Math. Ann. 319, 517–532 (2001) **Mathematische Annalen**

Digital Object Identifier (DOI) 10.1007/s002080000165

S-modules and symmetric spectra

Stefan Schwede

Received March 24, 2000 / Revised June 16, 2000 /
Published online October 30, 2000 – © Springer-Verlag 2000

Mathematics Subject Classification (1991): 55U35, 55P42

1. Introduction

Stable homotopy theory studies the homotopy category of spectra. This category has a symmetric monoidal smash product which allows the definition of ring spectra 'up to homotopy'. In recent years there was an increasing interest in more refined notions of ring spectra which are associative (and possibly commutative) up to coherent homotopy, and a complex machinery developed around this issue. The coherence questions can be avoided if there is a model for the category of spectra (not just its homotopy category) which admits a symmetric monoidal smash product. For a long time no such category was known, and there was even evidence that it might not exist [Lew91].

Then at approximately the same time, two categories of spectra with nice smash products were discovered. Elmendorf, Kriz, Mandell and May constructed the category of *S-modules* [EKMM], and Jeff Smith introduced *symmetric spectra* [HSS]. Both categories are Quillen model categories [Q, Hov] and have associated notions of ring and module spectra. However these two categories arise in completely different ways. And even though the homotopy categories are equivalent, it is not a priori clear if both frameworks give rise to the same homotopy theory of rings and modules. Both categories have their merits, described in detail in the introductions of [EKMM] and [HSS], and it is desirable to be able to translate results obtained in one category into conclusions valid in the other. The present paper describes a mechanism which makes such comparisons possible.

Below we define a lax symmetric monoidal functor $\Phi : \mathcal{M}_S \longrightarrow Sp^{\Sigma}$ from the category of S-modules to the category of symmetric spectra. The functor Φ preserves homotopy groups and has a strong symmetric monoidal left adjoint.

S. Schwede
Fakultät für Mathematik, Universität Bielefeld, 33615 Bielefeld, Germany
(e-mail: schwede@mathematik.uni-bielefeld.de)

Research supported by a BASF-Forschungsstipendium der Studienstiftung des deutschen Volkes

We show that the two functors induce inverse equivalences of the homotopy categories of spectra, ring spectra, commutative ring spectra and module spectra:

Main Theorem. *The functor Φ from the category $\mathcal{M}_S$ of S-modules to the category Sp^Σ of symmetric spectra passes to a symmetric monoidal equivalence of homotopy categories*

$$\mathrm{Ho}\,(\mathcal{M}_S) \xrightarrow{\cong} \mathrm{Ho}\,(Sp^\Sigma)\ .$$

Furthermore, Φ induces equivalences of homotopy categories

$$\mathrm{Ho}\,(S\text{-algebras}) \xrightarrow{\cong} \mathrm{Ho}\,(\text{symmetric ring spectra})\ ,$$

$$\mathrm{Ho}\,(\text{com. } S\text{-algebras}) \xrightarrow{\cong} \mathrm{Ho}\,(\text{com. symmetric ring spectra})\ ,$$

and

$$\mathrm{Ho}\,(\mathcal{M}_R) \xrightarrow{\cong} \mathrm{Ho}\,(\Phi(R)\text{-modules})$$

for any S-algebra R.

All equivalences in the main theorem follow from the fact that the functor Φ is the right adjoint of various Quillen equivalences [Hov, 1.3.12]. The precise statement is in Theorem 5.1, which also deals with some additional cases of structured spectra.

When using the terms ‘fibrations’ or ‘cofibrations’ in the context of S-modules, we refer to the model category structure of [EKMM, VII 4.6], where those notions are called ‘q-fibrations’ and ‘q-cofibrations’. In the context of symmetric spectra, we consider different kinds of stable model category structures: the ‘stable’ [HSS, 3.4.4] or [MMSS, 9.2], the ‘positive stable’ [MMSS, 14.2] and the ‘S’-model category structure [HSS, 5.3.6]. The S-model structure is only hinted at in [HSS], and we will not use the fact that the model category axioms are satisfied. The weak equivalences are always the same, they are the *stable equivalences* [HSS, 3.1.3] or [MMSS, 8.3]. In particular the different choices of cofibrations lead to the same stable homotopy category of symmetric spectra. The unit of the smash product (i.e., the sphere spectrum) is denoted ‘S’ in both [EKMM] and [HSS]. In order to distinguish between these two objects we use the notation S_Σ for the symmetric sphere spectrum.

In the preprint version of [HSS], symmetric spectra were treated based on both simplicial sets and topological spaces. The published version is written entirely simplicially, but for the comparison with S-modules it is convenient to use topological spaces. We will refer to [MMSS] for a published treatment of symmetric spectra of topological spaces; as one expects, the two versions are Quillen equivalent, see [MMSS, 19.3]. As in [MMSS], a *space* is a compactly generated weak Hausdorff space. One reference is [McC], and a more detailed treatment is given in the appendix of Lewis’ thesis [Lew78]. We denote by $\mathcal{T}$ the category of pointed spaces.

The present paper could not have been written without the collaboration with Mike Mandell, Peter May and Brooke Shipley. It is a spin-off of our joint effort to understand the relationship between various spectra categories, which ultimately led to the paper [MMSS]. I learned a lot of what I know about S-modules and symmetric spectra through our extensive discussions. I would also like to thank Mike Hopkins, Jeff Smith and Charles Rezk for several clarifying conversions on topics related to this paper.

