COHOMOLOGY OPERATIONS AND THE STEENROD ALGEBRA

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Contents

1. Cohomology operations

Definition 1.1. Let A and B be abelian groups and n, m natural numbers. A *cohomology operation* of type (A, n, B, m) is a natural transformation of set valued functors on the category of topological spaces

$$\tau \; : \; H^n(-, A) \longrightarrow H^m(-, B) \; .$$

Note that we do not demand that $\tau_X \colon H^n(X, A) \longrightarrow H^m(X, B)$ be additive. However, two cohomology operations of the same type can be added pointwise, so the set of all cohomology operations of a fixed type forms an abelian group, which we denote $\mathrm{Oper}(A, n, B, m)$.

As before, $K(A, n)$ denotes an Eilenberg–MacLane space of type (A, n), i.e., a based space equipped with an isomorphism $\varphi \colon \pi_n(K(A, n), *) \cong A$ and such that the group $\pi_i(K(A, n), *)$ is trivial for $i \neq n$. We also assume that $K(A, n)$ is a CW-complex. The *fundamental class* $\iota_{n,A} \in H^n(K(A, n), A)$ is the unique element such that the composite

$$\pi_n(K(A, n), *) \xrightarrow{\text{Hurewicz}} H_n(K(A, n); \mathbb{Z}) \xrightarrow{\Phi(\iota)} A$$

is the isomorphism $\varphi \colon \pi_n(K(A, n), *) \cong A$. Here $\Phi \colon H^n(X; A) \longrightarrow \mathrm{Hom}(H_n(X; \mathbb{Z}), A)$ is from the universal coefficient theorem. For $n = 0$ we make the convention that $K(A, 0)$ is the group A with the discrete topology, and ι is the cohomology class represented by the identity 0-cocycle.

Lemma 1.2. *The map*

$$\mathrm{Oper}(A, n, B, m) \longrightarrow H^m(K(A, n), B)$$

which takes a cohomology operation $\tau \colon H^n(-; A) \longrightarrow H^m(-; B)$ *to the image of the fundamental class* $\tau(\iota_{n,A}) \in H^m(K(A, n), B)$ *is an isomorphism from the group of cohomology operations of type* (A, n, B, m) *and the* m*-th cohomology group of* $K(A, n)$ *with coefficients in* B*.*

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Proof. On the homotopy category of CW-complexes, the cohomology functor $H^n(-;A)$ is representable by the Eilenberg-MacLane space $K(A,n)$, i.e., $H^n(-;A)$ is naturally isomorphic to $[-,K(A,n)]$, by evaluation at the fundamental class.

If F is any functor from the homotopy category of CW-complexes to the category of sets, then the Yoneda lemma says that the natural transformations from the representable functor $[-,K(A,n)]$ to F are in bijective correspondence with the set $F(K(A,n))$, by evaluation at $(K(A,n),\mathrm{Id})$. Taking $F = H^m(-;B)$ shows that there is a unique natural transformation

$$\tau \ = \ \{\tau_X : H^n(X,A) \longrightarrow H^m(X,B)\}_{X\,:\,\mathrm{CW}}$$

of functors *on the homotopy category of CW-complexes* with the property of the lemma.

Every space Y has a CW-approximation $f\colon X \xrightarrow{\sim} Y$, i.e., a weak homotopy equivalence from a CW-complex. Moreover, the CW-approximation is unique up to preferred isomorphism in the homotopy category. Singular cohomology takes weak homotopy equivalences to isomorphisms. So there is a unique way to extend the natural transformation from CW-complexes to arbitrary spaces: we must define τ_Y as the unique map that makes the following diagram commute:

$$\begin{tikzcd} H^n(Y,A) \arrow[r, "\tau_Y"] \arrow[d, "f^*"', "\cong"] & H^m(Y,B) \arrow[d, "\cong"', "f^*"] \\ H^n(X,A) \arrow[r, "\tau_X"'] & H^m(X,B) \end{tikzcd}$$

□

Example 1.3. (i) The space $K(A,n)$ is $(n-1)$-connected, so we have $H^0(K(A,n),B) \cong B$ and $H^m(K(A,n),B) \cong 0$ for $1 \le m \le n-1$. So the only cohomology operations of type $(A,n,B,0)$ are the constant operations associated to the elements of B, and there are no non-trivial operations of type (A,n,B,m) for $1 \le m \le n-1$.

(ii) Any homomorphism of coefficient groups $f\colon A \longrightarrow B$ induces a cohomology operation of type (A,n,B,n) for every n. Since a $K(A,n)$ is $(n-1)$-connected and $H_n(K(A,n);\mathbb{Z}) \cong \pi_n(A,*) \cong A$, the universal coefficient theorem yields an isomorphism

$$H^n(K(A,n),B) \ \cong \ \mathrm{Hom}(A,B) \ ,$$

which shows that the cohomology operations of type (A,n,B,n) all arise from coefficient homomorphisms.

(iii) The Bockstein homomorphism $\delta\colon H^n(X;A) \longrightarrow H^{n+1}(X;B)$ associated to a short exact sequence of abelian groups

$$0 \ \longrightarrow \ B \ \longrightarrow \ E \ \longrightarrow \ A \ \longrightarrow \ 0$$

is a cohomology operation of type $(A,n,B,n+1)$ for every n. This gives a map

$$\mathrm{Ext}(A,B) \ \longrightarrow \ \mathrm{Oper}(A,n,B,n+1) \ .$$

The universal coefficient theorem yields a short exact sequence

$$0 \ \longrightarrow \ \mathrm{Ext}(A,B) \ \longrightarrow \ H^{n+1}(K(A,n),B) \ \longrightarrow \ \mathrm{Hom}(H_{n+1}(K(A,n);\mathbb{Z}),B) \ \longrightarrow \ 0 \ ,$$

so this map is injective. Moreover, for $n \ge 2$, the homology group $H_{n+1}(K(A,n);\mathbb{Z})$ is trivial (see e.g. [EM, Thm. 20.5]), so in that case every cohomology operation of type $(A,n,B,n+1)$ is the Bockstein homomorphism of an abelian group extension.

(iv) The group $H_2(K(A,1);\mathbb{Z})$ is not generally trivial, so there are cohomology operations of type $(A,1,B,2)$ which do not come from short exact sequences of coefficient groups. Indeed, for any group G, not necessarily abelian, $H^2(K(G,1);B)$ classifies equivalence classes of *central group extension* of G by B, i.e., short exact sequences of groups

$$0 \ \longrightarrow \ B \ \longrightarrow \ E \ \longrightarrow \ G \ \longrightarrow \ 1 \tag{1.4}$$

such that B is contained in the center of E. Exercise 1.6 below explains how to construct a non-abelian Bockstein operation from such a central extension. A proof of the correspondence between $H^2(K(G,1);B)$ and classes of central extensions can be found in [McL, IV Thm. 6.2] (in the special case of trivial coefficient modules). If G is abelian, then the image of $\mathrm{Ext}(G,B)$ in $H^2(K(G,1);B)$ corresponds to those central extensions for which E is abelian.

As a specific example we look at the quaternion group Q, i.e., the finite subgroup of the unit group of the quaternion numbers $\mathbb{H}$, consisting of the elements

$$Q \;=\; \{\pm 1, \pm i, \pm j, \pm k\}\ .$$

The relations in this group are

$$i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i \quad \text{and} \quad ki = j\ ,$$

which forces other relations such as $ji = -k$. The center of Q consists of the elements ± 1, and modulo its center, every element of Q has order 2, so $Q/\{\pm 1\}$ is isomorphic to $(\mathbb{Z}/2)^2$. Since the group Q is not commutative, the operation of type $((\mathbb{Z}/2)^2, 1, \mathbb{Z}/2, 2)$ associated to the central extension

$$0 \longrightarrow \{\pm 1\} \longrightarrow Q \longrightarrow (\mathbb{Z}/2)^2 \longrightarrow 1$$

via Exercise 1.6 is not in the image of the (ordinary) Bockstein homomorphisms.

(v) Let R be any ring and $k \geq 0$. Then the cup product power operation

$$H^n(X;R) \longrightarrow H^{kn}(X;R)\ , \quad x \longmapsto x^k$$

is a cohomology operation of type (R, n, R, kn). In some cases the cup powers give all operations of a certain type. For example, $\mathbb{R}P^\infty$ is a $K(\mathbb{F}_2, 1)$, and the group $H^n(\mathbb{R}P^\infty;\mathbb{F}_2)$ is cyclic of order 2, generated by the n-th power of the fundamental class. So by the representability Lemma 1.2 the n-th cup-power operation is the only non-trivial cohomology operation of type $(\mathbb{F}_2, 1, \mathbb{F}_2, n)$. Similarly, $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$, and the integral cohomology algebra of $\mathbb{C}P^\infty$ is polynomial on the fundamental class, so there is only the trivial operation of type $(\mathbb{Z}, 2, \mathbb{Z}, n)$ for odd n, and all cohomology operations of type $(\mathbb{Z}, 2, \mathbb{Z}, 2k)$ are multiples of the k-th cup power operation. Rationally, there are no other cohomology operations whatsoever, besides multiples of cup powers. Indeed we shall see below that the cohomology algebra $H^*(K(\mathbb{Q}, n);\mathbb{Q})$ is polynomial on the fundamental class for even n, and is an exterior algebra on the fundamental class for odd n.

(v) Let R be a commutative ring. Some time ago in the proof of the homotopy-commutativity of the chain level cup product, we introduced the $\cup_1$-product

$$\cup_1 \;:\; C^n(X,R) \otimes C^m(X,R) \longrightarrow C^{n+m-1}(X,R)\ .$$

The $\cup_1$-product satisfies the coboundary formula

$$\delta(f \cup_1 g) \;=\; (\delta f) \cup_1 g \;+\; (-1)^n f \cup_1 (\delta g) \;-\; (-1)^{n+m} f \cup g \;-\; (-1)^{(n+1)(m+1)} (g \cup f)$$

which implies that if $f \in C^n(X,R)$ is a cocycle and n is even, then the $\cup_1$-square $f \cup_1 f$ is a cocycle whose cohomology class only depends on the class of f. If n is odd, then $f \cup_1 f$ is a mod-2 cocycle whose mod-2 cohomology class only depends on the class of f. In other words, the formula $\mathrm{Sq}_1[f] = [f \cup_1 f]$ defines cohomology operations

$$\begin{array}{llll} \mathrm{Sq}_1 \;:\; H^n(X;R) & \longrightarrow & H^{2n-1}(X;R) & \text{if } n \text{ is even, and} \\ \mathrm{Sq}_1 \;:\; H^n(X;R) & \longrightarrow & H^{2n-1}(X;R/2R) & \text{if } n \text{ is odd.} \end{array}$$

The $\cup_1$-square is the first in a sequence of cohomology operation which were introduced by Steenrod in the paper [St]. and which are called the *divided squaring operations*.

Cohomology groups are abelian groups, so operations that are additive are easier to deal with. The following proposition translated the additivity property of a cohomology operation into a property of the 'characteristic class' that determines the whole operation in the sense of Lemma 1.2.

Proposition 1.5. *Let τ be a cohomology operation of type (A, n, B, m), and let $u = \tau_{K(A,n)}(\iota_{A,n})$ be the classifying cohomology class in $H^m(K(A,n);B)$. Then the following two conditions are equivalent.*

(i) *The operation τ is additive.*
(ii) *The relation*
$$\mu^*(u) \;=\; p_1^*(u) + p_2^*(u)$$
holds in $H^m(K(A,n)\times K(A,n);B)$, where $\mu, p_1, p_2\colon K(A,n)\times K(A,n) \longrightarrow K(A,n)$ are the homotopy addition and the two projections, respectively.

