom-'sets' in $\mathrm{Ho}(\mathcal{C})$ may not be small, but rather proper classes. One way to deal with this is to work with universes in the sense of Grothendieck; the homotopy category of a cofibration category then always exists in a larger universe. Another way to address the set theory issues is to restrict attention to those cofibration categories that have 'enough fibrant objects'. An object of a cofibration category $\mathcal{C}$ is *fibrant* if every acyclic cofibration out of it has a retraction. If the object B is fibrant, then the map $\gamma : \mathcal{C}(A, B) \longrightarrow \mathrm{Ho}(\mathcal{C})(A, B)$ given by the localization functor is surjective: an arbitrary morphism from A to B in $\mathrm{Ho}(\mathcal{C})$ is of the form $\gamma(s)^{-1}\gamma(a)$ for some acyclic cofibration $s : B \longrightarrow Z$. Since B is fibrant, there is a retraction $r : Z \longrightarrow B$ with $rs = \mathrm{Id}_B$, and then $\gamma(s)^{-1}\gamma(a) = \gamma(ra)$. Moreover, if two $\mathcal{C}$-morphisms $f, g : A \longrightarrow B$ become equal after applying the functor γ, then there is an acyclic cofibration $s : B \longrightarrow \bar{B}$ such that sf is homotopic to sg. Composing with any retraction to s shows that f is already homotopic to g. So the map $\mathcal{C}(A, B)/\text{homotopy} \longrightarrow \mathrm{Ho}(\mathcal{C})(A, B)$ sending the class of f to $\gamma(f)$, is bijective. We say that the cofibration category $\mathcal{C}$ has *enough fibrant objects* if, for every object X, there is a weak equivalence $r : X \longrightarrow Z$ with fibrant target. For example, if $\mathcal{C}$ is the collection of cofibrant objects in an ambient Quillen model category, then it has enough fibrant objects.

If $r : X \longrightarrow Z$ is a weak equivalence with fibrant target, then for every other object A the two maps

$$\mathrm{Ho}(\mathcal{C})(A, X) \ \xrightarrow{\gamma(r)_*} \ \mathrm{Ho}(\mathcal{C})(A, Z) \ \xleftarrow{\ \gamma\ } \ \mathcal{C}(A, Z)/\text{homotopy}$$

are bijective, so the morphisms $\mathrm{Ho}(\mathcal{C})(A, X)$ form a set (as opposed to a proper class). So, if $\mathcal{C}$ has enough fibrant objects, then the homotopy category $\mathrm{Ho}(\mathcal{C})$ has small hom-sets (or is 'locally small').

Now that we have a good handle on the homotopy category of a cofibration category, we can use the calculus of fractions to derive additional desirable properties. The homotopy category will typically have only very few limits and colimits. But it always has finite coproducts, and general coproducts and finite products are inherited from the cofibration category.

In a cofibration category, finite coproducts of weak equivalences are automatically weak equivalences. So for finite indexing sets, the following proposition follows from Proposition 3.8 (ii) and the fact that the homotopy category $\mathrm{Ho}(\mathcal{C})$ is a localization of $\mathcal{C}$ be the weak equivalences. It is not generally true that the localization functor for a relative category preserves infinite coproducts, so here the calculus of fractions for cofibration categories comes in handy.

Proposition 4.15. *Let $(\mathcal{D}_i, \mathcal{W}_i)_{i\in I}$ be an I-indexed family of relative categories, and let $\gamma_i \colon \mathcal{D}_i \longrightarrow \mathcal{D}_i[\mathcal{W}_i^{-1}]$ be localization functors. If each pair $(\mathcal{D}_i, \mathcal{W}_i)$ can be extended to a cofibration structure on $\mathcal{D}_i$, then the product functor*

$$\prod_I \gamma_i \ : \ \prod_I \mathcal{D}_i \ \longrightarrow \ \prod_I \mathcal{D}_i[\mathcal{W}_i^{-1}]$$

is a localization of $\prod_I \mathcal{D}_i$ at the class of morphisms $\prod_I \mathcal{W}_i$.

Informally speaking, the previous proposition says that

$$(\prod_I \mathcal{D}_i)[(\prod_I \mathcal{W}_i)^{-1}] \ = \ \prod_I \mathcal{D}_i[\mathcal{W}_i^{-1}] \ ,$$

whenever all the relative categories $(\mathcal{D}_i, \mathcal{W}_i)$ underly cofibration categories. We refrain from spelling out a proof of Proposition 4.15 and content ourselves with a reference, namely [40, Theorem 7.1.1]. The reference shows that one can get away with a little less than cofibration categories in the sense of Definition 4.1.

Proposition 4.16. *Let $\mathcal{C}$ be a cofibration category, and let I be a set. Suppose that $\mathcal{C}$ has I-indexed coproducts, and that the classes of cofibrations and weak equivalences are closed under I-indexed coproducts. Then the localization functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ preserves I-indexed coproducts. In particular, the homotopy category $\mathrm{Ho}(\mathcal{C})$ has I-indexed coproducts.*

Proof. We offer two proofs. The first one is a bit more conceptual in that it generalizes the formal argument for finite coproducts in the world of relative categories that we presented in Proposition 3.8 (ii); the second proof is down-to-earth and explicit. Both proofs depend on the calculus of fractions from Theorem 4.13.

The first proof exploits that the cofibration structure makes the localization compatible with I-indexed products, in the sense of Proposition 4.15. In the commutative diagram of categories and functors

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\mathrm{diag}_I"] \arrow[d, "\gamma"'] & \prod_I \mathcal{D} \arrow[d, "\prod_I \gamma"] \\
\mathcal{D}[\mathcal{W}^{-1}] \arrow[r, "\mathrm{diag}_I"'] & \prod_I \mathcal{D}[\mathcal{W}^{-1}]
\end{tikzcd}$$

both vertical functors are localizations, and the upper diagonal functor is a right adjoint because $\mathcal{D}$ has I-indexed coproducts. The left adjoint to the upper diagonal functor takes an family $(f_i)_I$ of $\mathcal{D}$-morphisms to their coproduct $\coprod_I f_i$; this functor is homotopical for the class $\prod_I \mathcal{W}$ on the category $\prod_I \mathcal{D}$, by hypothesis. The diagonal functor is homotopical, too. As we explained in Construction 3.5, the adjunction thus descends to the localizations. In particular, the lower horizontal diagonal functor for $\mathcal{D}[\mathcal{W}^{-1}]$ has a left adjoint. But this precisely means that the category $\mathcal{D}[\mathcal{W}^{-1}]$ has I-indexed coproducts. And by design of how the adjunction descends, the localization functor commutes with the left adjoints, which means that it preserves I-indexed coproducts.

Here is the second, more explicit, proof. Since the localization functor γ can be arranged to be the identity on objects, we drop γ in front of objects to simplify the notation. Now we consider an I-indexed family $\{X_i\}_{i\in I}$ of $\mathcal{C}$-objects. We denote a coproduct of the family by $\bigvee_{i\in I} X_i$, and we write $\kappa_j \colon X_j \longrightarrow \bigvee_{i\in I} X_i$ for the universal morphisms. We must show that for every $\mathcal{C}$-object Y, the map

$$\mathrm{Ho}(\mathcal{C})(\bigvee_{i\in I} X_i, Y) \ \longrightarrow \ \prod\nolimits_{j\in I} \mathrm{Ho}(\mathcal{C})(X_j, Y) \ , \quad \psi \longmapsto \ (\psi \circ \gamma(\kappa_j))_{j\in I} \tag{4.17}$$

is bijective. For surjectivity we let $(\psi_j \colon X_j \longrightarrow Y)$ be any I-indexed family of morphisms in $\mathrm{Ho}(\mathcal{C})$. By the calculus of left fractions, we can write

$$\psi_j \ = \ \gamma(s_j)^{-1} \circ \gamma(f_j)$$

for some families of $\mathcal{C}$-morphisms $f_j \colon X_j \longrightarrow W_j$ and $s_j \colon Y \longrightarrow W_j$ such that the morphisms s_j are acyclic cofibrations. We choose a coproduct of the family $\{W_i\}_{i\in I}$ and a coproduct of the constant family $\{Y\}_{i\in I}$

of copies of Y. Then we form the $\mathcal{C}$-morphisms

$$\bigvee_{i\in I} X_i \xrightarrow{\bigvee f_i} \bigvee_{i\in I} W_i \xleftarrow[\sim]{\bigvee s_i} \bigvee_{i\in I} Y \xrightarrow{\nabla} Y ,$$

where ∇ denotes the fold morphism. Since coproducts of acyclic cofibrations are acyclic cofibrations, the middle morphism is an acyclic cofibration. So we can form the morphism

$$\gamma(\nabla)\circ\gamma(\bigvee s_i)^{-1}\circ\gamma(\bigvee f_i) \ : \ \bigvee_{i\in I} X_i \longrightarrow Y$$

in the homotopy category. Then for every $j \in J$, the following diagram commutes:

$$\begin{tikzcd}
X_j \arrow[r, "f_j"] \arrow[d, "\kappa_j"'] & W_j \arrow[d, "\kappa_j"'] & Y \arrow[l, tail, "s_j", "\sim"'] \arrow[d, "\kappa_j"'] \arrow[dr, equal] & \\
\bigvee_{i\in I} X_i \arrow[r, "\bigvee f_i"'] & \bigvee_{i\in I} W_i & \bigvee_{i\in I} Y \arrow[l, tail, "\sim", "\bigvee s_i"'] \arrow[r, "\nabla"'] & Y
\end{tikzcd}$$

Hence

$$\begin{aligned}
\gamma(\nabla)\circ\gamma(\bigvee s_i)^{-1}\circ\gamma(\bigvee f_i)\circ\gamma(\kappa_j) \ &= \ \gamma(\nabla)\circ\gamma(\bigvee s_i)^{-1}\circ\gamma((\bigvee f_i)\circ\kappa_j) \\
&= \ \gamma(\nabla)\circ\gamma(\bigvee s_i)^{-1}\circ\gamma(\kappa_j)\circ\gamma(f_j) \\
&= \ \gamma(\nabla)\circ\gamma(\kappa_j)\circ\gamma(s_j)^{-1}\circ\gamma(f_j) \ = \ \gamma(s_j)^{-1}\circ\gamma(f_j) \ = \ \psi_j \ .
\end{aligned}$$

So the map (4.17) sends the morphism $\gamma(\nabla)\circ\gamma(\bigvee s_i)^{-1}\circ\gamma(\bigvee f_i)$ to the original family $(\psi_j)_{j\in I}$, and thus the map (4.17) is surjective.

For injectivity we consider two morphisms $\psi, \psi' \colon \bigvee_{i\in I} X_i \longrightarrow Y$ in $\mathrm{Ho}(\mathcal{C})$ such that $\psi\circ\gamma(\kappa_j) = \psi'\circ\gamma(\kappa_j)$ for all $j \in I$. We start with the special case where $\psi = \gamma(f)$ and $\psi' = \gamma(f')$ for two $\mathcal{C}$-morphisms $f, f' \colon \bigvee_{i\in I} X_i \longrightarrow Y$. Because

$$\gamma(f\kappa_j) \ = \ \psi\circ\gamma(\kappa_j) \ = \ \psi'\circ\gamma(\kappa_j) = \gamma(f'\kappa_j) \ ,$$

the calculus of left fractions provides acyclic cofibrations $t_j \colon Y \longrightarrow \bar{Y}_j$ such that $t_j f\kappa_j \colon X_i \longrightarrow \bar{Y}_j$ is homotopic to $t_j f'\kappa_j \colon X_j \longrightarrow \bar{Y}_j$ for every $j \in I$. We choose a pushout:

$$\begin{tikzcd}
\bigvee_{i\in I} Y \arrow[r, "\nabla"] \arrow[d, tail, "\bigvee t_i"', "\sim"] & Y \arrow[d, tail, "t", "\sim"'] \\
\bigvee_{i\in I} \bar{Y}_i \arrow[r, "\nabla'"'] & Y'
\end{tikzcd}$$

Since coproducts of acyclic cofibrations are acyclic cofibrations, the left vertical morphism is an acyclic cofibration, and hence so is the right vertical morphism $t \colon Y \longrightarrow Y'$.

For each $j \in I$, we choose a cylinder object Z_j of X_j and a homotopy $H_j \colon Z_j \longrightarrow \bar{Y}_j$ from $t_j f\kappa_j$ to $t_j f'\kappa_j$. Since coproducts preserve cofibrations and acyclic cofibrations, the coproduct $\bigvee_{i\in I} Z_i$ is a cylinder object for $\bigvee_{i\in I} X_i$, where we leave the additional data of a cylinder object implicit. Moreover, the composite

$$\bigvee_{i\in I} Z_i \xrightarrow{\bigvee H_i} \bigvee_{i\in I} \bar{Y}_i \xrightarrow{\nabla'} Y'$$

is then a homotopy from tf to tf'. We conclude that $\gamma(tf) = \gamma(tf')$ in $\mathrm{Ho}(\mathcal{C})$. Since t is a weak equivalence, $\gamma(t)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$, and so $\gamma(f) = \gamma(f')$. This proves injectivity in the special case.

Now we treat the general case, and we let $\psi, \psi' \colon \bigvee_{i\in I} X_i \longrightarrow Y$ be arbitrary morphisms in $\mathrm{Ho}(\mathcal{C})$ such that $\psi\circ\gamma(\kappa_j) = \psi'\circ\gamma(\kappa_j)$ for all $j \in I$. The calculus of left fractions provides $\mathcal{C}$-morphisms $f \colon \bigvee_{i\in I} X_i \longrightarrow$

W, $f' \colon \bigvee_{i \in I} X_i \longrightarrow W'$, $s \colon Y \longrightarrow W$ and $s' \colon Y \longrightarrow W'$ such that s and s' are acyclic cofibrations and such that

$$\psi \;=\; \gamma(s)^{-1} \circ \gamma(f) \qquad \text{and} \qquad \psi' \;=\; \gamma(s')^{-1} \circ \gamma(f') \; .$$

We choose a pushout:

$$\begin{tikzcd} Y \arrow[r, tail, "s", "\sim"'] \arrow[d, tail, "s'"', "\sim"] & W \arrow[d, tail, "\sim"', "t"] \\ W' \arrow[r, tail, "\sim", "t'"'] & V \end{tikzcd}$$

Then t and t' are acyclic cofibrations because s and s' are. We now obtain the relation

$$\gamma(tf) \circ \gamma(\kappa_j) \;=\; \gamma(t) \circ \gamma(s) \circ \psi \circ \gamma(\kappa_j) \;=\; \gamma(t') \circ \gamma(s') \circ \psi' \circ \gamma(\kappa_j) \;=\; \gamma(t'f') \circ \gamma(\kappa_j)$$

for every $j \in I$. The special case treated above lets us conclude that $\gamma(tf) = \gamma(t'f')$. Thus

$$\gamma(ts) \circ \psi \;=\; \gamma(tf) \;=\; \gamma(t'f') \;=\; \gamma(t's') \circ \psi' \; .$$

Because the morphism $\gamma(ts) = \gamma(t's')$ is an isomorphism, also $\psi = \psi'$. This completes the proof. □

To establish further properties of the stable homotopy category we will exploit that the stable equivalences of sequential spectra participate a certain cofibration structure.