2. An adjoint functor pair

In order to define the functor Φ from the category $\mathcal{M}_S$ of S-modules to the category Sp^Σ of symmetric spectra we start by choosing a desuspension of the sphere S-module. To be specific we let S_c^{-1} be the S-module $S \wedge_{\mathcal{L}} \mathbb{L} S^{-1} = S \wedge_{\mathcal{L}} \mathbb{L} \Sigma_1^\infty S^0$ defined in [EKMM, II 1.7]. What matters is not the precise form of S_c^{-1}, but that it is a cofibrant desuspension of the sphere S-module, i.e., it comes with a weak equivalence $S_c^{-1} \wedge S^1 \longrightarrow S$, where S^1 denotes the circle. We use the notation S_c^{-1}, as opposed to S_S^{-1} used in [EKMM], to emphasize that this is a cofibrant model for the (-1)-sphere. For $n > 0$ we define S_c^{-n} to be the n-fold smash power of the S-module S_c^{-1}, endowed with the permutation action of the symmetric group on n letters. We set $S_c^0 = S$, the unit of the smash product; here the notation is slightly misleading since S_c^0 is *not* cofibrant. The functor Φ is then given by

$$\Phi(M)_n \;=\; \mathcal{M}_S(S_c^{-n}, M)$$

where the right hand side is the topological mapping space in the category of S-modules. The symmetric group acts on the mapping space through the permutation action of the source. The m-fold smash power of the desuspension map $S_c^{-1} \wedge S^1 \longrightarrow S$ induces a map

$$\mathcal{M}_S(S_c^{-n}, M) \longrightarrow \mathcal{M}_S(S_c^{-(n+m)} \wedge S^m, M) \cong \mathcal{T}(S^m, \mathcal{M}_S(S_c^{-(n+m)}, M))$$

whose adjoint

$$S^m \wedge \mathcal{M}_S(S_c^{-n}, M) \longrightarrow \mathcal{M}_S(S_c^{-(n+m)}, M)$$

makes $\Phi(M)$ into a symmetric spectrum. For $n \geq 1$, the S-module S_c^{-n} is a cofibrant model of the (-n)-sphere spectrum. So the functor Φ takes weak equivalences of S-modules to maps which are level equivalences above level 0, and the i-th homotopy group of the space $\Phi(M)_n$ is isomorphic to the $(i - n)$-th homotopy group of the S-module M by [EKMM, II 1.8]. In particular there is a natural isomorphism of stable homotopy groups $\pi_* \Phi(M) \cong \pi_* M$, and Φ takes equivalences of S-modules to stable homotopy equivalences of symmetric spectra.

The functor Φ is lax symmetric monoidal: smashing maps induces

$$\mathcal{M}_S(S_c^{-m}, M) \wedge \mathcal{M}_S(S_c^{-n}, N) \longrightarrow \mathcal{M}_S(S_c^{-(m+n)}, M \wedge N)$$

which assemble into a natural map $\Phi(M) \wedge \Phi(N) \longrightarrow \Phi(M \wedge N)$ of symmetric spectra. The unit map $S_\Sigma \longrightarrow \Phi(S)$ comes from the identity map of S which is a point in $\Phi(S)_0$ (S_Σ is our notation for the symmetric sphere spectrum).

Proposition 2.1. *The map $S_\Sigma \longrightarrow \Phi(S)$ is a stable equivalence of symmetric spectra.*

Proof. We choose an acyclic cofibration $i : S_\Sigma \longrightarrow QS_\Sigma$ to a fibrant object in the positive stable model structure of symmetric spectra [MMSS, 14.2]. Then QS_Σ is an Ω-spectrum from level one on [MMSS, 14.2], $(QS_\Sigma)_1$ is a model for the infinite loop space $\Omega^\infty \Sigma^\infty S^1$, and the map

$$i_1 : S^1 = (S_\Sigma)_1 \longrightarrow (QS_\Sigma)_1$$

is a generator for the infinite cyclic fundamental group of $(QS_\Sigma)_1$.

The symmetric spectrum $\Phi(S)$ is also an Ω-spectrum from level one on, hence fibrant in the positive stable model structure, and so we can choose a map $j : QS_\Sigma \longrightarrow \Phi(S)$ which extends the map $S_\Sigma \longrightarrow \Phi(S)$. The S-module S_c^{-1} was defined as $S \wedge_{\mathcal{L}} \mathbb{L} S^{-1} = S \wedge_{\mathcal{L}} \mathbb{L} \Sigma_1^\infty S^0$, so the space $\Phi(S)_1$ can be rewritten as

$$\Phi(S)_1 = \mathcal{M}_S(S_c^{-1}, S) \cong \mathcal{S}(\Sigma_1^\infty S^0, F_{\mathcal{L}}(S, S)) \cong F_{\mathcal{L}}(S, S)(\mathbb{R}^1)$$

using the adjunction [EKMM, II 1.3] and the fact that $\mathbb{L}$ is left adjoint to the forgetful functor from $\mathbb{L}$-spectra to spectra ($F_{\mathcal{L}}(S, S)$ is the function $\mathbb{L}$-spectrum of [EKMM, I 7.1]). By [EKMM, I 8.7] there is a weak equivalence from the underlying $\mathbb{L}$-spectrum of S to the function spectrum $F_{\mathcal{L}}(S, S)$; hence the space $\Phi(S)_1$ is weakly equivalent to $S(\mathbb{R}^1)$, which is another model for the infinite loop space $\Omega^\infty \Sigma^\infty S^1$. Moreover the map

$$S^1 = (S_\Sigma)_1 \longrightarrow \Phi(S)_1 = \mathcal{M}_S(S_c^{-1}, S)$$

is adjoint to the weak equivalence $S_c^{-1} \wedge S^1 \longrightarrow S$, so it also is a generator for the infinite cyclic fundamental group of $\Phi(S)_1$.

We conclude that $j_1 : (QS_\Sigma)_1 \longrightarrow \Phi(S)_1$ is an infinite loop map between two spaces which are each weakly equivalent to $\Omega^\infty \Sigma^\infty S^1$, and that j_1 induces an isomorphism on fundamental groups. Hence the map j_1 is a weak equivalence, so $j : QS_\Sigma \longrightarrow \Phi(S)$ is a level equivalence from level one on, hence a stable equivalence of symmetric spectra. Thus the original map $S_\Sigma \longrightarrow \Phi(S)$ is also a stable equivalence. □

Since the category of S-modules is enriched over the category of based spaces, there is a natural isomorphism

$$\Phi(M)^K_n \;=\; \mathcal{T}(K, \mathcal{M}_S(S_c^{-n}, M)) \;\cong\; \mathcal{M}_S(S_c^{-n}, M^K)$$

for K a based space, M an S-module and $n \geq 0$. For varying n these assemble into an isomorphism $\Phi(M)^K \cong \Phi(M^K)$. Taking adjoints gives a natural map $K \wedge \Phi(M) \longrightarrow \Phi(K \wedge M)$ and the special case of the unit interval induces a natural transformation

$$\mathrm{Cone}(\Phi(f)) \longrightarrow \Phi(\mathrm{Cone}(f)) ,$$

where $\mathrm{Cone}(f) = I \wedge X \cup_{1 \times X} Y$ is the mapping cone of a map $f : X \longrightarrow Y$ of symmetric spectra or S-modules (here the unit interval I is pointed by $0 \in I$).