Proof. We abbreviate $\iota = \iota_{A,n}$. In the proof of the representability of cohomology by Eilenberg–MacLane spaces we showed the relation
$$\mu^*(\iota) \;=\; p_1^*(\iota) + p_2^*(\iota)$$
holds in $H^n(K(A,n)\times K(A,n);A)$. So if the operation τ is additive, then
$$\begin{aligned}\mu^*(u) \;&=\; \mu^*(\tau(\iota)) \;=\; \tau(\mu^*(\iota))\\ &= \tau(p_1^*(\iota) + p_2^*(\iota))\\ &= \tau(p_1^*(\iota)) + \tau(p_2^*(\iota))\\ &= p_1^*(\tau(\iota)) + p_2^*(\tau(\iota)) \;=\; p_1^*(u) + p_2^*(u)\ .\end{aligned}$$
For the converse we now suppose that the relation (ii) holds. We let X be a CW-complex, and $x, y \in H^n(X;A)$. Then by representability there are continuous maps $f, g\colon X \longrightarrow K(A,n)$ such that $x = f^*(\iota)$ and $y = g^*(\iota)$. Moreover,
$$x + y \;=\; ([f]+[g])^*(\iota) \;=\; (\mu\circ(f,g))^*(\iota)\ .$$
So we obtain
$$\begin{aligned}\tau(x+y) \;&=\; \tau((\mu\circ(f,g))^*(\iota))\\ &=\; (\mu\circ(f,g))^*(u)\\ &=\; (f,g)^*(\mu^*(u))\\ \overset{\text{(ii)}}&{=}\; (f,g)^*(p_1^*(u)+p_2^*(u))\\ &=\; (f,g)^*(p_1^*(u)) + (f,g)^*(p_2^*(u))\\ &=\; f^*(u) + g^*(u)\\ &=\; \tau(f^*(\iota)) + \tau(g^*(\iota)) \;=\; \tau(x)+\tau(y)\ .\end{aligned}$$
Hence the operation τ is additive. □

Exercise 1.6. Given a central group extension
$$0 \longrightarrow B \longrightarrow E \longrightarrow G \longrightarrow 1$$
with G and B abelian, we define an operation
$$H^1(X;G) \longrightarrow H^2(X;B)$$
generalizing the Bockstein homomorphism for abelian extensions, where X is any simplicial set. Suppose $f\colon X_1 \longrightarrow G$ is a 1-cocycle, choose a lift $\bar f\colon X_1 \longrightarrow E$. Show that for every $x \in X_2$ the expression
$$(\delta\bar f)(x) \;=\; \bar f(d_0x)\cdot \bar f(d_1x)^{-1}\cdot \bar f(d_2x)$$
is contained in the subgroup B of E, and that it defines a 2-cocycle of X with values in B. Then show that the cohomology class of $\delta\bar f$ is independent of the choice of lift, and of the choice of cocycle f within its cohomology class.

Exercise 1.7. Let $Q = \{\pm1, \pm i, \pm j, \pm k\}$ denote the quaternion group. Exercise 1.6 associates a cohomology operation
$$Q_* \;:\; H^1(X;(\mathbb{Z}/2)^2) \longrightarrow H^2(X;\mathbb{Z}/2)$$
of type $((\mathbb{Z}/2)^2, 1, \mathbb{Z}/2, 2)$ to the central extension
$$0 \longrightarrow \{\pm1\} \longrightarrow Q \longrightarrow Q/\{\pm1\} \longrightarrow 1\ .$$

Show that under a suitable identification $Q/\{\pm 1\} \cong (\mathbb{Z}/2)^2$, this cohomology operation is given by the formula

$$Q_*(x) \;=\; \Pi^1_*(x) \cup \Pi^2_*(x)$$

where $\Pi_1, \Pi_2 \colon (\mathbb{Z}/2)^2 \longrightarrow \mathbb{Z}/2$ are the two projections.

Exercise 1.8. Show that the operation

$$\mathrm{Sq}_1 \;:\; H^3(B\mathbb{Z}/2;\mathbb{F}_2) \;\longrightarrow\; H^5(B\mathbb{Z}/2;\mathbb{F}_2)$$

is non-trivial. *(Hint: the group $H^n(B\mathbb{Z}/2;\mathbb{F}_2)$ is generated by the class ι_1^n, and ι_1 is represented by the identity 1-cochain $I \in C^1(B\mathbb{Z}/2,\mathbb{F}_2)$. Work out the formula for $I^{\cup 3} \cup_1 I^{\cup 3} \in C^5(B\mathbb{Z}/2,\mathbb{F}_2)$ and compare it to $I^{\cup 5}$.)*

2. Stable cohomology operations

Definition 2.1. A *reduced* cohomology operation of type (A,n,B,m) is a natural transformation

$$\tau \;:\; \tilde{H}^n(-;A) \;\longrightarrow\; \tilde{H}^m(-;B)$$

of reduced cohomology functors from the category of pointed spaces to the category of sets.

The set of reduced cohomology operations of a fixed type forms an abelian group. There is only a minor difference between reduced and (non-reduced) cohomology operations. Indeed as in Lemma 1.2, the Yoneda lemma implies that the map

$$\mathrm{redOper}(A,n,B,m) \;\longrightarrow\; \tilde{H}^m(K(A,n);B)$$

which takes a reduced cohomology operation $\tau\colon \tilde{H}^n(-;A) \longrightarrow \tilde{H}^m(-;B)$ to the image of the fundamental class $\tau(\iota_{n,A}) \in \tilde{H}^m(K(A,n),B)$ is an isomorphism. So the only difference is that the non-trivial constant operations of type $(A,n,B,0)$ cannot be extended to reduced cohomology operations.

Construction 2.2. In the following we will often consider two Eilenberg–MacLane spaces for the same group in adjacent dimensions. As we shall now explain, these are related by specific maps. For $n \geq 1$, we let (X,φ) and (Y,ϕ) be two Eilenberg-MacLane spaces of type $K(A,n)$ and $K(A,n+1)$, respectively. By an earlier theorem about realizability of homomorphisms of homotopy groups, there is a based continuous map

$$\rho \;:\; X \;\longrightarrow\; \Omega Y \;, \tag{2.3}$$

unique up to based homotopy, such that $\rho_*\colon \pi_n(X,*) \longrightarrow \pi_n(\Omega Y,*)$ equals the composite

$$\pi_n(X,*) \;\xrightarrow[\cong]{\varphi}\; A \;\xrightarrow{\phi^{-1}}\; \pi_{n+1}(Y,*) \;\cong\; \pi_n(\Omega Y,*) \;.$$

The unnamed isomorphism takes the homotopy class of $f\colon S^{n+1} \longrightarrow Y$ to the homotopy class of the adjoint $f^\flat\colon S^n \longrightarrow \mathrm{map}_*(S^1,Y)$ under the adjunction $(-\wedge S^1,\Omega)$. In other words:

$$f^\flat(x)(y) \;=\; f(x \wedge y) \;,$$

for $x \in S^n$ and $y \in S^1$. Since X and Y are path connected and ρ induces isomorphisms of all homotopy groups, so ρ is a weak homotopy equivalence.

The definition of the fundamental class of an Eilenberg–MacLane space refers to the Hurewicz homomorphism, which in turn uses an orientation $[S^n] \in H_n(S^n;\mathbb{Z})$ of the n-sphere. When comparing Eilenberg–MacLane spaces of different dimensions we insists that these orientations are chosen consistently, in the sense that the composite

$$H_n(S^n;\mathbb{Z}) \;\xrightarrow[\cong]{\Sigma}\; H_{n+1}(\Sigma S^n;\mathbb{Z}) \;\xrightarrow{\cong}\; H_{n+1}(S^{n+1};\mathbb{Z})$$

takes the chosen orientation of S^n to the chosen orientation of S^{n+1}. The unnamed isomorphism is induced by the preferred homeomorphism

$$\Sigma S^n \;=\; S^n \wedge S^1 \;\xrightarrow{\cong}\; S^{n+1} \;, \quad (x_1,\ldots,x_n)\wedge y \;\longmapsto\; (x_1,\ldots,x_n,y) \;.$$

Lemma 2.4. *Let (X, φ) and (Y, ϕ) be two Eilenberg-MacLane spaces of type $K(A, n)$ and $K(A, n+1)$, respectively, and let $\epsilon \colon \Sigma X \longrightarrow Y$ be adjoint to the preferred weak homotopy equivalence ρ from (2.3).*

(i) *The following diagram commutes:*

$$\begin{tikzcd}
\pi_n(X,*) \arrow[r, "\Sigma"] \arrow[drr, bend right, "\varphi"', "\cong"] & \pi_{n+1}(\Sigma X,*) \arrow[r, "\epsilon_*"] & \pi_{n+1}(Y,*) \arrow[d, "\phi", "\cong"'] \\
 & & A
\end{tikzcd}$$

(ii) *The fundamental classes $\iota_{A,n} \in H^n(X; A)$ and $\iota_{A,n+1} \in H^{n+1}(Y; A)$ satisfy the relation*

$$\Sigma(\iota_{A,n}) \;=\; \epsilon^*(\iota_{A,n+1})$$

in $H^{n+1}(\Sigma X; A)$, where Σ is the suspension isomorphism in the cohomology of X.

Proof. (i) The rectangle in the following diagram commutes because $\epsilon \colon \Sigma X \longrightarrow Y$ was defined as the adjoint of $\rho \colon X \longrightarrow \Omega Y$:

$$\begin{tikzcd}
\pi_n(X,*) \arrow[r, "\Sigma"] \arrow[d, "\rho_*"'] \arrow[dd, bend right=80, "\varphi"', "\cong"] & \pi_{n+1}(\Sigma X,*) \arrow[d, "\epsilon_*"] \\
\pi_n(\Omega Y,*) \arrow[r, "\cong"'] & \pi_{n+1}(Y,*) \arrow[dl, bend left, "\phi", "\cong"'] \\
A &
\end{tikzcd}$$

The other part commutes by the defining property of the map ρ.

(ii) We show that the class

$$\Sigma^{-1}(\epsilon^*(\iota_{A,n+1})) \;\in\; H^n(X; A)$$

has the defining property of the fundamental class $\iota_{A,n}$. In other words, we show that the composite

$$\pi_n(X,*) \xrightarrow{\text{Hurewicz}} H_n(X;\mathbb{Z}) \xrightarrow{\Phi(\Sigma^{-1}(\epsilon^*(\iota_{A,n+1})))} A$$

is the isomorphism $\varphi \colon \pi_n(X, *) \cong A$, where

$$\Phi \;:\; H^n(X; A) \;\longrightarrow\; \mathrm{Hom}(H_n(X;\mathbb{Z}), A)$$

is the map from the universal coefficient theorem.

We let $f \colon S^n \longrightarrow X$ be a based continuous map that represents a class in $\pi_n(X, *)$. Naturality of the maps Φ from the universal coefficient theorem means that the following diagram commutes:

$$\begin{tikzcd}
H^n(X;A) \arrow[r, "\Phi"] \arrow[d, "f^*"'] & \mathrm{Hom}(H_n(X;\mathbb{Z}), A) \arrow[d, "\mathrm{Hom}(f_*;A)"] \\
H^n(S^n;A) \arrow[r, "\Phi"'] & \mathrm{Hom}(H_n(S^n;\mathbb{Z}), A)
\end{tikzcd} \tag{2.5}$$

Moreover, the maps Φ in adjacent dimensions are compatible with the suspension isomorphisms in homology and cohomology, i.e., the following diagram commutes:

$$\begin{tikzcd}
H^{n+1}(\Sigma X;A) \arrow[r, "\Phi"] \arrow[d, "\Sigma^{-1}"'] & \mathrm{Hom}(H_{n+1}(\Sigma X;\mathbb{Z}), A) \arrow[d, "\mathrm{Hom}(\Sigma;A)"] \\
H^n(X;A) \arrow[r, "\Phi"'] & \mathrm{Hom}(H_n(X;\mathbb{Z}), A)
\end{tikzcd} \tag{2.6}$$

So

$$\begin{aligned}
\Phi(\Sigma^{-1}(\epsilon^*(\iota_{A,n+1})))(\text{Hurewicz}[f]) &= \Phi(\Sigma^{-1}(\epsilon^*(\iota_{A,n+1})))(f_*[S^n]) \\
{\scriptstyle (2.5)} &= \Phi(f^*(\Sigma^{-1}(\epsilon^*(\iota_{A,n+1}))))[S^n] \\
&= \Phi(\Sigma^{-1}((\epsilon\circ(\Sigma f))^*(\iota_{A,n+1})))[S^n] \\
{\scriptstyle (2.6)} &= \Phi((\epsilon\circ(\Sigma f))^*(\iota_{A,n+1}))(\Sigma[S^n]) \\
{\scriptstyle (2.5)} &= \Phi(\iota_{A,n+1})((\epsilon\circ(\Sigma f))_*[S^{n+1}]) \\
&= \Phi(\iota_{A,n+1})(\text{Hurewicz}(\epsilon\circ(\Sigma f))) \\
&= \phi[\epsilon\circ(\Sigma f)] \ =_{(\text{i})} \ \varphi[f] \ .
\end{aligned}$$

This verifies the desired property for the class $\Sigma^{-1}(\epsilon^*(\iota_{A,n+1}))$. □

Lemma 2.7. *Let τ and $\bar\tau$ be two reduced cohomology operations of type (A,n,B,m) and type $(A,n+1,B,m+1)$ respectively. Then the following four conditions are equivalent.*

(a) *For every pair of based spaces (X,Y) with the homotopy extension property, the diagram*

$$\begin{tikzcd}
\widetilde H^n(Y;A) \arrow[r, "\delta"] \arrow[d, "\tau"'] & \widetilde H^{n+1}(X/Y;A) \arrow[d, "\bar\tau"] \\
\widetilde H^m(Y;B) \arrow[r, "\delta"'] & \widetilde H^{m+1}(X/Y;B)
\end{tikzcd}$$

commutes, where the horizontal maps δ are the connecting homomorphisms.