Definition 4.18. A morphism of sequential spectra $i \colon A \longrightarrow B$ is an *h-cofibration* if it has the following homotopy extension property. For every morphism of sequential spectra $\varphi \colon B \longrightarrow X$ and every homotopy $H \colon A \wedge [0,1]_+ \longrightarrow X$ such that $H_0 = \varphi i$, there is a homotopy $\bar{H} \colon B \wedge [0,1]_+ \longrightarrow X$ such that $H_0 = \varphi$ and $\bar{H} \circ (i \wedge [0,1]_+) = H$.

Much like in the context of topological spaces, there is a universal test case for the homotopy extension problem, namely when X is the pushout:

$$\begin{tikzcd} A \arrow[r, "{(-,0)}"] \arrow[d, "i"'] & A \wedge [0,1]_+ \arrow[d, "H"] \\ B \arrow[r, "\varphi"'] & B \cup_i (A \wedge [0,1]_+) \end{tikzcd}$$

So a morphism $i \colon A \longrightarrow B$ is an h-cofibration if and only if the canonical morphism

$$B \cup_i (A \wedge [0,1]_+) \;\longrightarrow\; B \wedge [0,1]_+$$

has a retraction. Also, the adjunction between $- \wedge [0,1]_+$ and $(-)^{[0,1]}$ lets us rewrite any homotopy extension data (φ, H) in adjoint form as a commutative square:

$$\begin{tikzcd} A \arrow[r, "\hat{H}"] \arrow[d, "i"'] & X^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ B \arrow[r, "\varphi"'] & X \end{tikzcd}$$

A solution to the homotopy extension problem is adjoint to a lifting, i.e., a morphism $\lambda \colon B \longrightarrow X^{[0,1]}$ such that $\lambda i = \hat{H}$ and $\mathrm{ev}_0 \circ \lambda = \varphi$. So a morphism $i \colon A \longrightarrow B$ is an h-cofibration if and only if it has the left lifting property with respect to the evaluation morphisms $\mathrm{ev}_0 \colon X^{[0,1]} \longrightarrow X$ for all sequential spectra X.

The three equivalent characterizations of h-cofibrations quickly imply various closure properties: every class of morphisms that can be characterized by the left lifting property with respect to some other class has the closure properties listed, see Exercise E.9.

Proposition 4.19. (i) *The class of h-cofibrations of sequential spectra is closed under retracts, cobase change, coproducts and sequential compositions.*

(ii) *For every sequential spectrum X, the unique morphism $* \longrightarrow X$ from the initial sequential spectrum is an h-cofibration.*
(iii) *Every h-cofibration of sequential spectra is levelwise an h-cofibration of based spaces.*

Theorem 4.20. *The category $\mathcal{S}p^{\mathbb{N}}$ of sequential spectra is a cofibration category with respect to the h-cofibrations and the stable equivalences.*

Proof. Most of the axioms are straightforward from the definitions. For (C1) we note that clearly, all isomorphisms are h-cofibrations and stable equivalences. Any sequential spectrum all of whose levels are one-point spaces is an initial object of $\mathcal{S}p^{\mathbb{N}}$. And the unique morphism from an initial sequential spectrum to any other sequential spectrum is an h-cofibration by Proposition 4.19 (ii).

The class of stable equivalences clearly satisfies the 2-out-of-3-property (C2). The h-cofibrations are closed under cobase change by Proposition 4.19 (i); and h-cofibrations that are simultaneously stable equivalences are stable under cobase change by Proposition 1.27. So axiom (C3) holds.

Finally, a morphism $X \longrightarrow Y$ of sequential spectra factors in the category $\mathcal{S}p^{\mathbb{N}}$ as the composite of the mapping cylinder inclusion $- \wedge 0\colon X \longrightarrow X \wedge [0,1]_+ \cup_f Y$, followed by the projection $X \wedge [0,1]_+ \cup_f Y \longrightarrow Y$ to the 'end' of the cylinder. This projection is a homotopy equivalence of sequential spectra, and hence a stable equivalence. The following square is a pushout:

$$\begin{tikzcd} X \amalg X \arrow[r, "\mathrm{Id}_X \amalg f"] \arrow[d, tail, "(-\wedge 0)+(-\wedge 1)"'] & X \amalg Y \arrow[d, tail] \\ X \wedge [0,1]_+ \arrow[r] & X \wedge [0,1]_+ \cup_f Y \end{tikzcd}$$

The left vertical morphism is an h-cofibration, hence so is the right vertical morphism. Because the summand inclusion $X \longrightarrow X \amalg Y$ is an h-cofibration, too, we conclude that the mapping cylinder inclusion $-\wedge 0\colon X \longrightarrow X \wedge [0,1]_+ \cup_f Y$ is an h-cofibration. This verifies the factorization axiom (C4). □

Example 4.21. We consider the cofibration structure on the category of sequential spectra from Theorem 4.20. For every sequential spectrum X the inclusions of the endpoints of the interval and the unique map from $[0,1]$ to a one-point space induce a factorization

$$X \vee X \xrightarrow{i_0 + i_1} X \wedge [0,1]_+ \xrightarrow{p} X$$

as an h-cofibration followed by a homotopy equivalence of sequential specta. So the 'cylinder' $X \wedge [0,1]_+$ is indeed a cylinder object in the abstract sense of Definition 4.4, and morphisms of sequential spectra that are homotopic in the 'concrete' sense (where a homotopy is a morphism defined on $X \wedge [0,1]_+$) are also homotopic in the 'abstract' sense (i.e., where a homotopy is a morphism defined on a general cylinder object).

One should beware, however, that the converse is *not* true: if f and g are homotopic in the abstract sense of Definition 4.4, then there need not be a 'classical homotopy' defined on $X \wedge [0,1]_+$.

We already proved in Theorem 3.18 that the stable homotopy category is additive, which includes the existence of finite products and coproducts. Now we can show that the stable homotopy catgeory in fact has arbitrary product and coproducts.

Theorem 4.22.
(i) *The stable homotopy category has arbitrary coproducts, and the localization functor $\gamma\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{SH}$ preserves arbitrary coproducts.*
(ii) *The stable homotopy category has arbitrary products. The localization functor $\gamma\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{SH}$ preserves arbitrary products of families of Ω-spectra.*

Proof. (i) We have shown in Theorem 4.20 that the stable equivalences participate in a cofibration structure on the category of sequential spectra. And we proved in Proposition 1.27 (i) that arbitrary wedges of stable equivalences are stable equivalences. Proposition 4.16 then shows that $\mathcal{SH} = \mathrm{Ho}(\mathcal{S}p^{\mathbb{N}})$ has coproducts, and that the localization functor $\gamma\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{SH}$ preserves these.

(ii) We exploit that the full subcategory $\mathcal{S}p^{\mathbb{N}}_{\Omega}$ of Ω-spectra supports a *fibration structure*, i.e., the classes of level equivalences and levelwise Serre fibrations satisfy the dual axioms of Definition 4.1. More formally, the classes of level equivalences and levelwise Serre fibrations form a cofibration structure on the opposite category $(\mathcal{S}p^{\mathbb{N}}_{\Omega})^{\mathrm{op}}$. The verification of this claim is straightforward from the more basic fact that the classes of weak homotopy equivalences and Serre fibrations form a fibration structure on the category of compactly generated spaces.

Because the loop functor preserves all weak equivalences and commutes with arbitrary products, and because products in $\mathcal{S}p^{\mathbb{N}}$ are levelwise, the full subcategory $\mathcal{S}p^{\mathbb{N}}_{\Omega}$ of Ω-spectra is closed under arbitrary products in $\mathcal{S}p^{\mathbb{N}}$. In particular, $\mathcal{S}p^{\mathbb{N}}_{\Omega}$ admits arbitrary products. Moreover, arbitrary products of weak homotopy equivalences of spaces are again weak equivalences. So the class of level equivalences is closed under arbitrary products. Finally, the class of Serre fibrations is closed under arbitrary products. We can thus apply the dual of Proposition 4.16 to the fibration category $(\mathcal{S}p^{\mathbb{N}}_{\Omega}, \text{level equiv.}, \text{level Serre fibr.})$, and deduce that the localization

$$\mathrm{Ho}(\mathcal{S}p^{\mathbb{N}}_{\Omega}) \ = \ \mathcal{S}p^{\mathbb{N}}_{\Omega}[\text{level equiv}^{-1}]$$

admits arbitrary products, and that the localiztion functor preserves theses. By Corollary 2.22, the inclusion $\mathcal{S}p^{\mathbb{N}}_{\Omega} \longrightarrow \mathcal{S}p^{\mathbb{N}}$ descends to an equivalence of localizations

$$\mathcal{S}p^{\mathbb{N}}_{\Omega}[\text{level eq}^{-1}] \ \longrightarrow \ \mathcal{S}p^{\mathbb{N}}[\text{stable eq}^{-1}] = \mathcal{SH} \ .$$

So also the stable homotopy category has arbitrary products. And for families of Ω-spectra, the product in $\mathcal{SH}$ is given by the product in $\mathcal{S}p^{\mathbb{N}}_{\Omega}$. □

Example 4.23. In Remark 1.25 we exhibited a countably family $(X^i)_{i\in\mathbb{N}}$ of sequential spectra each of which is stably trivial, but whose product $\prod_{\mathbb{N}} X^i$ is not stably trivial. Any such example also shows that the localization functor $\gamma\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{SH}$ does not in general preserve infinite products. Indeed, every sequential spectrum with vanishing stable homotopy groups becomes isomorphic to the trivial spectrum in $\mathcal{SH}$, and is thus a terminal object in $\mathcal{SH}$. In any category, every product of terminal objects is again terminal. So the target of the limit exchange morphism

$$\gamma(\prod_{\mathbb{N}} X^i) \ \longrightarrow \ \prod_{\mathbb{N}} \gamma(X^i) \tag{4.24}$$

is a terminal object. We write $*$ for a trivial (i.e., terminal) sequential spectrum, and $j\colon \prod_{\mathbb{N}} X^i \longrightarrow *$ for the unique morphisms. If $\gamma(\prod_{\mathbb{N}} X^i)$ were terminal in $\mathcal{SH}$, then $\gamma(j)$ would be an isomorphism in $\mathcal{SH}$, and thus j would be stable equivalence by Theorem 3.18 (iii). This is a contradiction, so the limit exchange morphism (4.24) is not an isomorphism, and the localization functor γ does not preserve this particular product.

While it can generally be difficult to explicitly describe morphism sets in a localization of a category at a random class of morphisms, we can calculate certain morphisms sets in $\mathcal{SH}$ using only the universal property as a localization, see the following Theorem 4.27.

Corollary 4.25. *Let X and Y be sequential spectra. Every morphism $\phi\colon X \longrightarrow Y$ in the stable homotopy category $\mathcal{SH}$ is of the form $\phi = \gamma(h)^{-1} \circ \gamma(g)$ for some Ω-spectrum Z, and some morphisms $g\colon X \longrightarrow Z$ and $h\colon Y \longrightarrow Z$ such that h is a stable equivalence.*

Proof. The cofibration structure on the category of sequential spectra yields a calculus of fractions for the stable homotopy category $\mathcal{SH}$, as detailed in Theorem 4.13. In particular, the morphism ϕ is a left fraction of the form $\phi = \gamma(\tau)^{-1} \circ \gamma(f)$, where $f\colon X \longrightarrow E$ and $\tau\colon Y \longrightarrow E$ are morphisms of sequential sepctra with the same target, and τ is a stable equivalence. Theorem 2.19 provides a stable equivalence $i\colon E \longrightarrow Z$ whose target is an Ω-spectrum. Then $\gamma(i)$ is an isomorphism, and thus

$$\begin{aligned} \phi \ &= \ \gamma(\tau)^{-1} \circ \gamma(f) \\ &= \ \gamma(\tau)^{-1} \circ \gamma(i)^{-1} \circ \gamma(i) \circ \gamma(f) \\ &= \ \gamma(i \circ \tau)^{-1} \circ \gamma(i \circ f) \ . \end{aligned}$$

So the morphisms $h = i \circ \tau$ and $g = i \circ f$ have the desired properties. $\square$

Construction 4.26. We let E be a sequential spectrum, and we let A be a based space. We define a set $E\{A\}$ by

$$E\{A\} \;=\; \mathrm{colim}_n\, [S^n \wedge A, E_n]_* \;,$$

where $[-,-]_*$ denotes the set of based homotopy classes of based continuous maps. The colimit is taken along the maps

$$[S^n \wedge A, E_n]_* \xrightarrow{S^1\wedge -} [S^1 \wedge S^n \wedge A, S^1 \wedge E_n]_* \xrightarrow{(\sigma_n)_*} [S^{1+n} \wedge A, E_{1+n}]_* \;.$$

Some comments about the construction $E\{A\}$ are in order.