Lemma 2.2. *For every map $f : X \longrightarrow Y$ of S-modules the natural map* $\mathrm{Cone}(\Phi(f)) \longrightarrow \Phi(\mathrm{Cone}(f))$ *is a π_*-isomorphism of symmetric spectra. For every family $\{X_i\}_{i \in I}$ of S-modules the natural map $\bigvee_I \Phi(X_i) \longrightarrow \Phi\left(\bigvee_I X_i\right)$ is a π_*-isomorphism of symmetric spectra.*

Proof. By [MMSS, 7.4 (vi)] there is a long exact sequence relating the homotopy groups of source, target and mapping cone of a map of symmetric spectra. The same is true for S-modules by [EKMM, I 6.4] since mapping cones are defined on underlying $\mathbb{L}$-spectra.

The functor Φ preserves homotopy groups, so the statement about wedges follows once we know that in the categories of symmetric spectra and S-modules the homotopy groups of a wedge coincide with the direct sum of the homotopy groups of the wedge summands. For symmetric spectra this is shown in [MMSS, 7.4 (ii)]. We were unable to find a reference for the corresponding property in the category of S-modules, so we provide the argument: the homotopy groups of an S-module are the homotopy groups of the underlying coordinate free spectrum, and wedges are also formed on underlying spectra [EKMM, I 4.4 and II 1.4]. If $\{X_i\}_{i \in I}$ is a family of coordinate free spectra, then their coproduct $\bigvee_I X_i$ is obtained by applying the spectrification functor L of [LMS, I 2.2] to the coproduct of the underlying prespectra, which in turn is given by spacewise wedge. Even though the spacewise wedge is not a spectrum (i.e., the adjoints of the structure maps are usually not homeomorphisms), it is still an inclusion prespectrum in the sense of [LMS, I 2.1]. Hence its spectrification can be calculated via the formula

$$\left(\bigvee_I X_i\right)(V) \;=\; \mathrm{colim}_{V \subset W}\, \Omega^{W-V} \left(\bigvee_I X_i(W)\right)$$

(see [LMS, p. 13]), where V is a finite dimensional inner product sub vector space of the indexing universe and the colimit is over all vector spaces W which contain V. Hence it suffices to show that the homotopy groups of a (spacewise) wedge of prespectra are the sum of the homotopy groups, which is shown in [MMSS, 7.4 (ii)]. □

The functor Φ has a left adjoint functor $\Lambda : Sp^\Sigma \longrightarrow \mathcal{M}_S$ which is strong symmetric monoidal. The value of Λ on a symmetric spectrum A is given by the coequalizer of two maps of S-modules

$$\bigvee_{k,l\geq 0} (S_c^{-k} \wedge_{\Sigma_k} S^k) \wedge (S_c^{-l} \wedge_{\Sigma_l} A_l) \Longrightarrow \bigvee_{n\geq 0} S_c^{-n} \wedge_{\Sigma_n} A_n \ .$$

One of the maps is induced from the k-fold smash power of the desuspension map $S_c^{-1} \wedge S^1 \longrightarrow S$. The other map comes from the $(\Sigma_k \times \Sigma_l)$-equivariant structure map $S^k \wedge A_l \longrightarrow A_{k+l}$ of A.

The fact that Λ is strong symmetric monoidal now follows formally. The adjoint of the map $S_\Sigma \longrightarrow \Phi(S)$ of Proposition 2.1 is a map $\Lambda(S_\Sigma) \longrightarrow S$; that map is an isomorphism because both sides represent the same functor which sends an S-module M to the mapping space $\mathcal{M}_S(S, M) = \Phi(M)_0$.

Combining the units of the adjunction, an instance of the monoidal map $\Phi(M) \wedge \Phi(N) \longrightarrow \Phi(M \wedge N)$ and the counit of the adjunction yields a natural map

$$\begin{aligned}\Lambda(A \wedge B) \longrightarrow \Lambda(\Phi(\Lambda(A)) \wedge \Phi(\Lambda(B))) &\longrightarrow \Lambda(\Phi(\Lambda(A) \wedge \Lambda(B))) \\ &\longrightarrow \Lambda(A) \wedge \Lambda(B) \ .\end{aligned}$$

Since Φ is symmetric monoidal its adjoint Λ becomes a symmetric *co*monoidal functor with respect to these maps. We claim that moreover the map $\Lambda(A \wedge B) \longrightarrow \Lambda(A) \wedge \Lambda(B)$ is an isomorphism, so that Λ becomes a strong symmetric monoidal functor with respect to the inverse transformation. Every symmetric spectrum is a colimit of spectra of the form $S \otimes W$ where W is a pointed Σ_m-space, viewed as a symmetric sequence concentrated in level m, and $\otimes$ denotes the tensor product of symmetric sequences [HSS, 2.1.3]. Since Λ preserves colimits, we only have to prove the claim when A and B are of this special form. The S-module $\Lambda(S_\Sigma \otimes W)$ is isomorphic to $S_c^{-m} \wedge_{\Sigma_m} W$ since both represent the functor which sends an S-module M to the space of Σ_m-equivariant maps from W to $\Phi(M)_m$. In particular for a free symmetric spectrum $F_m K = S_\Sigma \otimes ((\Sigma_m)_+ \wedge K)$ [HSS, 2.2.5] we obtain an isomorphism $\Lambda(F_m K) \cong S_c^{-m} \wedge K$. If V is a Σ_n-space, viewed as a symmetric sequence concentrated in level n, then the S-modules

$$\Lambda((S \otimes W) \wedge (S \otimes V)) \ \cong \ \Lambda(S \otimes ((\Sigma_{m+n})_+ \wedge_{\Sigma_m \times \Sigma_n} (W \wedge V))$$

and

$$\begin{aligned}\Lambda(S \otimes W) \wedge \Lambda(S \otimes V) &\cong (S_c^{-m} \wedge_{\Sigma_m} W) \wedge (S_c^{-n} \wedge_{\Sigma_n} V) \\ &\cong S_c^{-(m+n)} \wedge_{\Sigma_m \times \Sigma_n} (W \wedge V)\end{aligned}$$

both represent the same functor, via the instance of the transformation $\Lambda(A \wedge B) \longrightarrow \Lambda(A) \wedge \Lambda(B)$, and hence they are isomorphic.