(b) *For every non-degenerately based space X the diagram*

$$\begin{tikzcd}
\widetilde H^n(X;A) \arrow[r, "\Sigma"] \arrow[d, "\tau"'] & \widetilde H^{n+1}(\Sigma X;A) \arrow[d, "\bar\tau"] \\
\widetilde H^m(X;B) \arrow[r, "\Sigma"'] & \widetilde H^{m+1}(\Sigma X;B)
\end{tikzcd}$$

commutes, where the horizontal maps Σ are the suspension isomorphisms of X.

(c) *For every non-degenerately based space X and every reduced cohomology class $x\in\widetilde H^n(X;A)$ we have*

$$\tau(x)\times\iota \ = \ \bar\tau(x\times\iota) \qquad \textit{in} \ \ \widetilde H^{m+1}(\Sigma X;B)$$

where $\iota\in\widetilde H^1(S^1;\mathbb{Z})$ is the fundamental class.

(d) *Let $K(A,n)$ and $K(A,n+1)$ be Eilenberg-MacLane spaces of type (A,n) and $(A,n+1)$ respectively, and let $\epsilon\colon \Sigma K(A,n)\longrightarrow K(A,n+1)$ be a based continuous map whose adjoint is in the preferred homotopy class (2.3) of weak homotopy equivalence. Then the relation*

$$\Sigma(\tau(\iota_{A,n})) \ = \ \epsilon^*(\bar\tau(\iota_{A,n+1}))$$

among the fundamental classes holds in $H^{m+1}(\Sigma K(A,n);B)$, where Σ is the suspension isomorphism of $K(A,n)$.

Proof. Condition (b) is a special case of (a) for the inclusion of X into its reduced cone, with quotient the suspension of X. Conditions (b) and (c) are equivalent since the suspension isomorphism coincides with exterior product by the fundamental class $\iota\in\widetilde H^1(S^1;\mathbb{Z})$.

To see that condition (b) implies condition (a) we use that fact that the connecting homomorphism for the pair (X,Y) factors as a composite

$$\widetilde H^n(Y;A) \ \xrightarrow{\Sigma} \ \widetilde H^{n+1}(\Sigma Y;A) \ \xrightarrow{\Pi^*} \ \widetilde H^{n+1}(X/Y;A)$$

of the suspension isomorphism and a map induced from the geometric connecting homomorphism $\Pi \in [X/Y, \Sigma Y]$ which features in the Puppe sequence of the pair (X, Y). In more detail: we have a commutative diagram of cofiber sequences

$$\begin{tikzcd}
Y \arrow[r, tail] \arrow[d, equal] & X \arrow[r] & Y/X \\
Y \arrow[r, tail, "\mathrm{incl}_0"] \arrow[d, equal] & Y \times [0,1] \cup_{Y \times 1} X \arrow[r] \arrow[u, "\sim"'] \arrow[d, "\text{collapse } X"] & CY \cup_{Y \times 1} X \arrow[u, "\sim"'] \arrow[d, "\text{collapse } X"] \\
Y \arrow[r, tail] & Y \times [0,1] \cup_{Y \times 1} * \arrow[r] & \Sigma Y
\end{tikzcd}$$

Since the boundary map in cohomology is functorial for maps of pairs, we obtain a commutative diagram of cohomology groups

$$\begin{tikzcd}
\widetilde{H}^n(Y;A) \arrow[r, "\delta"] \arrow[d, "\Sigma"'] \arrow[dr, "\delta"] & \widetilde{H}^{n+1}(X/Y;A) \arrow[d, "\cong"] \\
\widetilde{H}^{n+1}(\Sigma Y;A) \arrow[r] & \widetilde{H}^{n+1}(CY \cup_Y X;A)
\end{tikzcd}$$

in which the right vertical map is an isomorphism. Since the operations τ and $\bar{\tau}$ are natural for maps of pointed spaces, compatibility with the suspension isomorphism implies compatibility with arbitrary connecting homomorphisms.

For the equivalence of conditions (b) and (d) we consider the two reduced operations $\Sigma \circ \tau$ and $\bar{\tau} \circ \Sigma$ of type $(A, n, B, m+1)$. By the representability lemma for cohomology operations (Lemma 1.2), these two operations agree if and only if they agree on the fundamental class $\iota_{A,n}$. By Lemma 2.4 we obtain

$$\bar{\tau}(\Sigma(\iota_{A,n})) \;=\; \bar{\tau}(\epsilon^*(\iota_{A,n+1})) \;=\; \epsilon^*(\bar{\tau}(\iota_{A,n+1})).$$

So condition (b) holds if and only if we have $\Sigma(\tau(\iota_{A,n})) = \epsilon^*(\bar{\tau}(\iota_{A,n+1}))$. □

Definition 2.8. Let A and B be abelian groups and n a natural number. A *stable cohomology operation* of type (A, B) and of degree n is a family $\{\tau_i\}_{i\geq 0}$ of reduced cohomology operations of type $(A, i, B, n+i)$ which are compatible with suspension isomorphisms, i.e., for every based space X and every $i \geq 0$ and every $x \in \widetilde{H}^i(X;A)$ we have

$$\tau_i(x) \times \iota \;=\; \tau_{i+1}(x \times \iota) \qquad \text{in} \;\; H^{n+i+1}(\Sigma X; B)$$

where $\iota \in \widetilde{H}^1(S^1;\mathbb{Z})$ is the fundamental class. We denote by $\mathrm{StOp}(A, B, n)$ the abelian group of stable cohomology operations of type (A, B) and degree n.

If $\tau = \{\tau_i\}_{i\geq 0}$ is a stable cohomology operation of degree n and type (A, B) and $\lambda = \{\lambda_i\}_{i\geq 0}$ is a stable cohomology operation of degree m and type (B, C), then they compose to yield a stable cohomology operation

$$\lambda \circ \tau \;=\; \{\lambda_{n+i} \circ \tau_i\}_{i\geq 0}$$

of degree $n + m$ and type (A, C).

As an immediate consequence of the definition and of Lemma 2.7 we get the following representability result for stable cohomology operations. We choose a family of Eilenberg-MacLane spaces $\{K(A, i)\}_{i\geq 0}$; then there are preferred homotopy classes (2.3) of weak homotopy equivalences $K(A, i) \xrightarrow{\sim} \Omega K(A, i+1)$, whose adjoints are continuous based maps $\epsilon_i \colon \Sigma K(A, i) \longrightarrow K(A, i+1)$.

Corollary 2.9. *A family $\{\tau_i\}_{i\geq 0}$ of cohomology operations of type $(A, i, B, n+i)$ forms a stable cohomology operation if and only if for all $i \geq 0$ the relation*

$$\epsilon_i^*(\tau_{i+1}(\iota_{A,i+1})) \;=\; \Sigma(\tau_i(\iota_{A,i}))$$

holds in $H^{n+i+1}(\Sigma K(A,i);B)$. *Hence the assignment*

$$\begin{array}{rcl} \mathrm{StOp}(A,B,n) & \longrightarrow & \lim_i H^{n+i}(K(A,i);B)\ , \\ \tau=\{\tau_i\} & \longmapsto & \{\tau_i(\iota_{A,i})\} \end{array}$$

is an isomorphism between the group of stable cohomology operations of degree n *and type* (A,B) *and the sequences* $\{x_i\}_{i\geq 0}$ *of cohomology classes such that* $x_i \in H^{n+i}(K(A,i);B)$ *and*

$$\epsilon_i^*(x_{i+1}) \ = \ \Sigma(x_i)\ .$$

More specifically, the limit of the cohomology groups is taken along the homomorphisms

$$H^{n+i+1}(K(A,i+1);B) \xrightarrow{\epsilon_i^*} H^{n+i+1}(\Sigma K(A,i);B) \xrightarrow[\cong]{\Sigma^{-1}} H^{n+i}(K(A,i);B)\ .$$

Lemma 2.10. (i) *If* τ *is any reduced cohomology operation and* X *a based space, then the value of* τ *at the suspension* ΣX *is an additive map.*

(ii) *Let* $\tau=\{\tau_i\}_{i\geq 0}$ *be a stable cohomology operation of degree* n *and type* (A,B). *Then each individual cohomology operation* $\tau_i\colon H^i(-,A)\longrightarrow H^{n+i}(-,B)$ *is additive, and hence the class* $u_i=\tau_i(\iota_{A,i})$ *in* $H^{n+i}(K(A,i),B)$ *satisfies*

$$\mu^*(u_i) \ = \ p_1^*(u_i)+p_2^*(u_i)$$

in $H^{n+i}(K(A,i),B)^2$.

(iii) *Composition of stable cohomology operations is bi-additive.*

Proof. (i) Suppose that τ is an operation of type (A,n,B,m). We start by letting X be any non-degenerately based space, and we choose two elements $x,y\in \tilde H^n(X;A)$. We consider the class

(2.11) $$\tau(p_1^*(x)+p_2^*(y)) \ - \ p_1^*(\tau(x)) \ - \ p_2^*(\tau(y))$$

in $\tilde H^m(X\times X;B)$. If we restrict the class (2.11) along the first inclusions $j\colon X\longrightarrow X\times X$, $j(z)=(z,*)$, then we get

$$\begin{aligned} j^*(\tau(p_1^*(x)+p_2^*(y))-p_1^*(\tau(x))-p_2^*(\tau(y))) \ &= \ \tau(j^*(p_1^*(x))+j^*(p_2^*(y)))-j^*(p_1^*(\tau(x)))-j^*(p_2^*(\tau(y))) \\ &= \ \tau(x)-\tau(x) \ = \ 0\ , \end{aligned}$$

and similarly for the second inclusion. We exploited that $p_1\circ j$ is the identity, and that the operation $j^*\circ p_2^*=(p_2\circ j)^*$ vanishes on reduced cohomology classes because the map $p_2\circ j$ is constant. This means that the restriction of the element (2.11) to the wedge $X\vee X\subseteq X\times X$ is trivial, so the element (2.11) is in the image of the map

$$\Pi^* \ : \ \tilde H^m(X\wedge X;B) \ \longrightarrow \ \tilde H^m(X\times X;B)$$

from the reduced cohomology of the smash product $X\wedge X=(X\times X)/(X\vee X)$, where

$$\Pi \ : \ X\times X \ \longrightarrow \ X\wedge X$$

denotes the quotient projection. If $X=\Sigma Y$ is a suspension, then the composite map (reduced diagonal)

$$\bar\Delta \ : \ X \ \xrightarrow{\Delta} \ X\times X \ \xrightarrow{\Pi} \ X\wedge X$$

equals the composite

$$Y\wedge S^1 \ \xrightarrow{\bar\Delta\wedge\bar\Delta} \ Y\wedge Y\wedge S^1\wedge S^1 \ \xrightarrow[\cong]{\text{shuffle}} \ (Y\wedge S^1)\wedge(Y\wedge S^1)\ .$$

Since the reduced diagonal $\bar\Delta\colon S^1 \longrightarrow S^1\wedge S^1 = S^2$ is null-homotopic, so is the reduced diagonal of $Y\wedge S^1 = X$.

Since the reduced diagonal $\bar\Delta=\Pi\circ\Delta$ is X is null-homotopic and the class (2.11) is in the image of Π^*, the class (2.11) restricts to zero along the diagonal. This gives

$$\begin{aligned} 0 \ &= \ \Delta^*(\tau(p_1^*(x)+p_2^*(y))-p_1^*(\tau(x))-p_2^*(\tau(y))) \\ &= \ \tau(\Delta^*(p_1^*(x))+\Delta^*(p_2^*(y)))-\Delta^*(p_1^*(\tau(x)))-\Delta^*(p_2^*(\tau(y))) \\ &= \ \tau(x+y)-\tau(x)-\tau(y)\ . \end{aligned}$$

We have exploited that $p_1 \circ \Delta = p_2 \circ \Delta = \mathrm{Id}_X$.