- The construction $E\{A\}$ is covariantly functorial in the sequential spectrum E, and contravariantly functorial in the based space A.
- For $n \geq 2$, the set $[S^n \wedge A, E_n]_*$ has a natural abelian group structure by 'pinch sum', using any pinch map of S^n. In other words, if $f, g\colon S^n \wedge A \longrightarrow X$ represent two classes, then their sum is represented by the composite

  $$S^n \wedge A \xrightarrow{\text{pinch}} (S^n \vee S^n) \wedge A \;\cong\; (S^n \vee A) \wedge (S^n \vee A) \xrightarrow{f+g} X \;.$$

  The stabilization maps are homomorphisms, so the colimit $E\{A\}$ inherits an abelian group structure that is natural in E and in A.
- For $A = S^k$, the preferred homeomorphisms $S^n \wedge S^k \cong S^{n+k}$ let us identify $[S^n \wedge S^k, E_n]_*$ with $[S^{n+k}, E_n]_* = \pi_{n+k}(E_n)$; under these bijections, the stabilization maps coincide with the stabilization maps that define the stable homotopy group $\pi_k(E)$. So in the colimit over n, we obtain a natural isomorphism

  $$E\{S^k\} \;\cong\; \pi_k(E) \;.$$
- Under the adjunction bijections

  $$[S^n \wedge A, E_n]_* \;=\; [S^n, \mathrm{map}_*(A, E_n)]_* \;=\; \pi_n(\mathrm{map}_*(A, E)_n)$$

  the stabilization maps coincide with the stabilization maps that define the stable homotopy group $\pi_0(\mathrm{map}_*(A, E))$. So we obtain another natural isomorphism

  $$E\{A\} \;\cong\; \pi_0(\mathrm{map}_*(A, E)) \;.$$

We define the *tautological class*

$$\iota_A \;\in\; (\Sigma^\infty A)\{A\}$$

as the class represented by the identity of $S^0 \wedge A$.

Theorem 4.27. *Let A be a based space that admits the structure of a finite CW-complex.*

(i) *The functor*

$$(-)\{A\} \;:\; \mathcal{S}p^{\mathbb{N}} \;\longrightarrow\; (\text{sets})$$

takes stable equivalences to bijections.

(ii) *For every sequential spectrum E the evaluation map*

$$\mathcal{SH}(\Sigma^\infty A, E) \;\longrightarrow\; E\{A\} \;, \quad f \;\longmapsto f\{A\}(\iota_A)$$

is bijective. In other words, the pair $(\Sigma^\infty_+ A, \iota_A)$ represents the functor

$$(-)\{A\} \;:\; \mathcal{SH} \;\longrightarrow\; (\text{sets}) \;, \quad E \;\longmapsto\; E\{A\} \;.$$

Proof. (i) Since A admits the structure of a finite CW-complex, then $\mathrm{map}_*(A, -)$ preserves stable equivalences by Proposition 1.27 (vii). So the functor

$$\mathcal{S}p^{\mathbb{N}} \;\longrightarrow\; \mathcal{A}b \;, \quad E \;\longmapsto\; \pi_0(\mathrm{map}_*(A, E))$$

takes stable equivalences to isomorphisms. Because $E\{A\}$ is naturally isomorphic to $\pi_0(\mathrm{map}_*(A, E))$, this proves the first claim.

(ii) We start with the special case where E is an Ω-spectrum. We define map

$$\alpha \ : \ [A, E_0] \ \longrightarrow \ \mathcal{SH}(\Sigma^\infty A, E)$$

by sending the homotopy class of a based continuous map $f\colon A \longrightarrow E_0$ to $\gamma(f^\flat)$, the image under the localization functor of the adjoint $f^\flat\colon \Sigma^\infty A \longrightarrow E$ of f. We claim that α is surjective. To show this we represent any given morphism $\phi\colon \Sigma^\infty A \longrightarrow E$ in $\mathcal{SH}$ as $\phi = \gamma(\tau)^{-1} \circ \gamma(f)$ for some Ω-spectrum Z, some morphism of sequential spectra $f\colon \Sigma^\infty A \longrightarrow Z$ and some stable equivalence $\tau\colon E \longrightarrow Z$; this is possible by Corollary 4.25. As a stable equivalence between Ω-spectra, the morphism $\tau\colon E \longrightarrow Z$ is a level equivalence by Proposition 2.3. In particular, $\tau_0\colon E_0 \longrightarrow Z_0$ is a weak equivalence of based spaces. In the commutative square

$$\begin{tikzcd}
{[A, E_0]} \arrow[r, "\alpha"] \arrow[d, "{[A,\tau_0]}"', "\cong"] & \mathcal{SH}(\Sigma^\infty A, E) \arrow[d, "{\mathcal{SH}(\Sigma^\infty A, \gamma(\tau))}"] \\
{[A, Z_0]} \arrow[r, "\alpha"'] & \mathcal{SH}(\Sigma^\infty A, Z)
\end{tikzcd}$$

the left map is then bijective because A is a CW-complex. And the right vertical map is bijective because $\gamma(\tau)$ is an isomorphism. Since $\gamma(f) = \alpha[f_0]$ is in the image of the lower horizontal morphism, and because $\mathcal{SH}(\Sigma^\infty A, \gamma(\tau))$ takes $\phi = \gamma(\tau)^{-1} \circ \gamma(f)$ to $\gamma(f)$, the morphism ϕ is in the image of the upper horizontal map. This concludes the proof that the map α is surjective.

Since E is an Ω-spectrum, and since A admits a CW-structure, all the maps in the colimit system for $E\{A\}$ are bijective, and hence also the canonical map $[A, E_0] \longrightarrow E\{A\}$ is bijective. The composite

$$[A, E_0] \ \xrightarrow{\alpha} \ \mathcal{SH}(\Sigma^\infty A, E) \ \longrightarrow \ E\{A\}$$

is the canonical map from $[A, E_0]$ to the colimit that defines $E\{A\}$, and thus bijective. Since the first map α is surjective by the above, the second evaluation map is bijective.

To treat the general case we use the stable equivalence $i\colon E \longrightarrow QE$ from Theorem 2.19. In the commutative square

$$\begin{tikzcd}
\mathcal{SH}(\Sigma^\infty A, E) \arrow[r] \arrow[d, "{\mathcal{SH}(\Sigma^\infty A, \gamma(i))}"'] & E\{A\} \arrow[d, "i\{A\}"] \\
\mathcal{SH}(\Sigma^\infty A, QE) \arrow[r] & (QE)\{A\}
\end{tikzcd}$$

the two vertical maps are bijective by part (i), and because $\gamma(i)$ is an isomorphism in $\mathcal{SH}$. The lower horizontal map is bijective by the special case, because QE is an Ω-spectrum. So the upper horizontal maps is bijective, too. $\square$

Example 4.28 (Representing π_n)**.** Theorem 4.27 (ii) effectively says that we can identify the morphism group $\mathcal{SH}(\Sigma^\infty A, E)$ in the stable homotopy category with the group $E\{A\}$ that was defined in concrete terms from the based space A and the spaces in the sequential spectrum of E.

For $A = S^k$, the group $E\{S^k\}$ identifies with the homotopy group $\pi_k(E)$, in a way that identifies the tautological class ι_{S^k} with the class $1 \wedge S^k \in \pi_k(\Sigma^\infty S^k)$, the image of the multiplicative unit $1 \in \pi_0(\mathbb{S})$ under the iterated suspension isomorphism $- \wedge S^k\colon \pi_0(\mathbb{S}) \cong \pi_k(\Sigma^\infty S^k)$. So for $A = S^k$, Theorem 4.27 (ii) specializes to a bijection

$$\mathcal{SH}(\Sigma^\infty S^k, E) \ \cong \ \pi_k(E) \ , \quad \psi \ \longmapsto \psi_*(1 \wedge S^k)$$

that is natural in the sequential spectrum E.

5. Triangulated categories

We have seen that the stable homotopy category, is an additive category with arbitrary coproducts and finite products. In this section we make $\mathcal{SH}$ into a triangulated category. The arguments apply more generally to certain classes of cofibration categories, and we work in that generality. First we recall the definition.

Let $\mathcal{T}$ be a category equipped with an endofunctor $\Sigma\colon \mathcal{T} \longrightarrow \mathcal{T}$. A *triangle* in $\mathcal{T}$ (with respect to the functor Σ) is a triple (f, g, h) of composable morphisms in $\mathcal{T}$ such that the target of h is equal to Σ applied to the source of f. We will often display a triangle in the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \ .$$

A *morphism* from a triangle (f, g, h) to a triangle (f', g', h') is a triple of morphisms $a\colon A \longrightarrow A'$, $b\colon B \longrightarrow B'$ and $c\colon C \longrightarrow C'$ in $\mathcal{T}$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r, "h"] \arrow[d, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A'
\end{tikzcd}$$

commutes. A morphism of triangles is an isomorphism (i.e., has an inverse morphism) if and only all three components are isomorphisms in $\mathcal{T}$.

Definition 5.1. A *triangulated category* is an additive category $\mathcal{T}$ equipped with a self-equivalence $\Sigma\colon \mathcal{T} \longrightarrow \mathcal{T}$ and a collection of triangles, called *distinguished triangles*, which satisfy the following axioms (T0) – (T5).

We refer to the equivalence Σ of a triangulated category as the *suspension*, since that is what it will be in our main example. In algebraic contexts, this equivalence is often denoted $X \mapsto X[1]$ and called the 'shift'.

- **(T0)** The class of distinguished triangles is closed under isomorphism.
- **(T1)** Every morphism f is part of a distinguished triangle (f, g, h).
- **(T2)** For every object X the triangle $0 \longrightarrow X \xrightarrow{\mathrm{Id}} X \longrightarrow 0$ is distinguished.
- **(T3)** [Rotation] If a triangle (f, g, h) is distinguished, then so is the triangle $(g, h, -\Sigma f)$.
- **(T4)** [Completion of triangles] Given distinguished triangles (f, g, h) and (f', g', h') morphisms (a, b) satisfying $bf = f'a$, there exists a morphism c making the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r, "h"] \arrow[d, dotted, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A'
\end{tikzcd}$$

- **(T5)** [Octahedral axiom] For every pair of composable morphisms $f\colon A \longrightarrow B$ and $f'\colon B \longrightarrow D$ there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "f'"'] & C \arrow[r, "h"] \arrow[d, "x"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f'f"'] & D \arrow[r, "g''"'] \arrow[d, "g'"'] & E \arrow[r, "h''"'] \arrow[d, "y"] & \Sigma A \arrow[d, "\Sigma f"] \\
 & F \arrow[r, equal] \arrow[d, "h'"'] & F \arrow[r, "h'"'] \arrow[d, "(\Sigma g)\circ h'"] & \Sigma B \\
 & \Sigma B \arrow[r, "\Sigma g"'] & \Sigma C &
\end{tikzcd}$$

  such that the triangles (f, g, h), (f', g', h'), $(f'f, g'', h'')$ and $(x, y, (\Sigma g) \circ h')$ are distinguished.

The above formulation of the axioms appears to be weaker, at first sight, than the original axioms of Verdier [54, II.1]; however, we show in Proposition 5.16 below that the weaker axioms imply the stronger properties: part (iii) establishes an 'if and only if' in the rotation axiom (T3), and part (iv) is the octahedral axiom in its original form.

We recall that a category $\mathcal{C}$ is *pointed* if it has a *zero object*, i.e., an object that is simultaneously initial and terminal; we will denote zero objects by '$*$'. In pointed categories, we will denote a coproduct of two objects X and Y by $X \vee Y$.

Definition 5.2. Let $\mathcal{C}$ be a pointed cofibration category. A *functorial cone* consists of a functor $C\colon \mathcal{C} \longrightarrow \mathcal{C}$ and a natural transformation $\iota\colon \mathrm{Id} \longrightarrow C$ such that for all $\mathcal{C}$-objects X, the morphism $\iota_X\colon X \longrightarrow CX$ is a cofibration, and the unique morphism $CX \longrightarrow *$ is a weak equivalence.

The main example for our purposes is the category of sequential spectra. There, the standard cone $X \wedge [0,1]$ (which is levelwise the reduced cone), together with the 'end point inclusion' $- \wedge 1\colon X \longrightarrow X \wedge [0,1]$ form a functorial cone for the cofibration structure of Theorem 4.20, compare also Example 4.21. However, there are many other examples, see Example 5.12 or Example 5.14 below.

Construction 5.3. Let $\mathcal{C}$ be a pointed cofibration category with a functorial cone. Then the *suspension functor*

$$\Sigma \;:\; \mathcal{C} \;\longrightarrow\; \mathcal{C}$$

is defined by

$$\Sigma X \;=\; (CX)/\iota_X \;,$$

i.e., ΣX is defined as a pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail, "\iota_X"'] & * \arrow[d] \\ CX \arrow[r, "q_X"'] & \Sigma X \end{tikzcd}$$

Any morphism $CX \longrightarrow CY$ is a weak equivalence by the 2-out-of-3 property, because the unique morphisms $CX \longrightarrow *$ and $CY \longrightarrow *$ are weak equivalences. So if $f\colon X \longrightarrow Y$ is a weak equivalence, then the gluing lemma (Proposition 4.3) shows that the morphism $\Sigma f\colon \Sigma X \longrightarrow \Sigma Y$ is a weak equivalence.

Because the suspension functor preserves weak equivalences, it descends to a functor on the homotopy category $\Sigma\colon \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$, for which we use the same name. Indeed,the composite functor $\gamma \circ \Sigma\colon \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ takes weak equivalences to isomorphisms. The universal property of the localization functor $\gamma\colon \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ provides a unique functor

$$\Sigma \;:\; \mathrm{Ho}(\mathcal{C}) \;\longrightarrow\; \mathrm{Ho}(\mathcal{C})$$

that satisfies $\Sigma \circ \gamma = \gamma \circ \Sigma$. In particular, Σ is given on objects by the previous suspension functor, and the behavior on morphisms is as follows. Every morphism $X \longrightarrow Y$ in $\mathrm{Ho}(\mathcal{C})$ is of the form $\gamma(\tau)^{-1} \circ \gamma(f)$ for two $\mathcal{C}$-morphisms $f\colon X \longrightarrow Z$ and $\tau\colon Y \longrightarrow Z$ such that τ is a weak equivalence. Then $\Sigma\tau\colon \Sigma X \longrightarrow \Sigma Z$ is a weak equivalence, too, and

$$\Sigma(\gamma(\tau)^{-1} \circ \gamma(f)) \;=\; \gamma(\Sigma\tau)^{-1} \circ \gamma(\Sigma f) \;.$$

Construction 5.4. We introduce the distinguished triangles in the homotopy category of a pointed cofibration category with functorial cones. The *elementary distinguished triangle* associated to a $\mathcal{C}$-morphism $\psi\colon X \longrightarrow Y$ is the sequence

$$X \xrightarrow{\;\gamma(\psi)\;} Y \xrightarrow{\;\gamma(i)\;} C\psi \xrightarrow{\;\gamma(p)\;} \Sigma X \;.$$

Here $C\psi$ is the mapping cone of ψ, defined by a pushout square

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, tail, "\iota_X"'] & Y \arrow[d, tail, "i"] \\ CX \arrow[r] & C\psi \end{tikzcd}$$

The third morphism is $p = q_X \cup 0\colon C\psi = CX \cup_\psi Y \longrightarrow \Sigma X$.