3. Homotopical analysis

The functor Φ preserves and reflects weak equivalences between arbitrary S-modules, so it passes to a functor on homotopy categories. The left adjoint Λ is only homotopically well-behaved for certain symmetric spectra. In order to single out a big enough class of symmetric spectra for which Λ can be controlled homotopically, we use the notion of an *S-cofibrant* symmetric spectrum. In [HSS, 5.3.6], Hovey, Shipley and Smith introduce the S-model structure for symmetric spectra of simplicial sets. However, they defer the verification of the model category axioms to a future paper of Smith about commutative symmetric ring spectra. For lack of published reference we will not use the S-model structure explicitly, but rather recall and prove those aspects which are relevant for us.

A map $X \longrightarrow Y$ of symmetric spectra is an *S-acyclic fibration* if for all $m \geq 0$ its m-th level $X_m \longrightarrow Y_m$ is a Σ_m-acyclic fibration of spaces (i.e., a Serre fibration and weak equivalence on fixed points for all subgroups of the symmetric group Σ_m). A map of symmetric spectra is an *S-cofibration* if it has the left lifting property [Hov, 1.1.2] with respect to all S-acyclic fibrations. Every stable cofibration (see [HSS, 3.4.1]; stable cofibrations are called 'q-cofibrations' in [MMSS, 6.1 (vi)]) is an S-cofibration, but not vice versa.

Theorem 3.1. *For every S-cofibrant symmetric spectrum A, the unit $A \longrightarrow \Phi(\Lambda(A))$ of the adjunction is a stable equivalence of symmetric spectra.*

Proof. For the proof we need yet another class of cofibrations: a map $A \longrightarrow B$ of symmetric spectra is an *h-cofibration* if it has the homotopy extension property in the classical sense, i.e., if every map from $I_+ \wedge A \cup_A B$ can be extended to $I_+ \wedge B$ (I is the unit interval). Equivalently, a map is an h-cofibration if and only if it has the left lifting property with respect to the evaluation map $X^I \longrightarrow X$ (induced by the inclusion $\{0\} \longrightarrow I$) for all symmetric spectra X. Such an evaluation map is levelwise a Σ_m-acyclic fibration, hence every S-cofibration is an h-cofibration.

For the course of this proof we call a symmetric spectrum *good* if the unit of the adjunction is a stable equivalence for this spectrum. We prove a sequence of claims, the last of which gives the theorem.

(a) A symmetric spectrum is good if and only if its suspension is good.
(b) The mapping cone of any map between good symmetric spectra is good.
(c) If $A \longrightarrow B$ is an h-cofibration of symmetric spectra and if A and B/A are good, then so is B.
(d) The class of good spectra is closed under wedges.
(e) Given a sequence $X_n \longrightarrow X_{n+1}, n \geq 0$, of h-cofibrations of symmetric spectra such that every X_n is good, then the colimit of the sequence is also good.
(f) Let $m \geq 0$ and let H be a subgroup of the symmetric group Σ_m. Then the symmetric spectrum $S \otimes (\Sigma_m/H)_+$ is good, where $\otimes$ denotes the tensor

product of symmetric sequences [HSS, 2.1.3] and $(\Sigma_m/H)_+$ is viewed as symmetric sequence concentrated in level m.

(g) Let I_S be the set of maps of symmetric spectra

$$S \otimes (\Sigma_m/H \times S^{n-1})_+ \longrightarrow S \otimes (\Sigma_m/H \times D^n)_+$$

for $n, m \geq 0$ and H a subgroup of the symmetric group Σ_m. Every relative I_S-cell complex is good (i.e., every sequential colimit of pushouts of wedges of maps in I_S, see e.g. [MMSS, 5.4]).

(h) Every S-cofibrant symmetric spectrum is good.

(a) The functor Λ commutes with suspension, and the map $\Sigma\Phi(\Lambda(A)) \longrightarrow \Phi(\Sigma\Lambda(A)) \cong \Phi(\Lambda(\Sigma A))$ is a π_*-isomorphism, hence a stable equivalence, by Lemma 2.2. Claim (a) follows since the unit map $A \longrightarrow \Phi(\Lambda(A))$ is a stable equivalence if and only if its suspension $\Sigma A \longrightarrow \Sigma\Phi(\Lambda(A))$ is a stable equivalence, see [MMSS, 8.12 (i')].

(b) Let $f : A \longrightarrow B$ be a map between good symmetric spectra. The mapping cylinder of a map is homotopy equivalent to the target object, hence the map $\mathrm{Cyl}(f) \longrightarrow \mathrm{Cyl}(\Phi(\Lambda(f)))$ is a stable equivalence since B is good. Since A is good and the source inclusion of a mapping cylinder is an h-cofibration, the induced map on mapping cones $\mathrm{Cone}(f) \longrightarrow \mathrm{Cone}(\Phi(\Lambda(f)))$ is a stable equivalence by [MMSS, 8.12 (iv)]. Since Λ commutes with taking mapping cones, the mapping cone of f is good by Lemma 2.2.

(c) For an h-cofibration $i : A \longrightarrow B$ the map $\mathrm{Cone}(i) \longrightarrow B/A$ from the mapping cone to the quotient is a homotopy equivalence. We choose a homotopy inverse $B/A \longrightarrow \mathrm{Cone}(i)$ and let $f : B/A \longrightarrow \Sigma A$ be the composite map $B/A \longrightarrow \mathrm{Cone}(i) \longrightarrow \Sigma A$. By parts (a) and (b) the mapping cone of f is good. But the mapping cone of f is homotopy equivalent to the suspension of B, so ΣB is good and hence B is good by part (a).

(d) For any family $\{A_i\}_{i\in I}$ of symmetric spectra the map

$$\bigvee_{i\in I} \Phi(\Lambda(A_i)) \longrightarrow \Phi(\bigvee_{i\in I} \Lambda(A_i)) \cong \Phi(\Lambda(\bigvee_{i\in I} A_i))$$

is a π_*-isomorphism, thus a stable equivalence, by Lemma 2.2. Claim (d) follows since a wedge of stable equivalences is again a stable equivalence, see [MMSS, 8.12 (ii)].