(ii) We let X be a non-degenerately based spaces. The horizontal suspension isomorphism in the commutative square

$$\begin{tikzcd} \widetilde{H}^n(X;A) \arrow[r, "\Sigma", "\cong"'] \arrow[d, "\tau_i"'] & \widetilde{H}^{n+1}(\Sigma X;A) \arrow[d, "\tau_{i+1}"] \\ \widetilde{H}^m(X;B) \arrow[r, "\cong", "\Sigma"'] & \widetilde{H}^{m+1}(\Sigma X;B) \end{tikzcd}$$

are additive. Part (i) says that the operation τ_{i+1} is additive on ΣX. So the left vertical map in the diagram is also additive. The final property of the class u_i then follows from Proposition 1.5.

(iii) Since addition of cohomology operations is pointwise, it is clear from the definition that the assignment $(\lambda, \tau) \longmapsto \lambda \circ \tau$ is additive in λ. That composition is also additive in τ follows from the fact that all the individual operations λ_i are additive by part (ii). $\square$

Example 2.12. (i) By Example 1.3 (i) there are no stable cohomology operations of negative degree. If $f\colon A \longrightarrow B$ is a homomorphism of coefficient groups, then the associated cohomology operations of type (A, m, B, m) for every $m \geq 0$ form a stable cohomology operation. Indeed, the group all stable cohomology operations of type (A, B) of degree 0 is naturally isomorphic to $\mathrm{Hom}(A, B)$,

$$\mathrm{StOp}(A, B, 0) \;\cong\; \mathrm{Hom}(A, B)\ .$$

(ii) The Bockstein homomorphisms $\delta\colon H^n(X;A) \longrightarrow H^{n+1}(X;B)$ associated to a short exact sequence of abelian groups

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0\ .$$

for $n \geq 0$ form a stable cohomology operation of type (A, B) of degree 1. For $n \geq 2$, the homology group $H_{n+1}(K(A, n); \mathbb{Z})$ is trivial (see e.g. [EM, Theorem 20.5]), so the universal coefficient theorem implies that this construction gives all stable operations of type (A, B) of degree 1,

$$\mathrm{StOp}(A, B, 1) \;\cong\; \mathrm{Ext}(A, B)\ .$$

(iii) If R is a ring, then the cup product power operation $x \longmapsto x^k$ is usually not additive, and whenever it fails to be so, then as an operation of type (R, n, R, kn) is does not extend to a stable operation of degree $(k-1)n$ (by part (ii) of Lemma 2.10). However, if p is a prime number and R is an $\mathbb{F}_p$-algebra, then the p-th power operation $x \longmapsto x^p$ is additive. As we shall see in Example 3.6 below, the cup-square

$$H^i(X; \mathbb{F}_2) \;\longrightarrow\; H^{2i}(X; \mathbb{F}_2)\ ,\ x \longmapsto x^2$$

indeed extends to a *unique* stable mod-2 cohomology operation of degree i. This operation is denoted Sq^i and is called the i-th *Steenrod divided square operation.* If p is an odd prime, then in even dimensions, the p-th cup power

$$H^{2i}(X; \mathbb{F}_p) \;\longrightarrow\; H^{2ip}(X; \mathbb{F}_p)\ ,\ x \longmapsto x^p$$

extends to a stable mod-p cohomology operation of degree $2i(p-1)$, called the i-divided power operation and denoted P^i.

Definition 2.13. Let A be an abelian group. Then we denote by

$$\mathcal{A}(A)^n \;=\; \mathrm{StOp}(A, A, n)$$

the group of stable cohomology operations of degree n and type (A, A). By Lemma 2.10 (iii) the groups $\mathcal{A}(A)^*$ form a graded ring under composition, which is called the *Steenrod algebra* for the group A.

Since the components of a stable cohomology operation are always additive (Lemma 2.10 (ii)), the reduced cohomology $\widetilde{H}^*(X, A)$ of a based space X with coefficients in an abelian group A is tautologically a graded left module over the Steenrod algebra $\mathcal{A}(A)^*$ via

$$\tau \cdot x \;=\; \tau_i(x) \;\in\; \widetilde{H}^{n+i}(X; A)$$

for $\tau = \{\tau_i\}_{i\geq 0} \in \mathcal{A}(A)^n$ and $x \in \widetilde{H}^i(X;A)$. So cohomology with coefficients in A can be viewed as a functor

$$\widetilde{H}^*(-;A) \;:\; \mathrm{Ho}(\mathrm{Top}_*) \;\longrightarrow\; \mathcal{A}(A)^*\text{-mod} .$$

Moreover, the suspension isomorphism

$$\Sigma \;:\; \widetilde{H}^*(X;A)[1] \;\longrightarrow\; \widetilde{H}^*(\Sigma X;A)$$

is an isomorphism of graded $\mathcal{A}(A)^*$-modules, by the compatibility condition in the definition of a stable cohomology operation. Here the square brackets [1] denote the shift of a graded module. Similarly, if $Y \subset X$ is a subspace containing the basepoint, and such that (X,Y) has the homotopy extension property, then the boundary map of the pair

$$\delta \;:\; \widetilde{H}^*(Y;A)[1] \;\longrightarrow\; \widetilde{H}^*(X/Y;A)$$

is a homomorphism of graded $\mathcal{A}(A)^*$-modules (by part (i) of Lemma 2.7).

If A is a ring, then sending an element $a \in A$ to the map $\lambda_a \colon A \longrightarrow A$ given by left multiplication by a gives a ring homomorphism

$$A \;\longrightarrow\; \mathrm{Hom}(A,A) \;\cong\; \mathcal{A}(A)^0 .$$

If A is commutative, then the image of λ is central in the Steenrod-algebra $\mathcal{A}(A)^*$ so in this case $\mathcal{A}(A)^*$ is naturally an A-algebra.

All this is particularly useful when the structure of the Steenrod algebra $\mathcal{A}(A)^*$ is explicitly known. The aim of the next section is to describe the mod-p Steenrod algebra $\mathcal{A}_p^* = \mathcal{A}(\mathbb{F}_p)^*$ by generators (Steenrod's *divided power operations*) and relations (the *Adem relations*). Alongside we use this new algebraic structure to answer some geometric questions.

Remark 2.14. We have shown in Lemma 1.2 that unstable cohomology operations $\mathrm{Oper}(A,n,B,m)$ are in bijective correspondence with cohomology classes in $H^m(K(A,n),B)$, hence with homotopy classes of maps from the Eilenberg-Mac Lane space $K(A,n)$ to the Eilenberg-Mac Lane space $K(B,m)$. Something similar is true for stable operations, but only when we replace *spaces* by *spectra*: the stable operations $\mathrm{StOp}(A,B,n)$ are in bijective correspondence with homotopy classes of morphisms from the Eilenberg-Mac Lane *spectrum* HA to the shifted Eilenberg-Mac Lane *spectrum* $HB[n]$.

3. Cohomology in the stable range

Theorem 3.1. *Let X be an n-connected based space, for $n \geq 1$. Let $\epsilon \colon \Sigma(\Omega X) \longrightarrow X$ be the unit of the adjunction (Σ,Ω). Then for every abelian group B, the map*

$$\epsilon^* \;:\; H^i(X;B) \;\longrightarrow\; H^i(\Sigma(\Omega X);B)$$

is an isomorphism for all $0 \leq i \leq 2n$ and injective for $i = 2n+1$.

Proof. Since X is n-connected, the loop space ΩX is $(n-1)$-connected. By the Freudenthal suspension theorem, the suspension homomorphism

$$\Sigma \;:\; \pi_i(\Omega X,*) \;\longrightarrow\; \pi_{i+1}(\Sigma(\Omega X),*)$$

is an isomorphism for $1 \leq i \leq 2n-2$, and surjective for $i = 2n-1$. The composite

$$\pi_i(\Omega X,*) \xrightarrow{\;\Sigma\;} \pi_{i+1}(\Sigma(\Omega X),*) \xrightarrow{\;\epsilon_*\;} \pi_{i+1}(X,*)$$

implements the dimension-shifting isomorphism given by adjoining; it is thus bijective for all $i \geq 1$. In particular the suspension homomorphism is also injective, and hence bijective, for $i = 2n-1$. Hence also the homomorphism

$$\epsilon_* \;:\; \pi_{i+1}(\Sigma(\Omega X),*) \;\longrightarrow\; \pi_{i+1}(X,*)$$

is bijective for $1 \leq i \leq 2n-1$, and also surjective for $i \geq 2n$. Setting $j = i+1$ this shows that

$$\epsilon_* \;:\; \pi_j(\Sigma(\Omega X),*) \;\longrightarrow\; \pi_j(X,*)$$

is bijective for $1 \leq j \leq 2n$ and surjective for $j = 2n+1$. Relative CW-approximation thus provides a relative CW-complex $(Z, \Sigma(\Omega X))$ with all relative cells of dimensions $\geq 2n+2$, and a weak equivalence

$f\colon Z \xrightarrow{\sim} X$ that extends ϵ. The relative cohomology groups $H^i(Z,\Sigma(\Omega X);A)$ then vanish for all $i \leq 2n+1$, and the long exact sequence of this pair shows that the restriction map

$$H^i(Z;A) \longrightarrow H^i(\Sigma(\Omega X);A)$$

is an isomorphism for $i \leq 2n$, and it yields an exact sequence

$$0 \longrightarrow H^{2n+1}(Z;A) \xrightarrow{\text{incl}^*} H^{2n+1}(\Sigma(\Omega X);A) \xrightarrow{\partial} H^{2n+2}(Z,\Sigma(\Omega X);A)$$

Since $f\colon Z \xrightarrow{\sim} X$ is a weak equivalence that extends ϵ, this proves the claim for the map ϵ. $\square$

We apply the previous Theorem 3.1 to $X = K(A,n+1)$ for some $n \geq 1$. Then $\Omega X = \Omega K(A,n+1)$ is an Eilenberg–MacLane space of type (A,n). More precisely: if we have chosen some $K(A,n)$, there is a preferred homotopy class (2.3) of weak homotopy equivalence $K(A,n) \sim \Omega K(A,n+1)$. The previous theorem then specializes to:

Corollary 3.2. *Let $n \geq 1$, and let A and B be abelian groups. Let $\epsilon\colon \Sigma K(A,n) \longrightarrow K(A,n+1)$ be adjoint to the preferred homotopy class* (2.3) *of weak homotopy equivalence $K(A,n) \sim \Omega K(A,n+1)$. Then the map*

$$\epsilon^* \ : \ H^i(K(A,n+1);B) \longrightarrow H^i(\Sigma K(A,n);B)$$

is an isomorphism for all $0 \leq i \leq 2n$ and injective for $i = 2n+1$.