A *distinguished triangle* is any triangle (f, g, h) in $\mathrm{Ho}(\mathcal{C})$ which is isomorphic to an elementary distinguished triangle, i.e., such that there is a $\mathcal{C}$-morphism $\psi\colon X \longrightarrow Y$ and isomorphisms $a\colon X \longrightarrow A$, $b\colon Y \longrightarrow B$ and $c\colon C\psi \longrightarrow C$ in $\mathrm{Ho}(\mathcal{C})$ that make the diagram

$$\begin{tikzcd}
X \arrow[r, "\gamma(\psi)"] \arrow[d, "a"', "\cong"] & Y \arrow[r, "\gamma(i)"] \arrow[d, "b"', "\cong"] & C\psi \arrow[r, "\gamma(p)"] \arrow[d, "\cong"', "c"] & \Sigma X \arrow[d, "\cong"', "\Sigma a"] \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma A
\end{tikzcd}$$

commute.

Construction 5.5 (Distinguished triangles from cofibrations)**.** As we shall now explain, cofibrations are another source of distinguished triangles. Given any cofibration $j\colon A \longrightarrow B$ in $\mathcal{C}$, we write B/A for any cokernel of j, and $q\colon B \longrightarrow B/A$ for the projection. This is a slight abuse of notation, because B/A depends on the morphism j (and not just on its source and target). Applying the gluing lemma to the commutative diagram

$$\begin{tikzcd}
CA \arrow[d, "\sim"'] & A \arrow[l, tail, "\iota_A"'] \arrow[r, tail, "j"] \arrow[d, equal] & B \arrow[d, equal] \\
* & A \arrow[l] \arrow[r, tail, "j"'] & B
\end{tikzcd}$$

shows that the morphism

$$0 \cup q \;:\; Cj \;=\; CA \cup_j B \;\longrightarrow\; B/A$$

is a weak equivalence. We define the *connecting morphism* $\delta(j)\colon B/A \longrightarrow \Sigma A$ in $\mathrm{Ho}(\mathcal{C})$ as

$$\delta(j) \;=\; \gamma(p) \circ \gamma(0 \cup q)^{-1} \;\;: B/A \;\longrightarrow\; \Sigma A\ .$$

Here $p = (q_X \cup 0)\colon Cj \longrightarrow \Sigma A$ is the 'projection' that was already considered above. The following diagram commutes by definition:

$$\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, equal] & B \arrow[r, "\gamma(i)"] \arrow[d, equal] & Cj \arrow[r, "\gamma(p)"] \arrow[d, "\cong"', "\gamma(0\cup q)"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(j)"'] & B \arrow[r, "\gamma(q)"'] & B/A \arrow[r, "\delta(j)"'] & \Sigma A
\end{tikzcd}$$

The upper row is an elementary distinguished triangle and all vertical morphisms are isomorphisms. So the lower triangle is distinguished.

Proposition 5.6. *Let $\mathcal{C}$ be a pointed cofibration category with functorial cones. A triangle if $\mathrm{Ho}(\mathcal{C})$ is distinguished if and only if it is isomorphic to a triangle of the form $(\gamma(j), \gamma(q), \delta(j))$ for some cofibration $j\colon A \longrightarrow B$.*

Proof. We already showed that triangles of the form $(\gamma(j), \gamma(q), \delta(j))$ for cofibrations j are distinguished. For the reverse implication we consider any $\mathcal{C}$-morphism $\psi\colon X \longrightarrow Y$. We choose a factorization $\psi = \pi j$ for some cofibration $j\colon X \longrightarrow Z$ and some weak equivalence $\pi\colon Z \longrightarrow Y$. All vertical morphisms in the following commutative diagram are weak equivalences, either by definition of by the gluing lemma:

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, equal] & Z \arrow[r, "q"] \arrow[d, equal] & Z/X & \\
X \arrow[r, "j"] \arrow[d, equal] & Z \arrow[r, "i_j"] \arrow[d, "\sim"', "\pi"] & Cj \arrow[u, "\sim", "0\cup q"'] \arrow[r, "p_j"] \arrow[d, "\sim"', "\mathrm{Id}_{CX}\cup\pi"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "\psi"'] & Y \arrow[r, "i_\psi"'] & C\psi \arrow[r, "p_\psi"'] & \Sigma X
\end{tikzcd}$$

After applying the localization functor, we thus obtain a commutative diagram in $\mathrm{Ho}(\mathcal{C})$ in which all vertical morphisms are isomorphisms:

$$\begin{tikzcd} X \arrow[r, "\gamma(j)"] \arrow[d, equal] & Z \arrow[r, "\gamma(q)"] \arrow[d, "\gamma(\pi)", "\cong"'] & Z/X \arrow[r, "\delta(j)"] \arrow[d, "\gamma(\mathrm{Id}\cup\pi)\circ\gamma(0\cup q)^{-1}", "\cong"'] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "\gamma(\psi)"'] & Y \arrow[r, "\gamma(i_\psi)"'] & C\psi \arrow[r, "\gamma(p_\psi)"'] & \Sigma X \end{tikzcd}$$

This shows that the elementary distinguished triangle of $\psi\colon X \longrightarrow Y$ is isomorphic to one of the form $(\gamma(j), \gamma(q), \delta(j))$. $\square$

Proposition 5.7. *Let $\mathcal{C}$ be a pointed cofibration category with functorial cones, and suppose that $\mathrm{Ho}(\mathcal{C})$ is additive. Let $\psi\colon X \longrightarrow Y$ be a $\mathcal{C}$-morphism. Then the connecting homomorphism of the cofibration $i\colon Y \longrightarrow C\psi$ satisfies the relation*

$$\delta(i) \; = \; -\Sigma\gamma(\psi) \; : \; \Sigma X \; \longrightarrow \; \Sigma Y \; .$$

Proof. We start by showing that the two morphisms

$$q_Y \cup 0 \; , \; 0 \cup q_Y \; : \; CY \cup_Y CY \; \longrightarrow \; \Sigma Y$$

become additive inverses in the abelian monoid $\mathrm{Ho}(CY \cup_Y CY, \Sigma Y)$. To this end we consider the morphism $\xi\colon CY \cup_Y CY \longrightarrow \Sigma Y \vee \Sigma Y$ induced by taking horizontal pushouts of the commutative diagram

$$\begin{tikzcd} CY \vee CY \arrow[d, equal] & Y \vee Y \arrow[l, tail, "\iota_Y \vee \iota_Y"'] \arrow[r, "\nabla"] \arrow[d, equal] & Y \arrow[d, "\sim"] \\ CY \vee CY & Y \vee Y \arrow[l, tail, "\iota_Y \vee \iota_Y"] \arrow[r] & * \end{tikzcd}$$

Then

$$(\mathrm{Id}_{\Sigma Y} + 0) \circ \xi \; = \; q_Y \cup 0 \qquad \text{and} \qquad (0 + \mathrm{Id}_{\Sigma Y}) \circ \xi \; = \; 0 \cup q_Y \; .$$

This means that

$$\gamma(\xi) \; = \; \gamma(q_Y \cup 0) \perp \gamma(0 \cup q_Y)$$

as morphisms in $\mathrm{Ho}(\mathcal{C})$. The following square commutes in $\mathcal{C}$:

$$\begin{tikzcd} CY \cup_Y CY \arrow[r, "\xi"] \arrow[d, "\mathrm{Id}\cup\mathrm{Id}"'] & \Sigma Y \vee \Sigma Y \arrow[d, "\nabla"] \\ CY \arrow[r, "q_Y"'] & \Sigma Y \end{tikzcd}$$

Because the cone CY is weakly equivalent to the zero object, it becomes a zero object in $\mathrm{Ho}(\mathcal{C})$. We conclude that

$$\gamma(q_Y \cup 0) + \gamma(0 \cup q_Y) \; = \; \gamma(\nabla) \circ (\gamma(q_Y \cup 0) \perp \gamma(0 \cup q_Y)) \; = \; \gamma(\nabla) \circ \gamma(\xi) \; = \; 0$$

in the abelian monoid $\mathrm{Ho}(CY \cup_Y CY, \Sigma Y)$. This proves that $\gamma(q_Y \cup 0) = -\gamma(0 \cup q_Y)$

Now we consider the morphism $\zeta\colon CY \cup_i C\psi \longrightarrow CY \cup_Y CY$, induced by taking horizontal pushouts of the commutative diagram

$$\begin{tikzcd} CY \vee CX \arrow[d, "\mathrm{Id}\vee C(\psi)"'] & X \vee X \arrow[l, tail, "(\iota_Y \circ \psi) \vee \iota_X"'] \arrow[r, "\nabla"] \arrow[d, "\psi\vee\psi"'] & X \arrow[d, "\psi"] \\ CY \vee CY & Y \vee Y \arrow[l, tail, "\iota_Y \vee \iota_Y"] \arrow[r, "\nabla"'] & Y \end{tikzcd}$$

Then

$$(q_Y \cup 0) \circ \zeta \; = \; q_Y \cup 0 \qquad \text{and} \qquad (0 \cup q_Y) \circ \zeta \; = \; (\Sigma\psi) \circ (0 \cup p)$$

as $\mathcal{C}$-morphisms $CY \cup_i C\psi \longrightarrow \Sigma Y$ Thus

$$\begin{aligned}
\delta(i) &= \gamma(q_Y \cup 0) \circ \gamma(0 \cup p)^{-1} \\
&= \gamma(q_Y \cup 0) \circ \gamma(\zeta) \circ \gamma(0 \cup p)^{-1} \\
&= -\gamma(0 \cup q_Y) \circ \gamma(\zeta) \circ \gamma(0 \cup p)^{-1} \\
&= -\gamma(\Sigma\psi) \circ \gamma(0 \cup p) \circ \gamma(0 \cup p)^{-1} \ = \ -\gamma(\Sigma\psi) \ .
\end{aligned}$$

□

Now we can state and prove the main result of this section.

Theorem 5.8. *Let $\mathcal{C}$ be a pointed cofibration category with functorial cones. Suppose moreover that* $\mathrm{Ho}(\mathcal{C})$ *is additive, and that the suspension functor* $\Sigma\colon \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$ *is an autoequivalence of categories. Then the suspension functor and the class of distinguished triangles make the derived category* $\mathrm{Ho}(\mathcal{C})$ *into a triangulated category.*

Proof. It remains to prove the axioms (T0)–(T5).

(T0) By definition, the class of distinguished triangles is closed under isomorphism.

(T1) We let $f\colon A \longrightarrow B$ be a morphism in $\mathrm{Ho}(\mathcal{C})$. We appeal to the calculus of fractions (Theorem 4.13) to write $f = \gamma(s)^{-1} \circ \gamma(\psi)$ for two $\mathcal{C}$-morphisms $\psi\colon A \longrightarrow D$ and $s\colon B \longrightarrow D$ such that s is a weak equivalence. Then the following diagram of triangles commutes in $\mathrm{Ho}(\mathcal{C})$:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "\gamma(is)"] \arrow[d, "\gamma(s)"', "\cong"] & C\psi \arrow[r, "\gamma(p)"] \arrow[d, equal] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(\psi)"'] & D \arrow[r, "\gamma(i)"'] & C\psi \arrow[r, "\gamma(p)"'] & \Sigma A
\end{tikzcd}$$

All vertical morphisms in the diagram are isomorphisms, and the lower row is an elementary distinguished triangle. So the upper row is the desired distinguished triangle starting with f.

(T2) The identity $\mathrm{Id}_X\colon X \longrightarrow X$ is a cokernel of the unique morphism $0\colon * \longrightarrow X$. So the triangle $(0, \mathrm{Id}_X, 0)$ is distinguished by Construction 5.5, applied to the cofibration $0\colon * \longrightarrow X$.

(T3 – Rotation) We let (f, g, h) be a distinguished triangle; we need to show that the triangle $(g, h, -\Sigma f)$ is also distinguished. Since the class of distinguished triangles is closed under isomorphisms, it suffices to consider the elementary distinguished triangle $(\gamma(\psi), \gamma(i), \gamma(p))$ associated to a $\mathcal{C}$-morphism $\psi\colon X \longrightarrow Y$. The morphism $i\colon Y \longrightarrow C\psi$ is a cofibration, and the morphism $p = q_X \cup 0\colon C\psi = CX \cup_\psi Y \longrightarrow \Sigma X$ exhibits the suspension ΣX as a cokernel of i. So the triangle

$$Y \xrightarrow{\gamma(i)} C\psi \xrightarrow{\gamma(p)} \Sigma X \xrightarrow{\delta(i)} \Sigma Y$$

is distinguished, as explained in Construction 5.5. By Proposition 5.7, the connecting morphism $\delta(i)$ of the cofibration $i\colon Y \longrightarrow C\psi$ is the *additive inverse* of the morphism $\Sigma\gamma(\psi) = \gamma(\Sigma\psi)\colon \Sigma X \longrightarrow \Sigma Y$. So the rotated triangle $(\gamma(i), \gamma(p), -\Sigma\gamma(\psi))$ is distinguished.

(T4 – Completion of triangles) We are given two distinguished triangles (f, g, h) and (f', g', h') and two morphisms a and b in $\mathrm{Ho}(\mathcal{C})$ satisfying $bf = f'a$ as in the diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, dashed, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A'
\end{tikzcd}$$

We have to extend this data to a morphism of triangles, i.e., to find a morphism c making the entire diagram commute. If we can solve the problem for isomorphic triangles, then we can also solve it for the original triangles. By Proposition 5.6 we can thus assume that the triangles (f,g,h) and (f',g',h') are the distinguished triangle arising from two cofibrations $j\colon A \longrightarrow B$ and $j'\colon A' \longrightarrow B'$ via Construction 5.5.