(e) Since the maps $X_n \longrightarrow X_{n+1}$ are h-cofibrations, the colimit of the sequence is homotopy equivalent to the mapping telescope. The wedge of the spectra X_n includes by an h-cofibration into the mapping telescope, with quotient the wedge of the suspension of the spectra X_n. Hence the mapping telescope is good by parts (a), (c) and (d). Since both Λ and Φ preserve homotopy equivalences, the colimit of the sequence is also good.

(f) The case $m = 0$ is proved in Proposition 2.1, so we may assume $m \geq 1$. We consider the commutative diagram

$$\begin{array}{ccccc} S\otimes(\Sigma_m/H\times E\Sigma_m)_+ & \longrightarrow & \Phi(\Lambda(S\otimes(\Sigma_m/H\times E\Sigma_m)_+)) & \xrightarrow{\cong} & \Phi(S_c^{-m}\wedge_{\Sigma_m}(\Sigma_m/H\times E\Sigma_m)_+) \\ \downarrow & & \downarrow & & \downarrow \\ S\otimes(\Sigma_m/H)_+ & \longrightarrow & \Phi(\Lambda(S\otimes(\Sigma_m/H)_+)) & \xrightarrow[\cong]{} & \Phi(S_c^{-m}\wedge_{\Sigma_m}(\Sigma_m/H)_+) \end{array}$$

where $E\Sigma_m$ is a free and non-equivariantly contractible Σ_m-CW-complex and $\Sigma_m/H \times E\Sigma_m$ carries the diagonal action. Since $\Sigma_m/H \times E\Sigma_m \longrightarrow \Sigma_m/H$ is a weak equivalence of underlying spaces, the left vertical map is a levelwise equivalence, hence a stable equivalence, of symmetric spectra. Since S_c^{-1} is a cell S-module, the map

$$S_c^{-m}\wedge_{\Sigma_m}(\Sigma_m/H \times E\Sigma_m)_+ \longrightarrow S_c^{-m}\wedge_{\Sigma_m}(\Sigma_m/H)_+$$

is a weak equivalence of S-modules. Indeed the case $H = \Sigma_m$ is established in [EKMM, III 5.1], but the same argument works for a general subgroup. Hence the right vertical map is a stable equivalence and it suffices to show that the symmetric spectrum $S \otimes (\Sigma_m/H \times E\Sigma_m)_+$ is good.

The skeleton inclusions of $\Sigma_m/H \times E\Sigma_m$ are h-cofibrations, so by parts (c) and (e) it suffices to show the claim for the subquotients of the skeleton filtration, which are wedges of Σ_m-spaces of the form $(\Sigma_m)_+\wedge S^n$. By (a) we may assume $n = 0$. Because of the isomorphism $S \otimes (\Sigma_m)_+ \cong F_m S^0$ we only have to verify that $F_m S^0$ is good, or, by (a) again, that $F_m S^m$ is good. The map of symmetric spectra $F_m S^m \longrightarrow S_\Sigma$ which is the identity at level m is a stable equivalence. The functor Λ takes this stable equivalence to the m-fold power of the desuspension map $S_c^{-1}\wedge S^1 \longrightarrow S$, which is a weak equivalence of S-modules. The symmetric suspension spectrum S_Σ is good by Proposition 2.1, hence so is $F_m S^m$.

(g) By parts (a) and (f) the cofibers of the maps in I_S are good. All maps in I_S are h-cofibrations, so parts (c) and (d) show that good spectra are closed under pushout along a coproduct of maps in I_S. By part (e) the colimits of a sequence of such maps is good, so all I_S-cell complexes are good.

(h) Quillen's small object argument shows that every S-cofibrant spectrum is a retract of an I_S-cell complex, so part (g) gives the conclusion. More precisely, the domains of the maps in I_S are compact in the sense of [MMSS, 5.6] and I_S satisfies the Cofibration Hypothesis of [MMSS, 5.3]. Hence the small object argument in the formulation of [MMSS, 5.8] provides an I_S-cell complex A^c and a map $p : A^c \longrightarrow A$ which has the right lifting property for the maps in I_S. For our choice of I_S this means that $p_m : A^c_m \longrightarrow A_m$ is a Σ_m-acyclic fibration of spaces for all $m \geq 0$. Since A is S-cofibrant it has the left lifting property for p, and so A is indeed a retract of if the I_S-cell complex A^c. Since A^c is good, so is A. □

4. S-cofibrant symmetric spectra

Because of the good formal properties of the functors Λ and Φ, they induce adjoint functor pairs between the various categories of rings, modules and algebras based on symmetric spectra and S-modules. We can control the homotopical behavior of Λ in these cases because cofibrant objects in the respective categories of rings, modules and algebras are in particular S-cofibrant as symmetric spectra; this allows us to apply Theorem 3.1.

Let R be a symmetric ring spectrum. We call a map of R-modules an *S-cofibration* if it has the left lifting property with respect to all R-module homomorphisms which are S-acyclic fibrations of underlying symmetric spectra.

Lemma 4.1. *Let $f : R \longrightarrow P$ be a map of symmetric ring spectra. Then extension of scalars along f takes S-cofibrations of R-modules to S-cofibrations of P-modules. If P is S-cofibrant as an R-module, then restriction of scalars along f takes S-cofibrations of P-modules to S-cofibrations of R-modules.*

Proof. Since restriction of scalars does not change the underlying symmetric spectra, its left adjoint extension functor preserves S-cofibrations. For the second statement we exploit the fact that restriction of scalars along f also has a *right* adjoint of the form $\mathrm{Hom}_R(P, -)$. If P is S-cofibrant as an R-module, then this right adjoint preserves the property of being an S-acyclic fibration on underlying spectra (the proof is similar to [HSS, 5.3.9 (3)], which proves the case $R = S_\Sigma$). So the left adjoint restriction functor preserves S-cofibrations. □

By an unpublished theorem of Jeff Smith the category of commutative symmetric ring spectra supports a model structure with the stable equivalences as the weak equivalences. The model structure is created in the *positive* stable model structure of underlying symmetric spectra. A published account for symmetric spectra based on topological spaces can be found in Sect. 15 of [MMSS]. The key technical property from our present point of view is that commutative symmetric ring spectra which are cofibrant in this model structure are also S-cofibrant as symmetric spectra. The proof of the following slightly more general result is modeled on, and refers to, [MMSS, 15.9] where it is shown that positive cofibrations between commutative symmetric ring spectra are h-cofibrations.