By Corollary 3.2, all the cohomology suspensions morphisms in the sequence

$$\cdots \xrightarrow[\cong]{\sigma} H^{2n+k}(K(A,n+k);B) \xrightarrow[\cong]{\sigma} \cdots \xrightarrow[\cong]{\sigma} H^{2n+1}(K(A,n+1);B) \xrightarrow{\sigma} H^{2n}(K(A,n);B)$$

up to the group $H^{2n+1}(K(A,n+1);B)$ are isomorphisms. Moreover, the final cohomology suspension is injective. Since the group of stable operations of type (A,B,n) is isomorphic to the inverse limit of this sequence, we conclude that the map

$$\mathrm{StOp}(A,B,n) \longrightarrow H^{2n}(K(A,n);B) \ , \quad \tau \longmapsto \tau_n(\iota_{A,n})$$

defined by evaluation at the fundamental class $\iota_{A,n} \in H^n(K(A,n);A)$ is injective. Moreover, by Lemma 2.10 (ii) the class $u = \tau_n(\iota_{A,n}) \in H^{2n}(K(A,n);B)$ satisfies

$$\mu^*(u) \ = \ p_1^*(u) + p_2^*(u) \ .$$

We shall use without proof:

Theorem 3.3. *Let $n \geq 1$, and let A and B be abelian groups. Then the image of the monomorphism*

$$\mathrm{StOp}(A,B,n) \longrightarrow H^{2n}(K(A,n);B) \ , \quad \tau \longmapsto \tau_n(\iota_{A,n})$$

equals the set of element $u \in H^{2n}(K(A,n);B)$ that satisfy

$$\mu^*(u) \ = \ p_1^*(u) + p_2^*(u) \ .$$

Remark 3.4. We let R be a ring. Then the exterior product

$$\times \ : \ H^m(X;R) \times H^n(Y;R) \longrightarrow H^{m+n}(X \times Y;R)$$

was defined by

$$x \times y \ = \ p_1^*(x) \cup p_2^*(x) \ .$$

So for coefficients in a ring, the relation $\mu^*(x) = p_1^*(x) + p_2^*(x)$ from Theorem 3.3 can equivalently be formulated as

$$\mu^*(x) \ = \ x \times 1 \ + \ 1 \times x$$

We showed in Proposition 1.5 that a cohomology operation τ is additive if and only if its characteristic class $u = \tau(\iota_{A,n})$ satisfies the relation

$$\mu^*(u) \ = \ p_1^*(u) + p_2^*(u) \ .$$

So Theorem 3.3 is equivalent to:

Corollary 3.5. *Let $n \geq 1$, and let A and B be abelian groups. For every additive cohomology operation σ of type $(A, n, B, 2n)$ there is a unique stable cohomology operations τ of type (A, B, n) such that $\tau_n = \sigma$.*

Example 3.6 (Steenrod squares)**.** The cup product with coefficients in a ring R satisfies the relation

$$\begin{aligned}(x+y)^2 \;&=\; (x+y)\cup(x+y) \;=\; (x\cup y)+(x\cup y)+(y\cup x)+(y\cup y)\\ &=\; x^2 \;+\; (1+(-1)^n)\cdot(x\cup y) \;+\; y^2\ ,\end{aligned}$$

for $x, y \in H^n(X; R)$. So if n is odd or $2 = 0$ in the ring R, then the cup square is an additive operation. By Corollary 3.5, the cup square then extends to a stable cohomology operations.

Particularly important is the special case $R = \mathbb{F}_2$, in which case Corollary 3.5 provides a unique stable mod-2 cohomology operation

$$\mathrm{Sq}^n \;:\; H^i(X;\mathbb{F}_2) \;\longrightarrow\; H^{n+i}(X;\mathbb{F}_2)$$

of degree n satisfying $\mathrm{Sq}^n(x) = x^2$ for every n-dimensional cohomology class. This operation is called the *n-th Steenrod square.*

The zeroth Steenrod operation

$$\mathrm{Sq}^0 \;:\; H^i(X;\mathbb{F}_2) \;\longrightarrow\; H^i(X;\mathbb{F}_2)$$

is the identity operation, because $\iota_0^2 = \iota_0$ in $H^0(\mathbb{F}_2; \mathbb{F}_2)$.

The family of Bockstein operations

$$\beta \;:\; H^i(X;\mathbb{F}_2) \;\longrightarrow\; H^{i+1}(X;\mathbb{F}_2)$$

associated to the short exact sequence

$$0 \;\longrightarrow\; \mathbb{F}_2 \;\xrightarrow{\cdot 2}\; \mathbb{Z}/4 \;\xrightarrow{\mathrm{proj}}\; \mathbb{F}_2 \;\longrightarrow\; 0$$

form a stable mod-2 cohomology operation. We have seen earlier that the Bockstein operation

$$\beta \;:\; H^1(X;\mathbb{F}_2) \;\longrightarrow\; H^2(X;\mathbb{F}_2)$$

originating in dimension 1 equals the cup square, i.e., $\beta(x) = x^2$ for 1-dimensional cohomology classes x. So the first Steenrod square equal the Bockstein:

$$\mathrm{Sq}^1 = \beta \;:\; H^i(X;\mathbb{F}_2) \;\longrightarrow\; H^{i+1}(X;\mathbb{F}_2)\ .$$

We shall see later that the operations Sq^i for $i \geq 1$ generated the algebra mod stable mod-2 cohomology operations.

4. Steenrod's divided squaring operations

We saw in Example 3.6 that for every $i \geq 0$ there is a unique stable mod-2 cohomology operation Sq^i of degree i with the property that $\mathrm{Sq}^i(x) = x \cup x$ for every cohomology class x of dimension i. This operation is called the *i-th Steenrod square.* In this section we begin a more detailed study of the operations Sq^i. Eventually we will see that the Sq^i's generate the mod-2 Steenrod algebra $\mathcal{A}_2$, and we will give a complete list of relations between these operations, the *Adem relations.*

Theorem 4.1. *For each $i \geq 0$ there is a unique stable mod-2 cohomology operation Sq^i of degree i with the property that $\mathrm{Sq}^i(x) = x \cup x$ for every cohomology class x of dimension i. Moreover, these operations enjoy the following properties:*

(i) *The operation Sq^0 is the identity and Sq^1 coincides with the mod-2 Bockstein operation.*
(ii) *(Unstability condition) For $x \in H^n(X, \mathbb{F}_2)$ and $i > n$ we have $\mathrm{Sq}^i(x) = 0$.*
(iii) *(Cartan formula) For $x, y \in H^*(X, \mathbb{F}_2)$ and $i \geq 0$ we have*

$$\mathrm{Sq}^i(x\cup y) \;=\; \sum_{a+b=i} \mathrm{Sq}^a(x)\cup \mathrm{Sq}^b(y)\ .$$

Proof. Existence and uniqueness of Sq^i was established in Example 3.6, along with property (i).

For part (ii) we consider the iterated suspension isomorphism $\Sigma^{i-n}\colon H^n(X;\mathbb{F}_2) \longrightarrow H^i(\Sigma^{i-n}X;\mathbb{F}_2)$. Since Sq^i is a stable operation, we have

$$\Sigma^{i-n}(\mathrm{Sq}^i(x)) \;=\; \mathrm{Sq}^i(\Sigma^{i-n}(x)) \;=\; (\Sigma^{i-n}(x))^2 \;=\; 0$$

since cup products are trivial in the cohomology of a suspension. Since Σ^{i-n} is an isomorphism, this proves the relation $\mathrm{Sq}^i(x) = 0$.

The Cartan formula follows from the *external Cartan formula* which we prove as a separately in Theorem 4.2 below. To get from the external to the internal form, one simply takes $X = Y$ and applies the map $\Delta^*\colon H^*(X\times X,\mathbb{F}_2) \longrightarrow H^*(X,\mathbb{F}_2)$ induced by the diagonal $\Delta\colon X \longrightarrow X\times X$. □

Theorem 4.2. (External Cartan formula) *For all spaces X and Y, all cohomology classes $x \in H^n(X,\mathbb{F}_2)$ and $y \in H^m(Y,\mathbb{F}_2)$ and all $i \geq 0$ we have*

$$\mathrm{Sq}^i(x\times y) \;=\; \sum_{a+b=i} \mathrm{Sq}^a(x)\times \mathrm{Sq}^b(y)$$

in $H^{n+m}(X\times Y,\mathbb{F}_2)$.

Proof. In the proof we abbreviate $K(n,\mathbb{F}_2)$ to $K(n)$. For $i > n+m$, both sides of the Cartan formula are trivial since the squaring operations vanish on cohomology classes of lower dimensions (part (ii) of Theorem 4.1). For $i = n+m$, the same argument gives

$$\mathrm{Sq}^{n+m}(x\times y) \;=\; (x\times y)\cup(x\times y) \;=\; x^2\times y^2 \;=\; \mathrm{Sq}^n(x)\times\mathrm{Sq}^m(y) \;=\; \sum_{a+b=i}\mathrm{Sq}^a(x)\times\mathrm{Sq}^b(y)$$

where we also used the defining property of the squaring operations and the fact that $(x\times y)\cup(x'\times y') = (x\cup x')\times(y\cup y')$.

So it remains to treat the case where $i < n+m$ and here we use induction on $n+m$. By naturality it is enough to verify the formula for the fundamental classes, i.e., for $x = \iota_n \in H^n(K(n);\mathbb{F}_2)$ and $y = \iota_m \in H^m(K(m);\mathbb{F}_2)$. There is nothing to show for $n+m=0$, so we assume $n+m\geq 1$. For $p \leq 2n-1$, the restriction map

$$\epsilon^* \;:\; H^p(K(n);\mathbb{F}_2) \;\longrightarrow\; H^p(\Sigma K(n-1);\mathbb{F}_2)$$

induced by the map $\epsilon\colon \Sigma K(n-1) \longrightarrow K(n)$ is injective by Corollary 3.2. Similarly, the map $\epsilon^*\colon H^q(K(m);\mathbb{F}_2) \longrightarrow H^q(\Sigma K(m-1);\mathbb{F}_2)$ is injective for $q \leq 2m-1$. So by the Künneth theorem, the map

$$H^k(K(n)\times K(m);\mathbb{F}_2) \;\cong\; \bigoplus_{p+q=k} H^p(K(n);\mathbb{F}_2)\otimes H^q(K(m);\mathbb{F}_2) \xrightarrow{(\epsilon^*\otimes 1,\ 1\otimes\epsilon^*)}$$

$$\bigoplus_{p+q=*} (H^p(\Sigma K(n-1);\mathbb{F}_2)\otimes H^q(K(m);\mathbb{F}_2)\oplus(H^p(K(n);\mathbb{F}_2)\otimes H^q(\Sigma K(m-1);\mathbb{F}_2))$$

is injective in dimensions $k \leq 2n+2m-1$. This means that the Cartan formula holds if we can verifying it after applying the maps $(\epsilon\times 1)^*$ and $(1\times\epsilon)^*$ to both sides. In the first case we calculate

$$\begin{aligned}
(\epsilon\times 1)^*(\mathrm{Sq}^i(\iota_n\times\iota_m)) \;&=\; \mathrm{Sq}^i((\epsilon\times 1)^*(\iota_n\times\iota_m)) \;=\; \mathrm{Sq}^i(\epsilon^*(\iota_n)\times\iota_m)) \;=\; \mathrm{Sq}^i(\Sigma(\iota_{n-1})\times\iota_m))\\
&=\; \Sigma\left(\mathrm{Sq}^i(\iota_{n-1}\times\iota_m)\right) \;=\; \Sigma\left(\sum_{a+b=i}\mathrm{Sq}^a(\iota_{n-1})\times\mathrm{Sq}^b(\iota_m))\right)\\
&=\; \sum_{a+b=i}\Sigma(\mathrm{Sq}^a(\iota_{n-1}))\times\mathrm{Sq}^b(\iota_m) \;=\; \sum_{a+b=i}\mathrm{Sq}^a(\Sigma(\iota_{n-1}))\times\mathrm{Sq}^b(\iota_m)\\
&=\; \sum_{a+b=i}\mathrm{Sq}^a(\epsilon^*(\iota_n))\times\mathrm{Sq}^b(\iota_m) \;=\; (\epsilon\times 1)^*\left(\sum_{a+b=i}\mathrm{Sq}^a(\iota_n)\times\mathrm{Sq}^b(\iota_m)\right)\;.
\end{aligned}$$

We have used that $\epsilon^*(\iota_n) = \Sigma\iota_{n-1}$ and that Sq^i is a stable cohomology operation. The fourth equality uses the induction hypothesis, which applies since the dimension of ι_{n-1} is smaller than n. The second case is similar. □

Exercise 4.3. Show that for every 1-dimensional cohomology class x the following formula holds:

$$\mathrm{Sq}^i(x^n) \;=\; \binom{n}{i} x^{i+n} \;.$$

5. Examples and applications

An important problem in homotopy theory is the find ways of telling when a continuous map $f\colon X \longrightarrow Y$ is null-homotopic. A map which is not null-homotopic is called *essential*.

Sometimes a map can be shown to be essential by checking that it induces a non-trivial map on cohomology with suitable coefficients. If this does not help, then one can use the *mapping cone* $C(f)$ of a continuous map $f\colon X \longrightarrow Y$. The mapping cone is defined by

$$C(f) \;=\; * \cup_{X\times 0} X \times [0,1] \cup_{X\times 1} Y \;,$$

and it comes with an injection $i\colon Y \longrightarrow C(f)$ and a projection $C(f) \longrightarrow C(f)/Y \cong \Sigma X$. The mapping cone is designed so that the map f is null-homotopic if and only if i has a retraction, i.e., there is a map $\sigma\colon C(f) \longrightarrow Y$ such that the composite $\sigma \circ i$ is the identity of Y.