We start with the special case where $a = \gamma(\alpha)$ and $b = \gamma(\beta)$ for $\mathcal{C}$-morphisms $\alpha\colon A \longrightarrow A'$ and $\beta\colon B \longrightarrow B'$. Then $\gamma(j'\alpha) = \gamma(\beta j)$, so the calculus of fractions (Theorem 4.13 (ii)) provides an acyclic cofibration $s\colon B' \longrightarrow \bar{B}$, a cylinder object (I, i_0, i_1, p) for A and a homotopy $H\colon I \longrightarrow \bar{B}$ from $Hi_0 = sj'\alpha$ to $Hi_1 = s\beta j$. The following diagram of cofibrations on the left commutes in $\mathcal{C}$, so the diagram of distinguished triangles on the right commutes in $\mathrm{Ho}(\mathcal{C})$ by the naturality of the connecting morphisms:

$$\begin{tikzcd}
A \arrow[r, tail, "j"] \arrow[d, equal] & B \\
A \arrow[r, tail, "i_0"] \arrow[d, "\alpha"'] & I \cup_{i_1} B \arrow[u, "\sim", "jp\cup B"'] \arrow[d, "H\cup s\beta"] \\
A' \arrow[r, tail, "sj'"] \arrow[d, equal] & \bar{B} \\
A' \arrow[r, tail, "j'"'] & B' \arrow[u, "\sim", "s"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, equal] & B \arrow[r, "\gamma(q)"] & B/A \arrow[r, "\delta(j)"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(i_0)"] \arrow[d, "\gamma(\alpha)"'] & I \cup_{i_1} B \arrow[r, "\gamma(q)"] \arrow[u, "\gamma(jp\cup B)", "\cong"'] \arrow[d, "\gamma(H\cup s\beta)"'] & (I \cup_{i_1} B)/A \arrow[r, "\delta(i_0)"] \arrow[u, "\cong", "\gamma((jp\cup B)/A)"'] \arrow[d, "\gamma((H\cup s\beta)/\alpha)"] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\
A' \arrow[r, "\gamma(sj')"] \arrow[d, equal] & \bar{B} \arrow[r, "\gamma(\bar{q})"] & \bar{B}/A' \arrow[r, "\delta(sj')"] & \Sigma A' \arrow[d, equal] \\
A' \arrow[r, "\gamma(j')"'] & B' \arrow[r, "\gamma(q')"'] \arrow[u, "\gamma(s)", "\cong"'] & B'/A' \arrow[r, "\delta(j')"'] \arrow[u, "\cong", "\gamma(s/A')"'] & \Sigma A'
\end{tikzcd}$$

The canonical morphism $\kappa\colon B \longrightarrow I \cup_{i_1} B$ is right inverse to $jp \cup B\colon I \cup_{i_1} B \longrightarrow B$, so

$$\begin{aligned}
\gamma(s)^{-1} \circ \gamma(H \cup s\beta) \circ \gamma(jp \cup B)^{-1} &= \gamma(s)^{-1} \circ \gamma(H \cup s\beta) \circ \gamma(\kappa) \\
&= \gamma(s)^{-1} \circ \gamma(s\beta) \ = \ \gamma(\beta) \ = \ b\ .
\end{aligned}$$

So the morphism

$$c \ = \ \gamma(s/A')^{-1} \circ \gamma((H \cup s\beta)/\alpha) \circ \gamma((jp \cup B)/A)^{-1} \ : \ B/A \longrightarrow B'/A'$$

is the desired filler.

In the general case we write $a = \gamma(s)^{-1}\gamma(\alpha)$ where $\alpha\colon A \longrightarrow \bar{A}$ and $s\colon A' \longrightarrow \bar{A}$ are $\mathcal{C}$-morphisms and s is an acyclic cofibration. We choose a pushout

$$\begin{tikzcd}
\bar{A} \arrow[r, tail, "k"] & \bar{A} \cup_{A'} B' \\
A' \arrow[u, tail, "s", "\sim"'] \arrow[r, tail, "j'"'] & B' \arrow[u, tail, "\sim", "s'"']
\end{tikzcd}$$

Another application of the calculus of fractions lets us write $\gamma(s')b = \gamma(t)^{-1}\gamma(\beta)\colon B \longrightarrow \bar{A} \cup_{A'} B'$ where $\beta\colon B \longrightarrow \bar{B}$ and $t\colon \bar{A} \cup_{A'} B' \longrightarrow \bar{B}$ are $\mathcal{C}$-morphisms, and t is an acyclic cofibration. We then have

$$\gamma(tk)\gamma(\alpha) \ = \ \gamma(tk)\gamma(s)a \ = \ \gamma(ts')\gamma(j')a \ = \ \gamma(ts')b\gamma(j) \ = \ \gamma(\beta)\gamma(j)\ ,$$

so by the special case, applied to the cofibrations $j\colon A \longrightarrow B$ and $tk\colon \bar{A} \longrightarrow \bar{B}$ and the morphisms $\alpha\colon A \longrightarrow \bar{A}$ and $\beta\colon B \longrightarrow \bar{B}$, there exists a morphism $c\colon B/A \longrightarrow \bar{B}/\bar{A}$ in $\mathrm{Ho}(\mathcal{C})$ making the diagram

$$\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, "\gamma(\alpha)"'] & B \arrow[r, "\gamma(q)"] \arrow[d, "\gamma(\beta)"'] & B/A \arrow[r, "\delta(j)"] \arrow[d, dotted, "c"'] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\
\bar{A} \arrow[r, "\gamma(tk)"] & \bar{B} \arrow[r, "\gamma(\bar{q})"] & \bar{B}/\bar{A} \arrow[r, "\delta(tk)"] & \Sigma\bar{A} \\
A' \arrow[r, "\gamma(j')"'] \arrow[u, "\gamma(s)", "\cong"'] & B' \arrow[r, "\gamma(q')"'] \arrow[u, "\gamma(ts')", "\cong"'] & B'/A' \arrow[r, "\delta(j')"'] \arrow[u, "\cong", "\gamma(ts'/s)"'] & \Sigma A' \arrow[u, "\cong", "\Sigma\gamma(s)"']
\end{tikzcd}$$

commute (the lower part commutes by naturality of connecting morphisms). Since s is an acyclic cofibration, so is its cobase change s'. By the gluing lemma the weak equivalences $s\colon A' \longrightarrow \bar{A}$ and $ts'\colon B' \longrightarrow \bar{B}$ induce a weak equivalence $ts'/s\colon B'/A' \longrightarrow \bar{B}/\bar{A}$ on quotients and the composite

$$B/A \xrightarrow{\ c\ } \bar{B}/\bar{A} \xrightarrow{\gamma(ts'/s)^{-1}} B'/A'$$

in $\mathrm{Ho}(\mathcal{C})$ thus solves the original problem.

(T5 - Octahedral axiom) We start with the special case where $f = \gamma(j)$ and $f' = \gamma(j')$ for cofibrations $j\colon A \longrightarrow B$ and $j'\colon B \longrightarrow D$. Then the composite $j'j\colon A \longrightarrow D$ is a cofibration with $\gamma(j'j) = f'f$. The diagram

$$\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, equal] & B \arrow[r, "\gamma(q_j)"] \arrow[d, "\gamma(j')"'] & B/A \arrow[r, "\delta(j)"] \arrow[d, "\gamma(j'/A)"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(j'j)"'] & D \arrow[r, "\gamma(q_{j'j})"'] \arrow[d, "\gamma(q_{j'})"'] & D/A \arrow[r, "\delta(j'j)"'] \arrow[d, "\gamma(D/j)"] & \Sigma A \arrow[d, "\Sigma\gamma(j)"] \\
 & D/B \arrow[r, equal] \arrow[d, "\delta(j')"'] & D/B \arrow[r, "\delta(j')"'] \arrow[d, "\delta(j'/A)=(\Sigma\gamma(q_j))\delta(j')"] & \Sigma B \\
 & \Sigma B \arrow[r, "\Sigma\gamma(q_j)"'] & \Sigma(B/A) &
\end{tikzcd}$$

then commutes by naturality of connecting morphisms. Moreover, the four triangles in question are the distinguished triangles of the cofibrations j, j', $j'j$ and $j'/A\colon B/A \longrightarrow D/A$.

In the general case we write $f = \gamma(s)^{-1}\gamma(a)$ for $\mathcal{C}$-morphisms $a\colon A \longrightarrow B'$ and $s\colon B \longrightarrow B'$, such that s is a weak equivalence. Then a can be factored as $a = pj$ for a cofibration $j\colon A \longrightarrow \bar{B}$ and a weak equivalence $p\colon \bar{B} \longrightarrow B'$. Altogether we then have $f = \varphi \circ \gamma(j)$ where $\varphi = \gamma(s)^{-1} \circ \gamma(p)\colon \bar{B} \longrightarrow B$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. We can apply the same reasoning to the morphism $f'\varphi\colon \bar{B} \longrightarrow D$ and write it as $f' \circ \varphi = \psi \circ \gamma(j')$ for a cofibration $j'\colon \bar{B} \longrightarrow \bar{D}$ and an isomorphism $\psi\colon \bar{D} \longrightarrow D$ in $\mathrm{Ho}(\mathcal{C})$. The special case can then be applied to the cofibrations $j\colon A \longrightarrow \bar{B}$ and $j'\colon \bar{B} \longrightarrow \bar{D}$. The resulting commutative diagram that solves (T5) for $(\gamma(j), \gamma(j'))$ can then be translated back into a commutative diagram that solves (T5) for (f, f') by conjugating with the isomorphisms $\varphi\colon \bar{B} \longrightarrow B$ and $\psi\colon \bar{D} \longrightarrow D$. This completes the proof of the octahedral axiom (T5), and hence the proof of the theorem. $\square$

Remark 5.9. Theorem 5.8 is not best possible in the sense that the functoriality of the cone is unnecessary and the additivity requirement for $\mathrm{Ho}(\mathcal{C})$ is redundant. Indeed, in a pointed cofibration category, we can always choose a cone, i.e., a factorization of the unique morphism $X \longrightarrow *$ as a cofibration $\iota\colon X \longrightarrow C$ followed by a weak equivalence $C \longrightarrow *$. The suspension is then again the cokernel $\Sigma X = C/X$ of the cofibration ι. While such cones might not be functorial at the level of the cofibration category, one can show that the suspension construction descends to an endofunctor of the homotopy category, compare [42, Proposition A.4]. In this sense, functoriality of cones is an unnecessary assumption. Moreover, the assumption that the suspension functor is an autoequivalence of $\mathrm{Ho}(\mathcal{C})$ already implies that $\mathrm{Ho}(\mathcal{C})$ is additive, see [42, Proposition A.8]. In this sense, the additivity hypothesis is redundant. The reason why we require the unnecessary assumptions in Theorem 5.8 is that they simplify the proof substantially. The reader is invited to consult [42, Theorem A.12] for a proof that does not require functoriality of cones additivity.

Example 5.10 (Triangulated structure on $\mathcal{SH}$)**.** Now we specialize to the most important case for our purposes, the category of sequential spectra with the cofibration structure from Theorem 4.20. A functorial cone is given by smashing with the interval $[0,1]$, based at 0. We recall that $\mathbf{t}\colon [0,1] \longrightarrow S^1$ is the quotient map defined by $\mathbf{t}(x) = \frac{2x-1}{x(1-x)}$. The morphism

$$A \wedge t \ : \ A \wedge [0,1] \ \longrightarrow \ A \wedge S^1$$

witness the suspension $A \wedge S^1$ as a cokernel of the 'cone inclusion' $- \wedge 1 \colon A \longrightarrow A \wedge [0,1]$. So the suspension functor associated to this functorial cone is the usual suspension.

The *elementary distinguished triangle* in $\mathcal{SH}$ associated to a morphism $\psi \colon A \longrightarrow B$ of sequential spectra is thus the sequence

$$A \ \xrightarrow{\gamma(\psi)} \ B \ \xrightarrow{\gamma(i)} \ C\psi \ \xrightarrow{\gamma(p)} \ \Sigma A \ .$$

Here $C\psi = A \wedge [0,1] \cup_\psi B$ is the usual mapping cone of ψ, $i \colon B \longrightarrow C\psi$ the inclusion, and $p = (A \wedge \mathbf{t}) \cup 0 \colon C\psi \longrightarrow A \wedge S^1$.

A *distinguished triangle* is any triangle (f, g, h) in $\mathcal{SH}$ which is isomorphic to an elementary distinguished triangle, i.e., such that there is a morphism $\psi \colon A \longrightarrow B$ of sequential spectra and isomorphisms $a \colon A \longrightarrow X$, $b \colon B \longrightarrow Y$ and $c \colon C\psi \longrightarrow Z$ in $\mathcal{SH}$ that make the diagram

$$\begin{tikzcd}
A \arrow[r, "\gamma(\psi)"] \arrow[d, "a"'] & B \arrow[r, "\gamma(i)"] \arrow[d, "b"'] & C\psi \arrow[r, "\gamma(p)"] \arrow[d, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & \Sigma X
\end{tikzcd}$$

commute.

We showed in Theorem 3.18 that the stable homotopy category is additive. As explained above, the suspension functor $\Sigma \colon \mathcal{SH} \longrightarrow \mathcal{SH}$ is descended by localization from the suspension functor $- \wedge S^1 \colon \mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{S}p^{\mathbb{N}}$. The adjoint functor pair $(- \wedge S^1, \Omega)$ on $\mathcal{S}p^{\mathbb{N}}$ consists of homotopical functors, i.e., both $- \wedge S^1$ and Ω preserves all stable equivalences by Proposition 1.13 (i). So the pair $(- \wedge S^1, \Omega)$ descends to an adjoint pair of endofunctors on the stable homotopy category, as explained in Construction 3.5. Since the unit and counit of the original adjunction are natural stable equivalences by Proposition 1.13 (ii), they descend to natural isomorphisms at the level of the stable homotopy category. In particular, the descended suspension functor $\Sigma \colon \mathcal{SH} \longrightarrow \mathcal{SH}$ is an autoequivalence. So Theorem 5.8 applies, and yields:

Corollary 5.11. *The suspension functor and the class of distinguished triangles make the stable homotopy category $\mathcal{SH}$ into a triangulated category.*

We will later explain that a very similar reasoning provides a triangulated structure on the *derived category* $\mathcal{D}(R)$ of every orthogonal ring spectrum R. This derived category is defined as the localization of the category of orthogonal R-module spectra by the class of morphisms that are stable equivalences of underlying sequential spectra.