Cofibrant commutative symmetric ring spectra are built from free objects, so in a first step we analyze the 'free' or symmetric algebra functor $\mathbb{C}$ from symmetric spectra to commutative symmetric ring spectra. The symmetric algebra functor has the form

$$\mathbb{C}X \;=\; S_\Sigma \vee X \vee \mathrm{Sym}^2(X) \vee \ldots \vee \mathrm{Sym}^n(X) \vee \ldots$$

where $\mathrm{Sym}^n(X) = X^{\wedge n}/\Sigma_n$ is the n-th symmetric power functor.

Lemma 4.2. *Let X be a wedge of free symmetric spectra of the form $F_k S^q_+$ or $F_k S^0$ for $k, q \geq 0$. Then for all $n \geq 0$ the symmetric power* $\mathrm{Sym}^n(X)$ *is S-cofibrant.*

Proof. The symmetric smash power of a wedge is isomorphic to a wedge of smash products of smash powers (possibly with smaller exponents) of the summands. Since S-cofibrant symmetric spectra are closed under wedges and smash products [HSS, 5.3.7 (2)], it is enough to show that the symmetric smash power of every free symmetric spectrum of the form $F_k S^q_+$ or $F_k S^0$ is S-cofibrant. The only relevant property of S^q_+ and S^0 is that their n-th smash powers $(S^q_+)^{\wedge n} \cong (S^q)^n_+$ and $(S^0)^{\wedge n} \cong S^0$ are Σ_n-CW-complexes with respect to the permutation action; so for the rest of the proof we let A be any pointed space with this property and we show that $\mathrm{Sym}^n(F_k A)$ is S-cofibrant.

Since $F_k A \cong S \otimes ((\Sigma_k)_+ \wedge A)$, the n-th symmetric power of $F_k A$ is isomorphic to the symmetric spectrum $S \otimes (((\Sigma_k)_+ \wedge A)^{\otimes n} / \Sigma_n)$, where as before the tensor product is the tensor product of symmetric sequences, $(\Sigma_k)_+ \wedge A$ is viewed as a symmetric sequence concentrated in dimension k and Σ_n permutes the tensor powers. Since $S \otimes -$ is left adjoint to the forgetful functor from symmetric spectra to symmetric sequences, it suffices to show that the symmetric sequence $((\Sigma_k)_+ \wedge A)^{\otimes n} / \Sigma_n$ has the left lifting property for maps between symmetric sequences which are levelwise equivariant acyclic fibrations. This symmetric sequence is concentrated in dimension nk, so we are done if we can show that its only non-trivial space is a Σ_{nk}-CW-complex.

The nk-th level of $((\Sigma_k)_+ \wedge A)^{\otimes n}$ is isomorphic to $(\Sigma_{nk})_+ \wedge A^{\wedge n}$ with left Σ_{nk}-action on the left factor. Under this identification the permutation action of Σ_n becomes the diagonal action, permuting the n blocks of length k in Σ_{nk} and permuting the powers of A. So the orbit space by the diagonal Σ_n-action is of the form $(\Sigma_{nk})_+ \wedge_{\Sigma_n} A^{\wedge n}$ with Σ_{nk} still acting through the left factor. So the space in question is induced from the Σ_n-CW-complex $A^{\wedge n}$ along the homomorphism that injects Σ_n into Σ_{nk} as the block permutations, and so it is indeed a Σ_{nk}-CW-complex, which finishes the proof. □

Lemma 4.3. *Let $i : R \longrightarrow P$ be a cofibration in the positive model structure of commutative symmetric ring spectra* [MMSS, 15.1]. *Then i is an S-cofibration when viewed as a map of left R-modules.*

Proof. We proceed in steps. Suppose first that the map $i : R \longrightarrow P$ is the pushout of a generating cofibration $\mathbb{C}X \longrightarrow \mathbb{C}Y$ along some map $\mathbb{C}X \longrightarrow R$ of commutative symmetric ring spectra. Here $X \longrightarrow Y$ is a wedge of maps in the generating set F^+I of positive stable cofibrations [MMSS, Sect. 14]; in particular, X is a wedge of symmetric spectra of the form $F_k S^q_+$.

The pushout of a diagram $R \longleftarrow \mathbb{C}X \longrightarrow \mathbb{C}Y$ in the category of commutative symmetric ring spectra is given by the smash product $R \wedge_{\mathbb{C}X} \mathbb{C}Y$. By [MMSS, 15.10] the underlying R-module of $R \wedge_{\mathbb{C}X} \mathbb{C}Y$ can be written as the geometric realization of a certain simplicial R-module B_*. This simplicial R-module arises as a two-sided bar construction $B(R, \mathbb{C}X, \mathbb{C}T)$, where T is a wedge of free symmetric spectra of the form $F_k S^0$. However, all that matters for us is the following property: for every $k \geq 0$ the map from the R-module of degenerate k-simplices of B_* to B_k is isomorphic to the inclusion of a wedge summand whose complementary summand is a wedge of R-modules of the form

$$R \wedge \mathrm{Sym}^{n_1}(X) \wedge \ldots \wedge \mathrm{Sym}^{n_k}(X) \wedge \mathrm{Sym}^{n_{k+1}}(T) ;$$

for $k = 0$ the degenerate 0-simplices have to be interpreted as R and the splitting refers to an inclusion of R into the module of 0-simplices.

By Lemma 4.2 the symmetric spectra $\mathrm{Sym}^{n_j}(X)$ and $\mathrm{Sym}^{n_{k+1}}(T)$ are S-cofibrant. Since S-cofibrant spectra are closed under smash product [HSS, 5.3.7 (2)], all the complementary summands are S-cofibrant as R-modules. Hence the inclusions of the R-modules of degenerate simplices into B_k (for $k \geq 0$) are S-cofibrations of R-modules. S-cofibrations are closed under pushouts and sequential colimits, so by induction over simplicial skeleta the map from R to $|B_*| \cong R \wedge_{\mathbb{C}X} \mathbb{C}Y$ is an S-cofibration of R-modules.

If the map $i : R \longrightarrow P$ is a finite composition of pushouts along generating cofibrations $\mathbb{C}X \longrightarrow \mathbb{C}Y$, then we factor it as $i = i' \circ p$ where $p : R \longrightarrow R'$ is a single pushout along a generating cofibration $\mathbb{C}X \longrightarrow \mathbb{C}Y$, and $i' : R' \longrightarrow P$ is a finite, but shorter, composition of such pushouts. By induction the map i' is an S-cofibration of R'-modules. By the above the map p is an S-cofibration of R-modules; in particular, R' is S-cofibrant as an R-module. So by Lemma 4.1 the map i', and hence the original map $i = i' \circ p$, is also an S-cofibration of R-modules.