Now suppose that f is trivial in cohomology with coefficients in an abelian group A; then the long exact mod-p cohomology sequence yields an epimorphism

$$H^*(C(f), \mathbb{F}_p) \;\xrightarrow{i^*}\; H^*(Y, \mathbb{F}_p) \;,$$

where $i\colon Y \longrightarrow C(f)$ is the inclusion. If f is null-homotopic, then a choice of retraction $\sigma\colon C(f) \longrightarrow Y$ induces a map of graded abelian groups $\sigma^*\colon H^*(Y, \mathbb{F}_p) \longrightarrow H^*(C(f), \mathbb{F}_p)$ which is a section to the map i^*.

But such a section σ^* is induced by a geometric map, so it also respects all additional structure which is natural for continuous maps. For example, if A is a ring, then σ^* is compatible with the cup-product. In many cases, the original map f can be seen to be essential because there is no section to i^* which is multiplicative with respect to the cup-product.

Example 5.1. The Hopf maps $\eta\colon S^3 \longrightarrow S^2$, $\nu\colon S^7 \longrightarrow S^4$ and $\sigma\colon S^{15} \longrightarrow S^8$ are essential. The mapping cones of the Hopf maps η, ν and σ are isomorphic to the projective planes $\mathbb{C}\mathrm{P}^2, \mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$ over the complex numbers, the quaternions and the Cayley octaves respectively. The integral cohomology rings of these spaces are all of the form $\mathbb{Z}[x]/x^3$ where the dimension of the generator is 2, 4 or 8 respectively. Hence if $i\colon S^2 \cong \mathbb{C}\mathrm{P}^1 \longrightarrow \mathbb{C}\mathrm{P}^2$ is the inclusion, then there is no multiplicative section to the map

$$i^* \;:\; H^*(\mathbb{C}\mathrm{P}^2; \mathbb{Z}) \;\longrightarrow\; H^*(\mathbb{C}\mathrm{P}^1; \mathbb{Z}) \;,$$

and so the Hopf map η is essential. The same argument with $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$ shows that the Hopf maps ν and σ are essential.

The cup-product is useless for telling whether a map is *stably essential*, i.e., whether or not it becomes null-homotopic after some number of suspensions. This is because the cup-product is trivial on the reduced cohomology of any suspension. Indeed, if $f\colon X \longrightarrow Y$ is a map of spaces which is trivial in reduced mod-p cohomology, then we have $C(\Sigma f) \cong \Sigma C(f)$, and the map

$$H^*(\Sigma C(f); \mathbb{F}_p) \;\cong H^*(C(\Sigma f); \mathbb{F}_p) \;\xrightarrow{i^*}\; H^*(\Sigma Y; \mathbb{F}_p)$$

always has a multiplicative section.

In general, the more highly structured and calculable homotopy functors we find, the better chances we have to show that such a section cannot exist. For the problem at hand, instead of the cup-product we can use stable cohomology operations, which are still non-trivial after suspension. So if some suspension of $\Sigma^n f\colon \Sigma^n X \longrightarrow \Sigma^n Y$ is null-homotopic, then the cohomology $\tilde{H}^*(C(\Sigma^n f); A)$ is the direct sum, as a module over the Steenrod algebra $\mathcal{A}(A)$, of the cohomology groups of $\Sigma^{n+1} X$ and $\Sigma^n Y$ with coefficients in A. The mapping cone of a suspension is isomorphic to the suspension of the mapping cone. Since the Steenrod-algebra consists of stable operations, suspension amounts to reindexing the cohomology of a space, including the action of the Steenrod-algebra. In other words, if a map $f\colon X \longrightarrow Y$ becomes null-homotopic after some number of suspensions, then f is trivial on $H^*(-;A)$ and the map

$$i^* \ : \ \tilde{H}^*(C(f);A) \xrightarrow{i^*} \tilde{H}^*(Y;A)$$

has a section which is $\mathcal{A}(A)$-linear. We apply this strategy to the Hopf maps.

Example 5.2. The Hopf maps $\eta\colon S^3 \longrightarrow S^2$, $\nu\colon S^7 \longrightarrow S^4$ and $\sigma\colon S^{15} \longrightarrow S^8$ are stably essential. The mapping cones of the Hopf maps η, ν and σ are isomorphic to the projective planes $\mathbb{C}\mathrm{P}^2, \mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$ over the complex numbers, the quaternions and the Cayley octaves respectively. The mod-2 cohomology algebras of these spaces are all of the form $\mathbb{F}_2[x]/x^3$ where the dimension of the generator is 2, 4 or 8 respectively. Hence we have the relation

$$\mathrm{Sq}^2(x_2) \ = \ x_2^2 \ \neq \ 0 \ \in \ H^4(\mathbb{C}\mathrm{P}^2, \mathbb{F}_2) \ ,$$

and similarly the classes $\mathrm{Sq}^4(x_4) \in H^8(\mathbb{H}\mathrm{P}^2, \mathbb{F}_2)$ and $\mathrm{Sq}^8(x_8) \in H^{16}(\mathbb{O}\mathrm{P}^2, \mathbb{F}_2)$ are non-zero. So the mod-2 cohomologies of the mapping cones of η, ν and σ do not split as modules over the mod-2 Steenrod-algebra, hence these maps are stably essential.

Example 5.3. The degree 2 map of the mod-2 Moore space is stably essential. Let p be a prime and let

$$M(p) \ = \ S^1 \cup_p D^2$$

denote the mod-p Moore space of dimension 2, obtained by attaching a 2-cell to the circle along the degree p map $S^1 \longrightarrow S^1$. Note that $M(2)$ is homeomorphic to $\mathbb{R}\mathrm{P}^2$. Denote by $\times p\colon \Sigma M(p) \longrightarrow \Sigma M(p)$ the smash product of $M(p)$ with the degree p map of the circle. The degree p map induces multiplication by p in cohomology with any kind of coefficients, but the cohomology of $M(p)$, with any kind of coefficients, is annihilated by p. So $\times p$ induces the trivial map in cohomology, and we may ask whether this map is null-homotopic. The answer is different for the prime 2 and the odd primes: for odd p, the degree p map on $M(p)$ is stably nullhomotopic.

In contrast, for the prime 2 the degree 2 map of $\Sigma M(2)$ is stably essential. Since the degree 2 map of $\Sigma M(2)$ is obtained by smashing the $M(2)$ with the degree 2 map of S^1, its mapping cone of $C(\times 2)$ is isomorphic to the smash product of two copies of the Moore space,

$$C(\times 2\colon \Sigma M(2) \longrightarrow \Sigma M(2)) \ \cong \ M(2) \wedge M(2)$$

in such a way that the inclusion $\Sigma M(2) \longrightarrow C(\times 2)$ corresponds to the smash product of the inclusion $i\colon S^1 \longrightarrow M(2)$ with $M(2)$. Now the mod-2 cohomology of $M(2)$ has an $\mathbb{F}_2$-basis given by a class $x \in \tilde{H}^1(M(2);\mathbb{F}_2)$ and its square $x^2 \in \tilde{H}^2(M(2);\mathbb{F}_2)$. By the Künneth theorem, the cohomology of the smash product $M(2) \wedge M(2)$ is four-dimensional with basis given by the classes $x \otimes x$ in dimension 2, $x^2 \otimes x$ and $x^2 \otimes x$ in dimension 3, and $x^2 \otimes x^2$ in dimension 4. Also by the Künneth theorem, the map

$$(i \wedge M(2))^* \ : \ \tilde{H}^*(M(2) \wedge M(2);\mathbb{F}_2) \ \longrightarrow \ \tilde{H}^1(S^1 \wedge M(2);\mathbb{F}_2)$$

is given by

$$(i \wedge M(2))^*(x \otimes x) \ = \ \Sigma x \ , \quad \text{and} \quad (i \wedge M(2))^*(x \otimes x^2) \ = \ \Sigma(x^2) \ ,$$

and it vanishes on the classes $x^2 \otimes x$ and $x^2 \otimes x^2$. All cup products are trivial in the reduced cohomology of $S^1 \wedge M(2)$, but in the cohomology of $M(2) \wedge M(2)$, the cup-square of the two-dimensional class $x \otimes x$ in non-trivial. This shows that there is now section to $(i \wedge M(2))^*$ which is compatible with the cup-product, so the degree 2 map on $M(2)$ is essential.

However, after a single suspension, the cup products of both sides are trivial, so this argument does not give any hint as to whether the suspension of the degree 2 map on $M(2)$ is null-homotopic or not. However, we can calculate the action of the Steenrod-squares in the cohomology of $M(2) \wedge M(2)$. Note that the operation $\mathrm{Sq}^2(x)$ acts trivially on the cohomology of $S^1 \wedge M(2)$ for dimensional reasons. On the other hand, the Cartan-formula gives

$$\mathrm{Sq}^2(x \otimes x) \;=\; \mathrm{Sq}^2(x) \otimes x + \mathrm{Sq}^1(x) \otimes \mathrm{Sq}^1(x) + x \otimes \mathrm{Sq}^2(x) \;=\; x^2 \otimes x^2$$

in $\tilde{H}^4(M(2) \wedge M(2); \mathbb{F}_2)$. So there does not exist a section to $(i \wedge M(2))^*$ which is compatible with the action of the Steenrod-algebra. Hence we conclude that the degree 2 map of the mod-2 Moore space is stably essential.

6. The extended power construction

Usually the squaring operations Sq^i are introduced in a more geometric fashion using the symmetric square construction for spaces and the mod-2 cohomology of the real projective space $\mathbb{RP}^\infty$. We show in this section that our definition of the Sq^is agrees with the more traditional one, using the uniqueness part of Theorem 4.1. We also construct the *reduced power operations* P^i in mod-p cohomology for an odd prime p.

In this section we will define and study the *total power operation*

$$\mathcal{P}_p \;:\; H^n(X, \mathbb{F}_p) \;\longrightarrow\; H^{np}(X \times L(p), \mathbb{F}_p)$$

for a prime p and $n \geq 0$, where $L(p)$ is an infinite-dimensional lens space.

Construction 6.1. We write $S^\infty = \bigcup_{n \geq 0} S(\mathbb{C}^n)$ for the infinite dimensional complex unit sphere, with the weak topology by the filtration by the subspaces $S(\mathbb{C}^n) = \{v \in \mathbb{C}^n \colon |v| = 1\}$. We write

$$C_p \;=\; \{z \in \mathbb{C} \colon z^p = 1\}$$

for the multiplicative group of p-th roots of unity in $\mathbb{C}$, a cyclic group of order p. The group C_p acts freely on S^∞ by scalar multiplication. For $p = 2$, the generator of C_2 acts by the antipodal map, and the quotient space is

$$L(2) \;=\; S^\infty/(v \sim -v) \;=\; \mathbb{R}P^\infty \;.$$

When p is odd, the quotient space is $L(p) = S^\infty/C_p$ is an infinite-dimensional lens space. Since S^∞ is contractible and the C_p-action is free, the quotient map $S^\infty \longrightarrow S^\infty/C_p$ is a universal covering, and so S^∞/C_p is an Eilenberg–MacLane space of type $(\mathbb{Z}/p, 1)$.

The sphere S^∞ admits a CW-structure for which the C_p-action is cellular. The odd skeleta of this CW-structure are given by $S^\infty_{2k-1} = S(\mathbb{C}^k)$. The even skeleton S^∞_{2k} is the join inside $S(\mathbb{C}^{k+1})$ of the previous skeleton $S(\mathbb{C}^k \oplus 0)$ and the free C_p-orbit $\{(0, \ldots, 0, \zeta_p^i) \colon 1 \leq i \leq p\}$. This CW-structure has p cells in each dimension, and these cells are freely permuted by the group C_p.

For $p = 2$ we have calculated the mod-2 cohomology ring of $L(2) = \mathbb{R}P^\infty$ a long time ago. The calculation of $H^*(L(p); \mathbb{F}_p)$ can be done along similar lines, as follows. We have

$$C_p \;\cong\; \pi_1(L(p), *) \;\cong\; H_1(L(p); \mathbb{Z})$$

by Poincaré's theorem. And thus

$$H^1(L(p); \mathbb{F}_p) \;\cong\; \mathrm{Hom}(H_1(L(p); \mathbb{Z}), \mathbb{F}_p) \;\cong\; \mathrm{Hom}(C_p, \mathbb{F}_p)$$

by the universal coefficient theorem. We let $x \in H^1(L(p); \mathbb{F}_p)$ be the generator that corresponds to the isomorphism $C_p \cong \mathbb{F}_p$ that sends the generator $\zeta_p \in C_p$ to $1 \in \mathbb{F}_p$. We set

$$y \;=\; \beta(x) \;\in\; H^2(L(p); \mathbb{F}_p) \;.$$

We record that $x^2 = \beta(x) = y$ for $p = 2$, but $x^2 = 0$ for odd primes p, by graded-commutativity of the cup product.