Example 5.12 (Triangulated structure on $\mathcal{K}(\mathcal{A})$)**.** We let $\mathcal{A}$ be an additive category, and we write $\mathrm{Ch}(\mathcal{A})$ for the category of $\mathbb{Z}$-graded chain complexes in $\mathcal{A}$. We write $\mathcal{K}(A)$ for the algebraic homotopy category, with the same objects as $\mathrm{Ch}(\mathcal{A})$, and with chain homotopy classes of chain maps as morphisms. We call a chain map $f \colon A \longrightarrow B$ in $\mathrm{Ch}(\mathcal{A})$ a *cofibration* if it is dimensionwise a split monomorphism, i.e., for every $n \in \mathbb{Z}$ there is an $\mathcal{A}$-object C and an isomorphism $A_n \oplus C \cong B_n$ that restricts to $f_n \colon A_n \longrightarrow B_n$ on the first summand. Exercise E.10 (c) is devoted to showing that these cofibrations and the chain homotopy equivalences form a cofibration structure on the category $\mathrm{Ch}(\mathcal{A})$. And the content of Exercise E.11 is to check that the quotient functor $\pi \colon \mathrm{Ch}(\mathcal{A}) \longrightarrow \mathcal{K}(\mathcal{A})$ to the algebraic homotopy category is a localization at the class of chain homotopy equivalence. So in this particular case, the abstract notion of homotopy category coincides with the concrete notion.

The category $\mathrm{Ch}(\mathcal{A})$ is clearly pointed: any complex consisting only of zero objects is a zero object in $\mathrm{Ch}(\mathcal{A})$. The usual algebraic cone provides a functorial cone in the sense of Definition 5.2: the cone CA of a complex A is defined by

$$(CA)_n \ = \ A_n \oplus A_{n-1}$$

with differential

$$d_n^{CA}(a,a') \;=\; (d_n^A(a) + (-1)^n \cdot a',\; d_{n-1}^A(a'))\ .$$

The cone CA is chain contractible, and the inclusions of the first summand define a cofibration $\iota_A\colon A \longrightarrow CA$, so we have exhibited a functorial cone. The projections to the second summands witness the shift $A[1]$ as a cokernel of the 'cone inclusion' $\iota_A\colon A \longrightarrow CA$. So the suspension functor associated to this functorial cone is the shift functor.

The *elementary distinguished triangle* in $\mathcal{K}(A)$ associated to a chain morphism $\psi\colon A \longrightarrow B$ is thus the sequence

$$A \xrightarrow{\pi(\psi)} B \xrightarrow{\pi(i)} C\psi \xrightarrow{\pi(p)} A[1]\ .$$

Here $C\psi = CA \cup_\psi B$ is the algebraic mapping cone of ψ; in more explicit terms, this complex is given by

$$(CA)_n \;=\; B_n \oplus A_{n-1}$$

with differential

$$d_n^{C\psi}(b,a) = (d_n^B(b) + (-1)^n \cdot \psi_{n-1}(a), d_{n-1}^A(a))\ .$$

The morphism $i\colon B \longrightarrow C\psi$ the inclusion as the first summands, and $p\colon C\psi \longrightarrow A[1]$ is the projection to the second summands. A *distinguished triangle* is any triangle (f,g,h) in $\mathcal{K}(\mathcal{A})$ which is isomorphic to an elementary distinguished triangle.

In the previous example, already the cofibration category $\mathrm{Ch}(\mathcal{A})$ is additive, and so is the homotopy category $\mathcal{K}(\mathcal{A})$. Moreover, since the suspension functor is the shift functor, it is already invertible at the level of complexes, and hence also at the level of the homotopy category $\mathcal{K}(\mathcal{A})$. So Theorem 5.8 applies, and yields:

Corollary 5.13. *For every additive category $\mathcal{A}$, the shift functor and the class of distinguished triangles make the homotopy category of complexes $\mathcal{K}(\mathcal{A})$ into a triangulated category.*

Example 5.14 (Triangulated structure on $\mathcal{D}(S)$)**.** We let S be an associative and unital ring. We write $\mathrm{Ch}(S)$ for the category of $\mathbb{Z}$-graded chain complexes of left S-modules. We recall that a chain map in $\mathrm{Ch}(S)$ is a *quasi-isomorphism* if it induces isomorphisms of all homology groups. Exercise E.10 (d) is devoted to showing that the cofibrations from the previous Example 5.12 (i.e., dimensionwise split monomorphisms) and the quasi-isomorphism form another cofibration structure on the category $\mathrm{Ch}(S)$. We let $\gamma\colon\ \mathrm{Ch}(S) \longrightarrow \mathcal{D}(S)$ be a localization at the class of quasi-isomorphisms; the target $\mathcal{D}(S)$ is called the *derived category* of the ring S. Since every chain homotopy equivalence is in particular a quasi-isomorphism, this localization factors through the quotient $\pi\colon\ \mathrm{Ch}(S) \longrightarrow \mathcal{K}(S\text{-mod})$ discussed in the previous example, and the resulting functor $\bar{\gamma}\colon \mathcal{K}(S\text{-mod}) \longrightarrow \mathcal{D}(S)$ is also a localization at the quasi-isomorphisms (or rather their images in $\mathcal{K}(S\text{-mod})$).

The algebraic cone discussed in the previous example is also a functorial cone with respect to the quasi-isomorphisms; the shift functor preserves quasi-isomorphisms and thus descends to an invertible functor on $\mathcal{D}(S)$. The resulting distinguished triangles in the derived category $\mathcal{D}(S)$ can thus be described in two equivalent ways:

- as those triangles that are isomorphic to

  $$A \xrightarrow{\gamma(\psi)} B \xrightarrow{\gamma(i)} C\psi \xrightarrow{\gamma(p)} A[1]$$

  form some morphism $\psi\colon A \longrightarrow B$ in $\mathrm{Ch}(S)$;
- and as those triangles that are isomorphic to distinguished triangles in $\mathcal{K}(S\text{-mod})$ under the localization functor $\bar{\gamma}\colon \mathcal{K}(S\text{-mod}) \longrightarrow \mathcal{D}(S)$.

Corollary 5.15. *For every ring S, the shift functor and the class of distinguished triangles make the derived category $\mathcal{D}(S)$ into a triangulated category.*

In fact, the arguments leading up to Corollary 5.15 work just as well for any abelian category instead of the special case S-mod of the category of left S-modules; however, as we have not discussed abelian categories, we concentrated on the special case of modules above.

One reason why we might care about triangulated structures is that they are a source for many long exact sequences. We show this and some other elementary key properties of triangulated categories in the next proposition.

Proposition 5.16. *Let $\mathcal{T}$ be a triangulated category. Then the following properties hold.*

(i) *For every distinguished triangle (f,g,h) and every object X of $\mathcal{T}$, the two sequences of abelian groups*
$$\mathcal{T}(\Sigma A, X) \xrightarrow{\mathcal{T}(h,X)} \mathcal{T}(C,X) \xrightarrow{\mathcal{T}(g,X)} \mathcal{T}(B,X) \xrightarrow{\mathcal{T}(f,X)} \mathcal{T}(A,X)$$
and
$$\mathcal{T}(X,A) \xrightarrow{\mathcal{T}(X,f)} \mathcal{T}(X,B) \xrightarrow{\mathcal{T}(X,g)} \mathcal{T}(X,C) \xrightarrow{\mathcal{T}(X,h)} \mathcal{T}(X,\Sigma A)$$
are exact.

(ii) *Let (a,b,c) be a morphism of distinguished triangles. If two out of the three morphisms are isomorphisms, then so is the third.*

(iii) *Let (f,g,h) be a triangle such that the triangle $(g,h,-\Sigma f)$ is distinguished. Then the triangle (f,g,h) is distinguished.*

(iv) *Let (f_1,g_1,h_1), (f_2,g_2,h_2) and (f_3,g_3,h_3) be three distinguished triangles such that f_1 and f_2 are composable and $f_3 = f_2 f_1$. Then there exist morphisms $\bar{x}$ and $\bar{y}$ such that $(\bar{x}, \bar{y}, (\Sigma g_1)\circ h_2)$ is a distinguished triangle and the following diagram commutes:*
$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r, "g_1"] \arrow[d, "f_2"] & \bar{C} \arrow[r, "h_1"] \arrow[d, dashed, "\bar{x}"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f_3"'] & D \arrow[r, "g_3"'] \arrow[d, "g_2"] & \bar{E} \arrow[r, "h_3"'] \arrow[d, dashed, "\bar{y}"] & \Sigma A \arrow[d, "\Sigma f_1"] \\
 & \bar{F} \arrow[r, equal] \arrow[d, "h_2"] & \bar{F} \arrow[r, "h_2"'] \arrow[d, "(\Sigma g_1)\circ h_2"] & \Sigma B \\
 & \Sigma B \arrow[r, "\Sigma g_1"'] & \Sigma C &
\end{tikzcd}$$

(v) *For every distinguished triangle (f,g,h) the following three conditions are equivalent:*
- *The morphism $f\colon A \longrightarrow B$ has a retraction, i.e., there is a morphism r such that $rf = \mathrm{Id}_A$.*
- *The morphism $g\colon B \longrightarrow C$ has a section, i.e., there is a morphism s such that $gs = \mathrm{Id}_C$.*
- *The morphism $h\colon C \longrightarrow \Sigma A$ is zero.*

(vi) *Let (f,g,h) be a distinguished triangle and $s\colon C \longrightarrow B$ a morphism such that $gs = \mathrm{Id}_C$. Then the morphisms $f\colon A \longrightarrow B$ and $s\colon C \longrightarrow B$ make B into a coproduct of A and C.*

(vii) *Let I be a set and let (f_i,g_i,h_i) be a distinguished triangle for every $i \in I$. Then the triangles*
$$\bigoplus_I A_i \xrightarrow{\oplus f_i} \bigoplus_I B_i \xrightarrow{\oplus g_i} \bigoplus_I C_i \xrightarrow{\kappa\circ(\oplus h_i)} \Sigma(\bigoplus_I A_i)$$
and
$$\prod_I A_i \xrightarrow{\prod f_i} \prod_I B_i \xrightarrow{\prod g_i} \prod_I C_i \xrightarrow{\kappa^{-1}\circ(\prod h_i)} \Sigma(\prod_I A_i)$$
are distinguished, whenever the respective coproducts and products exist. Here $\kappa\colon \bigoplus_I \Sigma A_i \longrightarrow \Sigma(\bigoplus_I A_i)$ and $\kappa\colon \Sigma(\prod_I A_i) \longrightarrow \prod_I \Sigma A_i$ are the canonical isomorphisms.

(viii) *Let $A \oplus B$ be a coproduct of two objects A and B of $\mathcal{T}$ with respect to the morphisms $i_A\colon A \longrightarrow A\oplus B$ and $i_B\colon B \longrightarrow A \oplus B$. Then the triangle*
$$A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \xrightarrow{0} \Sigma A$$

is distinguished, where p_B is the morphism determined by $p_B i_A = 0$ and $p_B i_B = \mathrm{Id}_B$.

Proof. We start by showing that for every distinguished triangle (f, g, h) the composite gf is zero. Indeed, by (T4) applied to the pair (Id, f) there is a (necessarily unique) morphism from any zero object to C such that the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{Id}"] \arrow[d, equal] & A \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d, dotted] & \Sigma A \arrow[d, equal] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma A \end{tikzcd}$$

commutes, so $gf = 0$ (the upper row is distinguished by (T2) and (T3)).

(i) Since $gf = 0$ the image of $\mathcal{T}(g, X)$ is contained in the kernel of $\mathcal{T}(f, X)$. Conversely, let $\psi\colon B \longrightarrow X$ be a morphism in the kernel of $\mathcal{T}(f, X)$, i.e., such that $\psi f = 0$. Applying (T4) to the pair $(0, \psi)$ gives a morphism $\varphi\colon C \longrightarrow X$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d, "\psi"'] & C \arrow[r, "h"] \arrow[d, dotted, "\varphi"] & \Sigma A \arrow[d] \\ 0 \arrow[r] & X \arrow[r, "\mathrm{Id}"'] & X \arrow[r] & 0 \end{tikzcd}$$

commutes (the lower row is distinguished by (T1)). So the first sequence is exact at $\mathcal{T}(B, X)$. Applying this to the triangle $(g, h, -\Sigma f)$ (which is distinguished by (T3)), we deduce that the first sequence is also exact at $\mathcal{T}(X, C)$.

The argument for the other sequence is similar, but slightly more involved and depends on the assumption that the functor Σ is fully faithful. Since $gf = 0$, the image of $\mathcal{T}(X, f)$ is contained in the kernel of $\mathcal{T}(X, g)$. Conversely, let $\psi\colon X \longrightarrow B$ be a morphism in the kernel of $\mathcal{T}(X, g)$, i.e., such that $g\psi = 0$. Applying (T4) to the pair $(\psi, 0)$ gives a morphism $\bar{\varphi}\colon \Sigma X \longrightarrow \Sigma A$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[r, "-\mathrm{Id}"] \arrow[d, dotted, "\bar{\varphi}"] & \Sigma X \arrow[d, "\Sigma\psi"] \\ B \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma A \arrow[r, "-\Sigma f"'] & \Sigma B \end{tikzcd}$$

commutes (both rows are distinguished by (T1) and (T2)). Since shifting is full, there exists a morphism $\varphi\colon X \longrightarrow A$ such that $\bar{\varphi} = \Sigma\varphi$, and hence $\Sigma(f\varphi) = (\Sigma f)(\Sigma\varphi) = \Sigma\psi$. Since shifting is faithful we have $f\varphi = \psi$, so ψ is in the image of $\mathcal{T}(X, f)$. Altogether, the first sequence is exact at $\mathcal{T}(X, B)$. If we apply this to the triangle $(g, h, -\Sigma f)$ (which is distinguished by (T3)), we deduce that the first sequence is also exact at $\mathcal{T}(X, C)$.

(ii) We first treat the case where a and b are isomorphisms. If X is any object of $\mathcal{T}$ we have a commutative diagram

$$\begin{tikzcd} \mathcal{T}(X, A) \arrow[r, "f_*"] \arrow[d, "a_*"'] & \mathcal{T}(X, B) \arrow[r, "g_*"] \arrow[d, "b_*"'] & \mathcal{T}(X, C) \arrow[r, "h_*"] \arrow[d, "c_*"'] & \mathcal{T}(X, \Sigma A) \arrow[r, "(-\Sigma f)_*"] \arrow[d, "(\Sigma a)_*"] & \mathcal{T}(X, \Sigma B) \arrow[d, "(\Sigma b)_*"] \\ \mathcal{T}(X, A') \arrow[r, "f'_*"'] & \mathcal{T}(X, B') \arrow[r, "g'_*"'] & \mathcal{T}(X, C') \arrow[r, "h'_*"'] & \mathcal{T}(X, \Sigma A') \arrow[r, "(-\Sigma(f'))_*"'] & \mathcal{T}(X, \Sigma B') \end{tikzcd}$$

where we write f_* for $\mathcal{T}(X, f)$, etc. The top row is exact by part (i) applied to the distinguished triangles (f, g, h) and $(g, h, -\Sigma f)$. Similarly, the bottom row is exact. Since a and b (and hence Σa and Σb) are isomorphisms, all vertical maps except possibly the middle one are isomorphisms of abelian groups. So the five lemma says that c_* is an isomorphism. Since this holds for all objects X, the morphism $c\colon C \longrightarrow C'$ is an isomorphism.