The maps in the set $\mathbb{C}(F^+ I)$ generate the cofibrations in the positive model structure for commutative symmetric ring spectra. Hence by Quillen's small object argument (in the formulation of [MMSS, 5.8]) every cofibration of commutative symmetric ring spectra is a retract of a countable composition of pushouts along maps of the form $\mathbb{C}X \longrightarrow \mathbb{C}Y$ for $X \longrightarrow Y$ a wedge of maps in the set $F^+ I$. Sequential colimits over h-cofibrations of commutative symmetric ring spectra can be calculated on underlying spectra [MMSS, 15.3], so the statement holds for countable compositions of pushouts along generating cofibrations. Hence the claim holds for retracts of such maps, thus for arbitrary cofibrations of commutative symmetric ring spectra. □

5. Quillen equivalences

In this final section we show that the adjoint functors Φ and Λ form Quillen equivalences when considered as functors between various categories of spectra, ring spectra and module spectra. Quillen equivalences in particular give rise to equivalences of the associated homotopy categories, so Theorem 5.1 below implies our main theorem stated in the introduction.

The most prominent model structure for symmetric spectra is the *stable* structure of [HSS, 3.4.4] or [MMSS, 9.2]. However there is no strong symmetric monoidal functor from the category of symmetric spectra to the category of S-modules which is also a left Quillen functor with respect to the stable model category structure. The reason for this is quite simple: any strong monoidal functor has to take the cofibrant symmetric sphere S_Σ to the non-cofibrant S-module sphere. This can be easily remedied by slightly restricting the class of cofibrations of symmetric spectra and work with the *positive* stable model structure [MMSS, 14.2]. For this one keeps the stable equivalences as weak equivalences, and the *positive cofibrations* are those stable cofibrations which are a homeomorphism at level zero. The *positive stable fibrations* are those maps which have the right lifting property with respect to all positive cofibrations which are also stable equivalences. A symmetric spectrum is fibrant in the positive stable model structure if and only if it is an Ω-spectrum from level one upwards. Note that the symmetric sphere spectrum is no longer cofibrant in the positive stable model structure. Since the positive and the stable model structure have the same weak equivalences, then give rise to the same homotopy category. Indeed, the identity functor is a left Quillen equivalence from the positive to the stable model structure,

The positive stable model structure generates similar model structures for modules over a symmetric ring spectrum [MMSS, 14.5 (i)], algebras over a commutative symmetric ring spectrum [MMSS, 14.5 (iii)], and commutative algebras over a commutative symmetric ring spectrum [MMSS, 15.2 (i)]. In each case the weak equivalences and fibrations are the maps which are stable equivalences or positive stable fibrations respectively on underlying symmetric spectra.

The categories of P-modules and P-algebras (for commutative P) also admit model structures in which the weak equivalences and fibrations are defined on underlying symmetric spectra with respect to the stable (as opposed to the positive) model structure. Whenever this is the case, the stable and positive model structures have the same homotopy category and are Quillen equivalent. However, for *commutative* algebras over a commutative symmetric ring spectrum it is crucial to work relative to the positive model structure. For example the category of all commutative symmetric ring spectra does *not* form a model category with weak equivalences and fibrations defined in the stable model structure of underlying spectra [SS, 4.5].

Theorem 5.1. *The functors Φ and Λ are a Quillen equivalence when viewed as functors between any of the following pairs of model categories. (In all the categories of structured symmetric spectra the model structure under consideration is the positive stable structure.)*

(i) *The category of symmetric spectra and the category of S-modules.*
(ii) *The category of symmetric ring spectra and the category of S-algebras.*
(iii) *The category of commutative symmetric ring spectra and the category of commutative S-algebras.*
(iv) *The category of P-modules and the category of $\Lambda(P)$-modules for any symmetric ring spectrum P which is S-cofibrant as a symmetric spectrum.*
(v) *The category of P-algebras and the category of $\Lambda(P)$-algebras for any commutative symmetric ring spectrum P which is S-cofibrant as a symmetric spectrum.*
(vi) *The category of commutative P-algebras and the category of commutative $\Lambda(P)$-algebras for any commutative symmetric ring spectrum P which is S-cofibrant as a symmetric spectrum.*

Let R be an S-algebra and note that then the counit $\Lambda(\Phi(R)) \longrightarrow R$ of the adjunction is a map of S-algebras, or even of commutative S-algebras if R is commutative.

(vii) *The functors Φ and $R\wedge_{\Lambda(\Phi(R))}\Lambda$ are a Quillen equivalence between the category of R-modules and the category of $\Phi(R)$-modules.*
(viii) *If R is commutative, then the functors Φ and $R\wedge_{\Lambda(\Phi(R))}\Lambda$ are a Quillen equivalence between the category of R-algebras and the category of $\Phi(R)$-algebras.*

Proof. Parts (i), (ii) and (iii) are special cases of (iv), (v) and (vi) respectively for $P = S_\Sigma$. The positive cofibrations of symmetric spectra are generated by the set FI^+ of maps $F_m S_+^{n-1} \longrightarrow F_m D_+^n$ for positive integers m and $n \geq 0$ (with $S^{-1} = \emptyset$), see [MMSS, 14.2]. The left adjoint Λ takes a typical generating map to $S_c^{-m} \wedge S_+^{n-1} \longrightarrow S_c^{-m} \wedge D_+^n$ which is a cofibration of S-modules since S_c^{-m} is cofibrant for positive m. By a similar inspection, Λ takes the set K^+ [MMSS, Sect. 14] of generating positive acyclic cofibrations to acyclic cofibrations of S-modules. Hence Λ and Φ form a Quillen pair [Hov, 2.1.20] with respect to the positive stable model structure of symmetric spectra. In all other cases of rings, modules and algebras, the fibrations and weak equivalences are defined on underlying S-modules or symmetric spectra respectively. Since the right adjoint Φ preserves weak equivalences and fibrations, Φ and its respective left adjoints form in Quillen functor pair in all cases. The right adjoint Φ also detects all weak equivalences. So to show parts (iv), (v) and (vi) it suffices to prove [Hov, 1.3.16] that in each case the adjunction unit is a weak equivalence on the respective cofibrant objects. We now show that those cofibrant objects are S-cofibrant as symmetric spectra. Then Theorem 3.1 shows that the adjunction unit is a weak equivalence and thus finishes the proof.