Proposition 6.2. *For every prime p, the mod-p cohomology algebra of $L(p)$ is given by*

$$H^*(L(p); \mathbb{F}_p) \;=\; \begin{cases} \mathbb{F}_2[x] & \text{for } p = 2\text{; and} \\ \mathbb{F}_p[y] \otimes \Lambda(x) \;=\; \mathbb{F}_p[x, y]/(x^2) & \text{for } p \text{ odd.} \end{cases}$$

Proof. The case $p = 2$ was done a while ago, so we not treat the case of odd primes.

The cellular C_p-action makes the cellular chain complex $C_*^{\mathrm{cell}}(S^\infty)$ into a complex of $\mathbb{Z}[C_p]$-modules. Since there are p freely permuted cells in each dimension, $C_*^{\mathrm{cell}}(S^\infty)$ is free of rank 1 as a $\mathbb{Z}[C_p]$-module for each $k \geq 0$. After suitable choices of characteristic maps, we obtain additive generators

$$e_k^0 \ , \dots , e_k^{p-1}$$

of $C_*^{\mathrm{cell}}(S^\infty)$ such that $\zeta_p \cdot e_k^i = e_k^{i+1}$, with superscript '$i+1$' interpreted cyclically modulo p. The boundary map in the cellular chain complex satisfies

$$\partial(e_k^0) \ = \ \begin{cases} e_{k-1}^0 - e_{k-1}^1 & \text{for } k \text{ odd, and} \\ e_{k-1}^0 + \dots + e_{k-1}^{p-1} & \text{for } k \geq 2 \text{ even.} \end{cases}$$

Indeed, in the 1-skeleton, each 1-cell connects two adjacent 0-cells. And in higher dimensions, the boundary is forced up to a unit in $\mathbb{Z}[C_p]$ by the fact that the complex $C_*^{\mathrm{cell}}(S^\infty)$ is acyclic because S^∞ is contractible.

The cellular chain complex of $L(p) = S^\infty/C_p$ is obtained from that of S^∞ by equalizing the C_p-action, i.e.,

$$C_*^{\mathrm{cell}}(L(p)) \ = \ C_*^{\mathrm{cell}}(S^\infty/C_p) \ \cong \ C_*^{\mathrm{cell}}(S^\infty) \otimes_{\mathbb{Z}[C_p]} \mathbb{Z} \ .$$

So $C_*^{\mathrm{cell}}(L(p))$ is free of rank 1 in every dimension, generated by $e_k = [e_k^0]$, with boundary map

$$\partial(e_k) \ = \ \begin{cases} 0 & \text{for } k \text{ odd, and} \\ p \cdot e_{k-1} & \text{for } k \geq 2 \text{ even.} \end{cases}$$

We conclude that both the mod-p homology groups, and the mod-p cohomology groups, are 1-dimensional over $\mathbb{F}_p$ in every dimension. In particular, the additive structure of $H^*(L(p); \mathbb{F}_p)$ is as claimed.

Now we determine the multiplicative structure of $H^*(L(p); \mathbb{F}_p)$. We write $L(p)_l = S_l^\infty/C_p$ for the l-skeleton. We show by induction on k that

$$H^*(L(p)_{2k-1}; \mathbb{F}_p) \ = \ \mathbb{F}_p[x, y]/(x^2, y^k) \ .$$

The induction starts with $k = 1$: the 1-skeleton $L(p)_1$ is a circle, and hence $H^*(L(p)_1; \mathbb{F}_p) = \mathbb{F}_p[x]/(x^2)$, as claimed.

The space $L(p)_{2k-1} = S(\mathbb{C}^k)/C_p$ is the quotient of a free and orientation-preserving action of a finite group on an closed, connected and orientable $(2k-1)$-manifold. So $L(p)_{2k-1}$ is also a closed, connected and orientable $(2k-1)$-manifold, and thus satisfies Poincaré duality. Moreover, we know the multiplicative structure of its cohomology up to dimension $2k-3$ by induction. For the whole multiplicative structure to satisfy Poincaré duality, the multiplication

$$\cup \ : \ H^i(L(p)_{2k-1}; \mathbb{F}_p) \otimes H^{2k-1-i}(L(p)_{2k-1}; \mathbb{F}_p) \ \longrightarrow \ H^{2k-1}(L(p)_{2k-1}; \mathbb{F}_p)$$

is a perfect pairing. In particular, multiplication by $y \in H^2(L(p)_{2k-1}; \mathbb{F}_p)$ is an isomorphism

$$y\cdot \ : \ \mathbb{F}_p\{xy^{k-2}\} = H^{2k-3}(L(p)_{2k-1}; \mathbb{F}_p) \ \xrightarrow{\cong} \ H^{2k-1}(L(p)_{2k-1}; \mathbb{F}_p) \ ;$$

so the class xy^{k-1} generates $H^{2k-1}(L(p)_{2k-1}; \mathbb{F}_p)$. This in particular implies that y^{k-1} is non-zero, and hence it generates $H^{2k-2}(L(p)_{2k-1}; \mathbb{F}_p)$. □

The *p-th extended power* of a space X is

$$D_p(X) \ = \ X^p \times_{C_p} S^\infty \ ,$$

the quotient space of $X^p \times S^\infty$ by the equivalence relation generated by

$$(x_1, \dots, x_p; v) \ \sim \ (x_2, \dots, x_p, x_1; \zeta_p \cdot v) \ ,$$

where $\zeta_p = e^{2\pi i/p}$ generates the group C_p. If X is pointed, then the *reduced extended power* is

$$\tilde{D}_p(X) \ = \ \left(X^{\wedge p} \wedge S_+^\infty\right)/C_p \ ,$$

the quotient space by the analogous equivalence relation. If X is unpointed, then there is a natural homeomorphism

$$\tilde{D}_p(X_+) \cong D_p(X)_+ \ .$$

Proposition 6.3. *Let Y be a pointed $(n-1)$-connected CW-complex equipped with a continuous C_p-action, and let A be an abelian coefficient group. Then the space*

$$Y \wedge_{C_p} S^\infty_+ = (Y \wedge S^\infty_+)/C_p = (Y \wedge S^\infty_+)/(y \wedge v \sim (\zeta_p \cdot y) \wedge (\zeta_p \cdot v))$$

is $(n-1)$-connected and the map

$$j : Y \longrightarrow Y \wedge_{C_p} S^\infty_+ , \quad y \longmapsto [y \wedge (1, 0, \dots)]$$

induces an isomorphism

$$j_* : \tilde{H}_n(Y; A)/C_p \xrightarrow{\cong} \tilde{H}_n(Y \wedge_{C_p} S^\infty_+; A)$$

from the quotient of the group $\tilde{H}_n(Y; A)$ by the induced C_p-action. And the previous map induces an isomorphism

$$j^* : \tilde{H}^n(Y \wedge_{C_p} S^\infty_+; A) \xrightarrow{\cong} \tilde{H}^n(Y; A)^{C_p}$$

to the subgroup of fixed elements under the induced C_p-action on $\tilde{H}^n(Y; A)$.

Proof. The subquotients of the skeleton filtration are isomorphic to

$$S^\infty_k / S^\infty_{k-1} \cong (C_p)_+ \wedge S^k ,$$

where the C_p-action is by translation on the left factor. The induced filtration of $Y \wedge_{C_p} S^\infty_+$ by the subspaces $Y \wedge_{C_p} (S^\infty_k)_+$ has subquotients isomorphic to

$$Y \wedge_{C_p} (S^\infty_k / S^\infty_{k-1}) \cong Y \wedge_{C_p} ((C_p)_+ \wedge S^k) \cong Y \wedge S^k \ .$$

This shows that the subquotient $Y \wedge_{C_p} (S^\infty_k / S^\infty_{k-1})$ is $(k+n-1)$-connected.

In particular, the quotient $Y \wedge_{C_p} (S^\infty / S^\infty_2)$ is $(n+1)$-connected, so the inclusion of the first filtration

$$Y \wedge S(\mathbb{C})_+ = Y \wedge_{C_p} (S^\infty_1)_+ \longrightarrow Y \wedge_{C_p} S^\infty_+$$

induces an isomorphism on (co-)homology in dimension n. The cofiber sequence of spaces

$$Y \cong Y \wedge_{C_p} (C_p)_+ \longrightarrow Y \wedge_{C_p} S(\mathbb{C})_+ \longrightarrow Y \wedge_{C_p} (S(\mathbb{C})/C_p) \cong Y \wedge S^1$$

gives rise to an exact sequence of reduced homology groups

$$\tilde{H}_n(Y; A) \cong \tilde{H}_{n+1}(Y \wedge S^1; A) \xrightarrow{\delta} \tilde{H}_n(Y; A) \longrightarrow \tilde{H}_n(Y \wedge_{C_p} S(\mathbb{C})_+; A) \longrightarrow 0 \tag{6.4}$$

Indeed, the last map is surjective since $Y \wedge S^1$ is n-connected. The two boundary points of the fundamental 1-cell in the CW-structure in $S(\mathbb{C})$ are attached to 1 and ζ_p, respectively. So the boundary homomorphism δ in the sequence (6.4) becomes the map

$$\tilde{H}_n(Y; A) \longrightarrow \tilde{H}_n(Y; A) , \quad y \longmapsto y - (\zeta_p)_*(y) \ .$$

So the exact sequence (6.4) shows that the group $\tilde{H}_n(Y \wedge_{C_p} S^\infty_+; A) \cong \tilde{H}_n(Y \wedge_{C_p} S(\mathbb{C})_+; A)$ is isomorphic to the quotient of $\tilde{H}_n(Y; A)$ by the C_p-action.

Since Y is $(n-1)$-connected, the universal coefficient theorem provides an isomorphism

$$\tilde{H}^n(Y; A) \xrightarrow{\cong} \mathrm{Hom}(\tilde{H}_n(Y; \mathbb{Z}), A) \ .$$

This isomorphism is natural, so it restricts to an isomorphism of the fixed points of the C_p-action:

$$\left(\tilde{H}^n(Y; A)\right)^{C_p} \xrightarrow{\cong} \mathrm{Hom}(\tilde{H}_n(Y; \mathbb{Z}), A)^{C_p} \cong \mathrm{Hom}(\tilde{H}_n(Y; \mathbb{Z})/C_p, A) \ .$$

Since $Y \wedge_{C_p} S^\infty_+$ is $(n-1)$-connected, the universal coefficient theorem provides an isomorphism

$$\tilde{H}^n(Y \wedge_{C_p} S^\infty_+; A) \xrightarrow{\cong} \mathrm{Hom}(\tilde{H}_n(Y \wedge_{C_p} S^\infty_+; \mathbb{Z}), A) \ .$$

All these data participates in a commutative diagram:

$$\begin{tikzcd}
\tilde{H}^n(Y \wedge_{C_p} S^\infty_+; A) \arrow[rr, "\cong"] \arrow[d, "j^*"'] & & \mathrm{Hom}(\tilde{H}_n(Y \wedge_{C_p} S^\infty_+; \mathbb{Z}), A) \arrow[d, "\cong"', "\mathrm{Hom}(j_*, A)"] \\
\tilde{H}^n(Y; A)^{C_p} \arrow[r, "\cong"'] & \mathrm{Hom}(\tilde{H}_n(Y;\mathbb{Z}), A)^{C_p} \arrow[r, "\cong"'] & \mathrm{Hom}(\tilde{H}_n(Y;\mathbb{Z})/C_p, A)
\end{tikzcd}$$

in which the right vertical map is an isomorphism by the first part. Hence the left vertical map is also an isomorphism. □

Part (ii) of the next proposition refers to the continuous map

$$j \ : \ X^{\wedge p} \ \longrightarrow \ X^{\wedge p} \wedge_{C_p} S^\infty_+ = \tilde{D}_p(X) \ , \quad j(x_1 \wedge \ldots \wedge x_p) \ = \ [x_1 \wedge \ldots \wedge x_p \wedge (1, 0, \ldots)] \ .$$

Proposition 6.5. *Let p be a prime, and let $n \geq 1$.*

(i) *For every based space X and every reduced cohomology class $x \in \tilde{H}^n(X; \mathbb{F}_p)$, the class*

$$x \wedge \ldots \wedge x \ \in \ \tilde{H}^{np}(X^{\wedge p}, \mathbb{F}_p)$$

is invariant under the automorphism induced by the cyclic permutation of smash factors in $X^{\wedge p}$.