If b and c are isomorphisms, we apply the previous argument to the triple $(b, c, \Sigma a)$. This is a morphism from the distinguished (by (T3)) triangle $(g, h, -\Sigma f)$ to the distinguished triangle $(g', h', -\Sigma f')$. By the above, Σa is an isomorphism, hence so is a since shifting is an equivalence of categories. The third case is similar.

(iii) If the triangle $(g, h, -\Sigma f)$ is distinguished, then so is $(-\Sigma f, -\Sigma g, -\Sigma h)$ by two applications of (T3). Axiom (T1) provides a distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{\bar{g}} \bar{C} \xrightarrow{\bar{h}} \Sigma A$$

and by three applications of (T3), the triangle $(-\Sigma f, -\Sigma\bar{g}, -\Sigma\bar{h})$ is distinguished. By (T4) there is a morphism $\bar{c}\colon \Sigma C \longrightarrow \Sigma\bar{C}$ such that the diagram

$$\begin{tikzcd}
\Sigma A \arrow[r, "-\Sigma f"] \arrow[d, equal] & \Sigma B \arrow[r, "-\Sigma g"] \arrow[d, equal] & \Sigma C \arrow[r, "-\Sigma h"] \arrow[d, "\bar{c}"] & \Sigma^2 A \arrow[d, equal] \\
\Sigma A \arrow[r, "-\Sigma f"'] & \Sigma B \arrow[r, "-\Sigma\bar{g}"'] & \Sigma\bar{C} \arrow[r, "-\Sigma\bar{h}"'] & \Sigma^2 A
\end{tikzcd}$$

commutes. By part (ii), c is an isomorphism. Since suspension is an equivalence of categories, we have $\bar{c} = \Sigma c$ for a unique isomorphism $c\colon C \longrightarrow \bar{C}$. Then $(\mathrm{Id}_A, \mathrm{Id}_B, c)$ is an isomorphism from the triangle (f, g, h) to the distinguished triangle $(f, \bar{g}, \bar{h})$. So the triangle (f, g, h) is itself distinguished.

(iv) Axiom (T5) provides a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "f_2"] & C \arrow[r, "h"] \arrow[d, "x"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f_3"'] & D \arrow[r, "g''"'] \arrow[d, "g'"] & E \arrow[r, "h''"'] \arrow[d, "y"] & \Sigma A \arrow[d, "\Sigma f_1"] \\
 & F \arrow[r, equal] \arrow[d, "h'"] & F \arrow[r, "h'"'] \arrow[d, "(\Sigma g)\circ h'"] & \Sigma B \\
 & \Sigma B \arrow[r, "\Sigma g"'] & \Sigma C &
\end{tikzcd}$$

such that the triangles (f_1, g, h), (f_2, g', h'), (f_3, g'', h'') and $(x, y, (\Sigma g)\circ h')$ are distinguished. By (T4) there is a morphism $\varphi\colon \bar{C} \longrightarrow C$ that makes $(\mathrm{Id}_A, \mathrm{Id}_B, \varphi)$ a morphism of triangles from (f_1, g_1, h_1) to (f_1, g, h); this morphism is an isomorphism by part (ii). Similarly, there is an morphism $\psi\colon \bar{F} \longrightarrow F$ such that $(\mathrm{Id}_B, \mathrm{Id}_D, \psi)$ an isomorphism of triangles from (f_2, g_2, h_2) to (f_2, g', h'). Finally, there is an morphism $\nu\colon \bar{E} \longrightarrow E$ such that $(\mathrm{Id}_A, \mathrm{Id}_D, \nu)$ an isomorphism of triangles from (f_3, g_3, h_3) to (f_3, g'', h''). If we set

$$\bar{x} \;=\; \nu^{-1}x\varphi \;:\; \bar{C} \;\longrightarrow\; \bar{E} \qquad \text{and} \qquad \bar{y} \;=\; \psi^{-1}y\nu \;:\; \bar{E} \;\longrightarrow\; \bar{F}\;,$$

then the desired diagram commutes. Moreover, the triple (φ, ν, ψ) is an isomorphism from the triangle $(\bar{x}, \bar{y}, (\Sigma g_1)h_2)$ to the triangle $(x, y, (\Sigma g)h')$. Since the latter triangle is distinguished, so is the former.

(v) By part (i), the composite of two adjacent morphism in any distinguished triangle is zero. So if s is a section to g, then $h = hgs = 0$. Similarly, if r is a retraction to f, then $h = (-\Sigma r)(-\Sigma f)h = 0$ because the triangle $(g, h, -\Sigma f)$ is distinguished. Conversely, if $h = 0$, then the sequence

$$\mathcal{T}(C, B) \xrightarrow{\mathcal{T}(C,g)} \mathcal{T}(C, C) \longrightarrow 0$$

is exact by part (i), and any preimage of the identity of C is a section to g. Similarly, the sequence

$$\mathcal{T}(\Sigma B, \Sigma A) \xrightarrow{\mathcal{T}(-\Sigma f, \Sigma A)} \mathcal{T}(\Sigma A, \Sigma A) \longrightarrow 0$$

is exact because the triangle $(g, h, -\Sigma f)$ is distinguished. So there is a morphism $\bar{r}\colon \Sigma B \longrightarrow \Sigma A$ such that $-\bar{r} \circ \Sigma f = \mathrm{Id}_{\Sigma A}$. Since Σ is full, there is a morphism $r\colon B \longrightarrow A$ such that $\Sigma r = -\bar{r}$, hence $\Sigma(rf) = (\Sigma r)(\Sigma f) = \mathrm{Id}_{\Sigma A}$. Since Σ is faithful, r is a retraction to f.

(vi) Since s is a section to g, the morphism $\mathcal{T}(g, X)$ is injective. By part (v) the morphism f has a retraction, so $\mathcal{T}(f, X)$ is surjective. The first exact sequence of part (i) thus becomes a short exact sequence of abelian groups

$$0 \longrightarrow \mathcal{T}(C, X) \xrightarrow{\mathcal{T}(g,X)} \mathcal{T}(B, X) \xrightarrow{\mathcal{T}(f,X)} \mathcal{T}(A, X) \longrightarrow 0 \ .$$

Because $\mathcal{T}(s, X)$ is a section to the first map, the map $(\mathcal{T}(f, X), \mathcal{T}(s, X))\colon \mathcal{T}(B, X) \longrightarrow \mathcal{T}(A, X) \times \mathcal{T}(C, X)$ is bijective, i.e., the morphisms f and s make B a coproduct of A and B.

(vii) We choose a distinguished triangle:

$$\bigoplus\nolimits_I A_i \xrightarrow{\ \oplus f_i\ } \bigoplus\nolimits_I B_i \xrightarrow{\ g\ } C \xrightarrow{\ h\ } \Sigma(\bigoplus\nolimits_I A_i) \ .$$

We apply axiom (T3) to the canonical morphisms $\kappa_j\colon A_j \longrightarrow \bigoplus_I A_i$ and $\kappa'_j\colon B_j \longrightarrow \bigoplus_I B_i$ and obtain morphisms $\varphi_j\colon C_j \longrightarrow C$ such that the diagrams

$$\begin{tikzcd}
A_j \arrow[r, "f_j"] \arrow[d, "\kappa_j"'] & B_j \arrow[r, "g_j"] \arrow[d, "\kappa'_j"'] & C_j \arrow[r, "h_j"] \arrow[d, dotted, "\varphi_j"'] & \Sigma A_j \arrow[d, "\Sigma\kappa_j"] \\
\bigoplus_I A_i \arrow[r, "\oplus f_i"'] & \bigoplus_I B_i \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma(\bigoplus_I A_i)
\end{tikzcd}$$

commute. We claim that then the morphisms $\varphi_i\colon C_i \longrightarrow C$ make C into a coproduct of the objects C_i. For this we observe that the diagram

$$\begin{tikzcd}
\mathcal{T}(\Sigma(\bigoplus_I B_i), X) \arrow[r, "-(\Sigma\oplus f_i)_*"] \arrow[d, "((\Sigma\kappa_i)_*)"'] & \mathcal{T}(\Sigma(\bigoplus_I A_i), X) \arrow[r, "h_*"] \arrow[d, "((\Sigma\kappa'_i)_*)"'] & \mathcal{T}(C, X) \arrow[r, "g_*"] \arrow[d, "((\varphi_i)_*)"'] & \mathcal{T}(\bigoplus_I B_i, X) \arrow[r, "(\oplus f_i)_*"] \arrow[d, "((\kappa'_i)_*)"] & \mathcal{T}(\bigoplus_I A_i, X) \arrow[d, "((\kappa_i)_*)"] \\
\prod_I \mathcal{T}(\Sigma B_i, X) \arrow[r, "-\prod \Sigma(f_i)_*"'] & \prod_I \mathcal{T}(\Sigma A_i, X) \arrow[r, "\prod(h_i)_*"'] & \prod_I \mathcal{T}(C_i, X) \arrow[r, "\prod(g_i)_*"'] & \prod_I \mathcal{T}(B_i, X) \arrow[r, "\prod(f_i)_*"'] & \prod_I \mathcal{T}(A_i, X)
\end{tikzcd}$$

commutes by construction of the morphisms φ_i. The top row is exact by part (i), the bottom row is exact as a product of exact sequences. The four outer vertical maps are isomorphisms by the universal property of coproducts, so the middle vertical map is an isomorphism by the 5-lemma. This shows that C is a coproduct of the C_i's in a way that makes $g = \oplus g_i\colon \bigoplus_I B_i \longrightarrow C$ and $h = \kappa \circ (\oplus h_i)\colon C \longrightarrow \Sigma(\bigoplus_I A_i)$.

The statement about products of triangles can be proved in an analogous fashion. Alternatively, one can reduce to the first case by exploiting that products in $\mathcal{T}$ are coproducts in the opposite category $\mathcal{T}^{\mathrm{op}}$, which is triangulated with respect to the opposite triangulation (compare Exercise E.14).

(viii) This the special case of part (vii) for the two exact triangles

$$A \xrightarrow{\mathrm{Id}_A} A \longrightarrow 0 \longrightarrow \Sigma A \qquad \text{and} \qquad 0 \longrightarrow B \xrightarrow{\mathrm{Id}_B} B \longrightarrow 0$$

whose sum is the triangle in question, which is thus distinguished. □

Exercises

Exercise E.1 (Coordinatized orthogonal spectra)**.** A *coordinatized orthogonal spectrum* consists of the following data:

- a sequence of based spaces X_n for $n \geq 0$,
- a based continuous left $O(n)$-action on X_n for each $n \geq 0$,
- based maps $\sigma_n : S^1 \wedge X_n \longrightarrow X_{1+n}$ for $n \geq 0$.

This data is subject to the following condition: for all $m, n \geq 0$, the iterated structure map $S^m \wedge X_n \longrightarrow X_{m+n}$ defined as the composition

$$S^m \wedge X_n \xrightarrow{S^{m-1}\wedge\sigma_n} S^{m-1} \wedge X_{1+n} \xrightarrow{S^{m-2}\wedge\sigma_{1+n}} \quad \cdots \quad \xrightarrow{\sigma_{m-1+n}} X_{m+n}$$

is $(O(m)\times O(n))$-equivariant. Here the orthogonal group $O(m)$ acts on S^m as the one-point compactification of the tautological action on $\mathbb{R}^m$, and $O(m) \times O(n)$ acts on the target by restriction, along orthogonal sum, of the $O(m+n)$-action.

A *morphism* $f : X \longrightarrow Y$ of coordinatized orthogonal spectra consists of $O(n)$-equivariant based maps $f_n : X_n \longrightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (S^1 \wedge f_n)$ for all $n \geq 0$.

Let X be a coordinatized orthogonal spectrum.

(a) Let W be an inner product space of dimension n. We define a based space

$$X^\flat(W) \;=\; (\mathbf{L}(\mathbb{R}^n, W)_+ \wedge X_n) \,/\sim\,,$$

the quotient space of $\mathbf{L}(\mathbb{R}^n, W)_+ \wedge X_n$ by the equivalence relation

$$(\varphi \circ A) \wedge x \;\sim\; \varphi \wedge (A \cdot x)$$

for all linear isometries $\varphi : \mathbb{R}^n \cong W$ and all $A \in O(n)$ and $x \in X_n$. Show that $X^\flat(W)$ is homeomorphic to X_n.

(b) Let U be another inner product space of the same dimension as W. Show that the continuous map

$$\begin{aligned} \mathbf{L}(U, W)_+ \wedge \mathbf{L}(\mathbb{R}^n, U)_+ \wedge X_n &\longrightarrow \mathbf{L}(\mathbb{R}^n, W) \wedge X_n \\ \psi \wedge \varphi \wedge x &\longmapsto (\psi \circ \varphi) \wedge x \end{aligned}$$

factors through a continuous map

$$X^\flat(U, W) \;:\; \mathbf{L}(U, W)_+ \wedge X^\flat(U) \;\longrightarrow\; X^\flat(W)\,.$$

(c) Let V be another inner product space of dimension m, and let $\psi : \mathbb{R}^m \cong V$ be a linear isometry. Show that the continuous map

$$\begin{aligned} S^V \wedge \mathbf{L}(\mathbb{R}^n, W)_+ \wedge X_n &\longrightarrow \mathbf{L}(\mathbb{R}^{m+n}, V \oplus W)_+ \wedge X_{m+n} \\ v \wedge \varphi \wedge x &\longmapsto (\psi \oplus \varphi) \wedge \sigma^m(\psi^{-1}(v) \wedge x) \end{aligned}$$

factors through a continuous map

$$\sigma_{V,W} \;:\; S^V \wedge X^\flat(W) \;\longrightarrow\; X^\flat(V \oplus W)$$

that is independent of the choice of ψ.

(d) Show that there is a unique orthogonal spectrum $X^\flat$ whose values are the spaces $X^\flat(W)$ from (a), whose functoriality in isometries is as in (b), and whose structure maps are as in (c).