Suppose that P is any *commutative* symmetric ring spectrum. If A is a cofibrant object in the positive model structure of P-algebras, then A is also cofibrant in the absolute stable model structure of [MMSS, 12.1 (iv)]. Hence A is stably cofibrant as a P-module [MMSS, 12.1 (v)], thus S-cofibrant as a P-module. If A is a cofibrant in the positive model structure of *commutative* P-algebras, then the unit map $P \longrightarrow A$ is a cofibration of commutative symmetric ring spectra, thus an S-cofibration of P-modules by Lemma 4.3. So in cases (v) and (vi) the respective cofibrant algebras are S-cofibrant as P-modules. If P is a not necessarily commutative symmetric ring spectrum, then every P-module which is cofibrant in the positive stable model structure is also S-cofibrant as a P-module. Lemma 4.1 shows that if the ring spectrum P is S-cofibrant as a symmetric spectrum, then every S-cofibrant P-module is S-cofibrant as a symmetric spectrum. This takes care of parts (iv), (v) and (vi).

For parts (vii) and (viii) the left adjoint of Φ is not just given by Λ anymore, which takes $\Phi(R)$-modules or $\Phi(R)$-algebras to $\Lambda(\Phi(R))$-modules or $\Lambda(\Phi(R))$-algebras respectively. In addition one has to extend scalars along the counit $\Lambda(\Phi(R)) \longrightarrow R$ of the adjunction. However the precise form of the left adjoint is not relevant since we argue by showing that in each case the functor Φ passes to an equivalence of homotopy categories. For part (vii) we choose a stable equivalence of symmetric ring spectra $P \longrightarrow \Phi(R)$ with P cofibrant in the positive stable model structure of symmetric ring spectra. Since Φ and Λ are a Quillen equivalence for this model structure by part (ii), the adjoint $\Lambda(P) \longrightarrow R$ is a weak equivalence of S-algebras. We consider the commutative diagram of model categories and right Quillen functors

$$\begin{array}{ccc} \mathcal{M}_R & \xrightarrow{\text{restr.}} & \mathcal{M}_{\Lambda(P)} \\ \Phi \downarrow & & \downarrow \Phi \\ \Phi(R)\text{-mod} & \xrightarrow[\text{restr.}]{} & P\text{-mod} \end{array}$$

in which the horizontal functors are restriction of scalars along the weak equivalences of ring spectra $\Lambda(P) \longrightarrow R$ and $P \longrightarrow \Phi(R)$ respectively. Restriction of scalars along a weak equivalence induces an equivalence of homotopy categories (see [EKMM, III 4.2] for S-algebras and [HSS, 5.4.5] or [MMSS, 12.1 (vi)] for symmetric ring spectra). Furthermore the right vertical functor induces an equivalence of homotopy categories by the already established part (iv). So Φ, viewed as a functor from R-modules to $\Phi(R)$-modules, passes to an equivalence of homotopy categories, hence together with its left adjoint it forms a Quillen equivalence by [Hov, 1.3.13].

For part (viii) we choose a stable equivalence of *commutative* symmetric ring spectra $P \longrightarrow \Phi(R)$ with P cofibrant as a commutative symmetric ring spectrum. By part (iii), the adjoint $\Lambda(P) \longrightarrow R$ is then a weak equivalence of commutative S-algebras. We claim that restriction of scalars along a weak equivalence of commutative algebras passes to an equivalence of the homotopy categories of the algebras over the commutative algebras; given this, the same argument as in the previous paragraph reduces to the already established part (v). The claim is is shown in [HSS, 5.4.5] and [MMSS, 12.1 (vii)] for symmetric ring spectra. The corresponding statement for S-algebras does not appear explicitly in [EKMM], but we argue as follows. Suppose $R \longrightarrow P$ is a weak equivalence of commutative S-algebras. If A is cofibrant in the model category of R-algebras, then by [EKMM, VII 6.2] the unit map $R \longrightarrow A$ is a cofibration of R-modules. Hence the quotient A/R is a cofibrant R-module, so a retract of a cell R-module, and the map $A/R \cong R \wedge_R A/R \longrightarrow P \wedge_R A/R$ is a weak equivalence by [EKMM, III 3.8]. For a cofibrations of modules over an S-algebra, the quotient module is homotopy equivalent to the mapping cone. Since mapping cones are formed on underlying $\mathbb{L}$-spectra, [EKMM, I 6.4] gives a long exact sequences relating the homotopy groups of source, target and quotient. Hence the five lemma shows that the map $A \longrightarrow P \wedge_R A$ is a weak equivalence, and so restriction and extension of scalars form a Quillen equivalence. □

References

[EKMM] A. D. Elmendorf, I. Kriz, M. A. Mandell J. P. May. Rings, modules, and algebras in stable homotopy theory. Mathematical Surveys and Monographs 47. Amer. Math. Soc., Providence, RI, 1997. With an appendix by M. Cole.

[Hov] M. Hovey, Model categories, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999, xii+209 pp.

[HSS] M. Hovey, B. Shipley, J. H. Smith. Symmetric spectra. J. Amer. Math. Soc. **13** (2000), 149–208.

[Lew78] L. G. Lewis. The stable category and generalized Thom spectra. PhD thesis, University of Chicago (1978).

[Lew91] L. G. Lewis. Is there a convenient category of spectra ? J. Pure Appl. Algebra 73 (1991), 233-246.

[LMS] L. G. Lewis, J. P. May, M. Steinberger. Equivariant stable homotopy theory. With contributions by J. E. McClure. Lecture Notes in Math. 1213, Springer, 1986. x+538 pp.

[MMSS] M. A. Mandell, J. P. May, S. Schwede, B. Shipley. Model categories of diagram spectra. Proc. London Math. Soc., to appear.

[McC] M. C. McCord. Classifying spaces and infinite symmetric products. Trans. Amer. Math. Soc. **146** (1969), 273–298.

[Q] D. G. Quillen. Homotopical Algebra. Lecture Notes in Math. 43, Springer, 1967.

[SS] S. Schwede, B. Shipley, Algebras and modules in monoidal model categories, Proc. London Math. Soc. **80** (2000), 491–511.