(e) Extend the assignment $X \mapsto X^\flat$ to functor

$$(-)^\flat \;:\; \mathcal{S}p^{\mathrm{coord}} \;\longrightarrow\; \mathcal{S}p\,.$$

(f) A *forgetful functor*

$$\mathcal{S}p \;\longrightarrow\; \mathcal{S}p^{\mathrm{coord}}$$

is defined on objects by $(UX)_n = X(\mathbb{R}^n)$, where $\mathbb{R}^n$ is endowed with the standard inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. The structure map is

$$\sigma_n \;=\; \sigma_{\mathbb{R},\mathbb{R}^n} \;:\; S^1 \wedge X_n \;=\; S^{\mathbb{R}} \wedge X(\mathbb{R}^n) \;\longrightarrow\; X(\mathbb{R}^{1+n}) \;=\; X_{1+n}\,.$$

On morphisms, the forgetful functor evaluates at $\mathbb{R}^n$. Show that the forgetful functor is an equivalence of categories by exhibiting natural isomorphisms $U(X^\flat) \cong X$ and $Y \cong (UY)^\flat$.

Exercise E.2. Find a family $\{X^i\}_{i\in I}$ of orthogonal spectra for which the natural map

$$\pi_0\left(\prod_{i\in I} X^i\right) \longrightarrow \prod_{i\in I} \pi_0(X^i)$$

is not surjective.

Exercise E.3. We recall that the m-th stable homotopy group $\pi^{\mathrm{s}}_m(K)$ of a based space K is defined as the colimit of the sequence of abelian groups

$$\pi_m(K) \xrightarrow{S^1\wedge -} \pi_{1+m}(S^1\wedge K) \xrightarrow{-\wedge S^1} \pi_{2+m}(S^2\wedge K) \xrightarrow{S^1\wedge -} \cdots .$$

Smashing with the identity of S^1 from the right provides an isomorphism $-\wedge S^1 : \pi^{\mathrm{s}}_m(K) \longrightarrow \pi^{\mathrm{s}}_{m+1}(K\wedge S^1)$, a special case of the suspension isomorphism (see Proposition 1.13) for the suspension spectrum of K.

Show that the homotopy groups of an orthogonal spectrum X can also be calculated from the system of *stable* as opposed to *unstable* homotopy groups of the individual spaces X_n: exhibit $\pi_k(X)$ as a colimit of the sequence

$$\pi^{\mathrm{s}}_{k+n}(X_n) \xrightarrow{-\wedge S^1} \pi^{\mathrm{s}}_{k+n+1}(X_n\wedge S^1) \xrightarrow{(\sigma^{\mathrm{op}}_{\mathbb{R}^n,\mathbb{R}})_*} \pi^{\mathrm{s}}_{k+n}(X_{n+1}) .$$

Exercise E.4. Exhibit a left adjoint and a right adjoint to the shift functor $\mathrm{sh}^V : \mathcal{S}p \longrightarrow \mathcal{S}p$ for orthogonal spectra, introduced in Example **??**

Exercise E.5. Define orthogonal ring spectra via $\iota_V : S^V \longrightarrow R(V)$ and multiplication maps $\mu_{V,W} : R(V)\wedge R(W) \longrightarrow R(V\oplus W)$, requiring associativity, unit and centrality.

Exercise E.6. Let M be a monoid and A a ring. The *monoid ring* $A[M]$ is the A-linearization of the underlying set of M, endowed with the multiplication by the A-bilinear extension of the multiplication of M:

$$\left(\sum\nolimits_i a_i\cdot m_i\right)\cdot\left(\sum\nolimits_j a'_j\cdot m'_j\right) \;=\; \sum\nolimits_{i,j}(a_i\cdot a'_j)\cdot(m_i\cdot m'_j) .$$

This construction extends degreewise from rings to graded rings.

Now we let R be an orthogonal ring spectrum, and we endow the M with the discrete topology. Exhibit an isomorphism of graded rings

$$\pi_*(R)[M] \;\cong\; \pi_*(RM) ,$$

where RM is the monoid ring spectrum introduced in Example **??**.

Exercise E.7. Let $A = \{A_k\}_{k\in\mathbb{Z}}$ be a graded ring, and $m \geq 1$. We define the *graded matrix ring* $M_m(A)$ by taking $(m\times m)$-matrices degreewise: the abelian group $(M_m(A))_k$ is the group of $(m\times m)$-matrices with entries in A_k, and the multiplication maps $(M_m(A))_k \times (M_m(A))_l \longrightarrow (M_m(A))_{k+l}$ are defined by the usual matrix multiplication, using the multiplication of the graded ring A.

Let R be an orthogonal ring spectrum. Exhibit an isomorphism of graded rings

$$\pi_*(M_m(R)) \;\cong\; M_m(\pi_*(R)) ,$$

where $M_m(R)$ is the $(m\times m)$-matrix ring spectrum introduced in Example **??**.

Exercise E.8. Let $A = \{A_k\}_{k\in\mathbb{Z}}$ be a graded ring. The *graded-opposite ring* A^{op} has the same underlying graded abelian group as A, but the multiplication in A^{op} is defined by

$$x\cdot_{\mathrm{op}} y \;=\; (-1)^{kl}\cdot y\cdot x ,$$

where $x\in A_k$, $y\in A_l$ and the right hand side is the multiplication in A.

Let R be an orthogonal ring spectrum. Show that $\pi_*(R^{\mathrm{op}}) = (\pi_*(R))^{\mathrm{op}}$.

Exercise E.9. We let $i : A \longrightarrow B$ and $p : X \longrightarrow Y$ be morphisms in a category $\mathcal{C}$. Then i has the *left lifting property* with respect to p if the following holds: for all morphisms $\alpha : A \longrightarrow X$ and $\beta : B \longrightarrow Y$

such that $p\alpha = \beta i$, there is a morphism $\lambda : B \longrightarrow X$ such that $\lambda i = \alpha$ and $p\lambda = \beta$:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"'] \arrow[ur, dotted, "\lambda"] & Y \end{tikzcd}$$

For a class $\mathcal{E}$ of $\mathcal{C}$-morphisms, we denote by $\mathcal{E}^{\perp}$ the class of all $\mathcal{C}$-morphisms that have the left lifting property with respect to all morphisms in $\mathcal{E}$. Show that the class $\mathcal{E}^{\perp}$ has the following closure properties:

(a) The class $\mathcal{E}^{\perp}$ is closed under composition: for all composable morphisms $i : A \longrightarrow B$ and $j : B \longrightarrow C$ in $\mathcal{E}^{\perp}$, the composite ji belongs to $\mathcal{E}^{\perp}$.

(b) The class $\mathcal{E}^{\perp}$ is closed under cobase change: for every pushout square in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "j"] \\ B \arrow[r] & D \end{tikzcd}$$

such that $i \in \mathcal{E}^{\perp}$, the morphism j belongs to $\mathcal{E}^{\perp}$.

(c) The class $\mathcal{E}^{\perp}$ is closed under retracts: for every commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} C \arrow[r, "s"] \arrow[d, "j"'] & A \arrow[r, "r"] \arrow[d, "i"] & C \arrow[d, "j"] \\ D \arrow[r, "t"'] & B \arrow[r, "u"'] & C \end{tikzcd}$$

such that $rs = \mathrm{Id}_C$ and $ut = \mathrm{Id}_D$, if i belongs to $\mathcal{E}^{\perp}$, then so does j.

(d) The class $\mathcal{E}^{\perp}$ is closed under sequential composition: for every sequence of morphisms in $\mathcal{E}^{\perp}$

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots$$

that has a colimit in $\mathcal{C}$, the canonical morphism $A_0 \longrightarrow \mathrm{colim}_n A_n$ belongs to $\mathcal{E}^{\perp}$.

Exercise E.10. Show that the following classes of cofibrations and weak equivalences define cofibration structures on the respective categories.

(a) The category of compactly generated spaces with respect to the h-cofibrations and the weak homotopy equivalences.

(b) The category of simplicial sets with respect to the monomorphisms (i.e., morphisms that are dimensionwise injective) and the weak equivalences.

(c) The category of $\mathbb{Z}$-graded chain complexes in an additive category $\mathcal{A}$, with respect to the chain maps that are dimensionwise split monomorphisms, and the chain homotopy equivalences.

(d) For a ring S, the category of $\mathbb{Z}$-graded chain complexes of S-modules with respect to the chain maps that are dimensionwise split monomorphisms, and the quasi-isomorphisms.

(e) For a ring S, the category of S-modules with respect to monomorphism of S-modules and the *op-stable equivalences*. Here a morphism $f : M \longrightarrow N$ of S-modules is an op-stable equivalence if there exists a morphism $g : N \longrightarrow M$ of S-modules such that the morphism $gf - \mathrm{Id}_M : M \longrightarrow M$ and $fg - \mathrm{Id}_N : N \longrightarrow N$ each factor through an injective S-module.

In each example, find a class of morphisms $\mathcal{E}$ such that the respective cofibrations are characterized by the left lifting property with respect to all morphisms in $\mathcal{E}$.

Exercise E.11. Let $\mathcal{A}$ be an additive category. We consider the cofibration structure on the category $\mathrm{Ch}(\mathcal{A})$ of $\mathbb{Z}$-graded chain complexes in $\mathcal{A}$ from Exercise E.10 (c).

(a) Show that two chains maps are homotopic in this cofibration structure if and only if they are chain homotopic.

(b) We define a category $\mathcal{K}(\mathcal{A})$ with objects all $\mathbb{Z}$-graded chain complexes in $\mathcal{A}$, and with morphisms the chain homotopy classes of chain maps. Show that the quotient functor from $\mathrm{Ch}(\mathcal{A}) \longrightarrow \mathcal{K}(\mathcal{A})$ is a localization at the class of chain homotopy equivalence.

Exercise E.12. Let E be an orthogonal Ω-*spectrum*, i.e., the adjoint $\tilde{\sigma}_n : E_n \longrightarrow \Omega(E_{1+n})$ of the structure map is a weak homotopy equivalence for every $n \geq 0$. Show that for every based CW-complex A (not necessarily finite), the map

$$[A, E_0] \longrightarrow \mathcal{SH}(\Sigma^\infty A, E)$$

that sends a continuous based map $A \longrightarrow E_0$ to the image of the adjoint $\Sigma^\infty A \longrightarrow E$ under the localization functor is an isomorphism of abelian groups. (Hint: induction over a CW-structure on A)

Exercise E.13. Let $\mathcal{T}$ be a triangulated category. We call a triangle (f, g, h) in $\mathcal{T}$ *anti-distinguished* if the triangle $(-f, -g, -h)$ is distinguished in the original triangulation of $\mathcal{T}$. Show that the class of anti-distinguished triangles is also a triangulation of $\mathcal{T}$ (with respect to the same suspension functor).

Exercise E.14. Let $\mathcal{T}$ be a triangulated category and $\Sigma^{-1} : \mathcal{T} \longrightarrow \mathcal{T}$ a quasi-inverse to the suspension functor, i.e., a functor endowed with a natural isomorphism $\psi_A : A \cong \Sigma(\Sigma^{-1}A)$. We call a triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma^{-1}A$$

in the opposite category $\mathcal{T}^{\mathrm{op}}$ *op-distinguished* if the triangle

$$\Sigma^{-1}A \xrightarrow{h} C \xrightarrow{g} B \xrightarrow{\psi_A \circ f} \Sigma(\Sigma^{-1}A)$$

is distinguished in the original triangulation of $\mathcal{T}$. Show that the opposite category $\mathcal{T}^{\mathrm{op}}$ is a triangulated category with respect to the functor $\Sigma^{-1} : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{T}^{\mathrm{op}}$ as suspension functor and the class of op-distinguished triangles.

Exercise E.15. Let $\mathcal{T}$ be a triangulated category and $f_n : X_n \longrightarrow X_{n+1}$ a sequence of composable morphism for $n \geq 0$. Let $(\bar{X}, \varphi_n)$ and $(\bar{X}', \varphi'_n)$ be two homotopy colimits of the sequence (X_n, f_n). Construct an isomorphism $\psi : \bar{X} \longrightarrow \bar{X}'$ satisfying $\psi\varphi_n = \varphi'_n$ and commuting with the connecting morphisms to the suspension of $\bigoplus_{n\geq 0} X_n$. To what is extent it the isomorphism ψ unique?

Exercise E.16. Let $\mathcal{T}$ be a triangulated category with countable sums. Let X be any object of $\mathcal{T}$ and $e : X \longrightarrow X$ an idempotent endomorphism. Show that e splits in the following sense: there are objects eX and $(1-e)X$ and an isomorphism between X and the sum $eX \oplus (1-e)X$ under which $e : X \longrightarrow X$ corresponds to the endomorphism $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ of $eX \oplus (1-e)X$. (Hint: use that homotopy colimits exist in $\mathcal{T}$ and construct eX as the homotopy colimit of the sequence of e's).

Exercise E.17. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms in a triangulated category $\mathcal{T}$ such that the composite $gf : X \longrightarrow Z$ zero and the group $[\Sigma X, Z]$ is trivial. Show that there is at most one morphism $h : Z \longrightarrow \Sigma X$ such that (f, g, h) is a distinguished triangle.

Exercise E.18. Let $\psi : X \longrightarrow Y$ be a morphism of orthogonal spectra that is levelwise a Serre fibration. Let $\iota : F \longrightarrow X$ denote the inclusion of the strict fiber of ψ.

(a) Show that the morphism $l : F \wedge S^1 \longrightarrow C\psi$ induced by the pullback square

$$\begin{tikzcd} F \arrow[r, "\iota"] \arrow[d] & X \arrow[d, "\psi"] \\ * \arrow[r] & Y \end{tikzcd}$$

on the vertical mapping cones is a stable equivalence.

(b) As before we let $i : Y \longrightarrow C\psi$ denote the inclusion into the mapping cone. Show the triangle

$$F \xrightarrow{\gamma(\iota)} X \xrightarrow{\gamma(\psi)} Y \xrightarrow{-\gamma(l)^{-1} \circ \gamma(i)} \Sigma F$$

is distinguished in the stable homotopy category.

Exercise E.19. Let X be any space. Show that the homomorphism

$$\mathcal{N}_k(X) \longrightarrow H_k(X;\mathbb{F}_2) , \quad [M,h] \longmapsto h_*[M]$$

that evaluates a singular manifold at the mod-2 fundamental class is an isomorphism for $k = 0$, and surjective for $k = 1$.

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