(ii) *There is a unique class*

$$\tilde{\iota}_{n,p} \ \in \ H^{np}(\tilde{D}_p(K(\mathbb{F}_p, n)), \mathbb{F}_p)$$

such that

$$j^*(\tilde{\iota}_{n,p}) \ = \ \iota_n \wedge \ldots \wedge \iota_n \ \in \ H^{np}(K(\mathbb{F}_p, n)^{\wedge p}, \mathbb{F}_p) \ .$$

Proof. (i) We recall that for based spaces X and Y and cohomology classes $x \in \tilde{H}^k(X; \mathbb{F}_p)$ and $y \in \tilde{H}^l(Y; \mathbb{F}_p)$, the relation

$$x \wedge y \ = \ (-1)^{k \cdot l} \cdot \tau^*_{X,Y}(y \wedge x) \tag{6.6}$$

holds in $\tilde{H}^{k+l}(X \wedge Y; \mathbb{F}_p)$, where $\tau_{X,Y} \colon X \wedge Y \longrightarrow Y \wedge X$ is swapping the smash factors.

Now we consider $m \geq 2$ and based spaces $X_1, \ldots, X_m$. We write

$$c_m \ : \ X_1 \wedge X_2 \wedge \ldots \wedge X_m \ \longrightarrow \ X_2 \wedge \ldots \wedge X_m \wedge X_1$$

for the cyclic permutation of smash factors. We claim that

$$c^*_m(x_2 \wedge \ldots \wedge x_m \wedge x_1) \ = \ (-1)^{k_1 \cdot (k_2 + \cdots + k_m)} x_1 \wedge x_2 \wedge \ldots \wedge x_m \ ,$$

where k_i is the degree of the class x_i, i.e., $x_i \in H^{k_i}(X_i; \mathbb{F}_p)$. We prove this claim by induction on m, the case $m = 2$ being (6.6). For $m \geq 3$ we have

$$c_m \ = \ (X_2 \wedge \ldots \wedge X_{m-2} \wedge \tau_{X_1, X_m}) \circ (c_{m-1} \wedge X_m) \ ,$$

and so

$$\begin{aligned}
c^*_m(x_2 \wedge \ldots \wedge x_m \wedge x_1) \ &= \ (c_{m-1} \wedge X_m)^*((X_2 \wedge \ldots \wedge X_{m-2} \wedge \tau_{X_1, X_m})^*(x_2 \wedge \ldots \wedge x_m \wedge x_1) \\
&= \ (-1)^{k_1 k_m} \cdot (c_{m-1} \wedge X_m)^*((x_2 \wedge \ldots \wedge x_{m-1} \wedge x_1 \wedge x_m) \\
&= \ (-1)^{k_1 k_m} \cdot c^*_{m-1}(x_2 \wedge \ldots \wedge x_{m-1} \wedge x_1) \wedge x_m \\
&= \ (-1)^{k_1 k_m} \cdot (-1)^{k_1 \cdot (k_2 + \cdots + k_{m-1})} \cdot (x_1 \wedge x_2 \wedge \ldots \wedge x_{m-1}) \wedge x_m \\
&= \ (-1)^{k_1 \cdot (k_2 + \cdots + k_{m-1} + k_m)} \cdot x_1 \wedge x_2 \wedge \ldots \wedge x_{m-1} \wedge x_m \ .
\end{aligned}$$

Now we specialize to the case where $m = p$ is a prime, $X_1 = X_2 = \cdots = X_p = X$, and where $x_1 = x_2 = \cdots = x_p = x$, of degree n. Then the formula becomes

$$c^*_p(x \wedge \ldots \wedge x) \ = \ (-1)^{(p-1)n^2} \cdot x \wedge \ldots \wedge x \ .$$

If $p = 2$, then $-1 = 1$. If p is odd, then $p - 1$ is even, and $(-1)^{(p-1)n^2} = 1$. This proves claim (i).

(ii) Because $K(\mathbb{F}_p, n)$ is $(n-1)$-connected, its p-th smash power $K(\mathbb{F}_p, n)^{\wedge p}$ is $(np-1)$-connected. Proposition 6.3 thus shows that the map

$$j \ : \ K(\mathbb{F}_p, n)^{\wedge p} \longrightarrow K(\mathbb{F}_p, n)^{\wedge p} \wedge_{C_p} S^\infty_+ \ = \ \tilde{D}_p(K(\mathbb{F}_p, n))$$

induces an isomorphism

$$j^* \ : \ H^{np}(\tilde{D}_p(K(\mathbb{F}_p, n)), \mathbb{F}_p) \xrightarrow{\cong} \left(H^{np}(K(\mathbb{F}_p, n)^{\wedge p}, \mathbb{F}_p)\right)^{C_p} .$$

The class $\iota_n \wedge \ldots \wedge \iota_n$ in the target is invariant under the C_p-action by (i). So there is a unique class $\tilde{\iota}_{n,p}$ in the source that maps to $\iota_n \wedge \ldots \wedge \iota_n$. □

Construction 6.7. We let $\Pi \colon D_p(K(\mathbb{F}_p, n)) \longrightarrow \tilde{D}_p(K(\mathbb{F}_p, n))$ denote the projection from the unreduced to the reduced extended power. We set

$$\iota_{n,p} \ = \ \Pi^*(\tilde{\iota}_{n,p}) \ \in \ H^{np}(D_p(K(\mathbb{F}_p, n)); \mathbb{F}_p) \ . \tag{6.8}$$

The following square commutes:

$$\begin{array}{ccc} K(\mathbb{F}_p, n)^p & \xrightarrow{\text{proj}} & K(\mathbb{F}_p, n)^{\wedge p} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ D_p(K(\mathbb{F}_p, n)) & \xrightarrow[\Pi]{} & \tilde{D}_p(K(\mathbb{F}_p, n)) \end{array}$$

So we deduce the relation

$$j^*(\iota_{n,p}) \ = \ j^*(\Pi^*(\tilde{\iota}_{n,p})) \ = \ \text{proj}^*(j^*(\tilde{\iota}_{n,p})) \ = \ \text{proj}^*(\iota_n \wedge \ldots \wedge \iota_n) \ = \ \iota_n \times \cdots \times \iota_n \ . \tag{6.9}$$

Now we let X be a CW-complex. The diagonal map

$$\Delta \ : \ X \longrightarrow X^p \ , \quad \Delta(x) \ = \ (x, \ldots, x) \ ,$$

is C_p-equivariant with respect to the trivial action on the source and the permutation action on the target. So the diagonal induces a map

$$\Delta_X \ : \ X \times L(p) \ \cong \ X \times_{C_p} S^\infty \xrightarrow{\Delta \times_{C_p} S^\infty} X^p \times_{C_p} S^\infty = D_p(X) \ , \quad (x, [v]) \ \longmapsto \ [x, \ldots, x, v] \ .$$

The *p-th total power operation*

$$\mathcal{P}_p \ : \ H^n(X, \mathbb{F}_p) \ \longrightarrow \ H^{np}(X \times L(p), \mathbb{F}_p) \tag{6.10}$$

is then defined by

$$\mathcal{P}_p(x) \ = \ \mathcal{P}_p(f^*(\iota_n)) \ = \ \Delta_X^*(D_p(f)^*(\iota_{n,p})) \ .$$

In other words, if $x \in H^n(X, \mathbb{F}_p)$ is represented by a continuous based map $f \colon X \longrightarrow K(\mathbb{F}_p, n)$, then $\mathcal{P}_p(x)$ is defined as the restriction of the class $\iota_{n,p} \in H^{np}(D_p(K(\mathbb{F}_p, n)); \mathbb{F}_p)$ constructed in (6.8) along the composite map

$$X \times L(p) \ \xrightarrow{\Delta_X} \ D_p(X) \ \xrightarrow{D_p(f)} \ D_p(K(\mathbb{F}_p, n)) \ .$$

Said yet another way: $\mathcal{P}_p$ is the unique natural transformation such that $\mathcal{P}_p(\iota_n) = \Delta^*_{K(\mathbb{F}_p,n)}(\iota_{n,p})$.

In the following lemma we use the natural map

$$j \ : \ X \ \longrightarrow \ X \times L(p) \ , \quad x \ \longmapsto (x, [1, 0, \ldots]) \ .$$

Because the space $L(p)$ is path-connected, any point other than $[1, 0, \ldots] \in L(p)$ would yield a homotopic map.

Lemma 6.11.

(i) *The composite map*

$$H^n(X; \mathbb{F}_p) \ \xrightarrow{\mathcal{P}_p} \ H^{np}(X \times L(p); \mathbb{F}_p) \ \xrightarrow{j^*} \ H^{np}(X; \mathbb{F}_p)$$

sends a cohomology class to its p-th cup power.

(ii) *The total power operation and the exterior product are related by the formula*

$$\mathcal{P}_p(x \times y) \;=\; \Delta^*(\mathcal{P}_p(x) \times \mathcal{P}_p(y))$$

in $H^*(X \times Y \times L(p); \mathbb{F}_p)$ *for cohomology classes* $x \in H^*(X; \mathbb{F}_p)$ *and* $y \in H^*(Y; \mathbb{F}_p)$*, where*

$$\Delta \;:\; X \times Y \times L(p) \;\longrightarrow\; (X \times L(p)) \times (Y \times L(p))$$

is given by $\Delta(x, y, z) = ((x, z), (y, z))$.

Proof. (i) By naturality it suffices to check the universal example, i.e., we may take $X = K(\mathbb{F}_p, n)$ and evaluate on the fundamental class ι_n. In this case $\mathcal{P}_p(\iota_n)$ is the restriction of the class $\iota_{n,p} \in H^{np}(D_p(K(\mathbb{F}_p, n)); \mathbb{F}_p)$ along the lower map in the following commutative diagram:

$$\begin{tikzcd} K(\mathbb{F}_p, n) \arrow[r, "\Delta"] \arrow[d, "j"'] & K(\mathbb{F}_p, n)^p \arrow[d, "j"] \\ K(\mathbb{F}_p, n) \times L(p) \arrow[r, "\Delta_{K(\mathbb{F}_p,n)}"'] & K(\mathbb{F}_p, n)^p \times_{C_p} S^\infty = D_p(K(\mathbb{F}_p, n)) \end{tikzcd}$$

So we deduce that

$$\begin{aligned} j^*(\mathcal{P}_p(\iota_n)) \;&=\; j^*(\Delta^*_{K(\mathbb{F}_p,n)}(\iota_{n,p})) \;=\; \Delta^*(j^*(\iota_{n,p})) \\ \overset{(6.9)}{=}\; &\Delta^*(\iota_n \times \cdots \times \iota_n) \;=\; \iota_n \cup \cdots \cup \iota_n \;=\; \iota_n^p \,. \end{aligned}$$

(ii) By naturality it suffices to check the universal example, i.e., we may take $X = K(\mathbb{F}_p, n)$, $Y = K(\mathbb{F}_p, m)$, $x = \iota_n$ and $y = \iota_m$, We simplify the notation by writing $K(n)$ for $K(\mathbb{F}_p, n)$ and $K(m)$ for $K(\mathbb{F}_p, m)$. We consider the map

$$\tilde{\Delta} \;:\; \tilde{D}_p(X \wedge Y) \;\longrightarrow\; \tilde{D}_p(X) \wedge \tilde{D}_p(Y) \;, \quad [x, y; v] \;\longmapsto\; ([x, v], [y, v]) \tag{6.12}$$

that arises from the diagonal map $S^\infty \longrightarrow S^\infty \times S^\infty$. It makes the following square commute:

$$\begin{tikzcd} (X \wedge Y)^{\wedge p} \arrow[r, "\text{shuffle}", "\cong"'] \arrow[d, "j"'] & X^{\wedge p} \wedge Y^{\wedge p} \arrow[d, "j \wedge j"] \\ \tilde{D}_p(X \wedge Y) \arrow[r, "\tilde{\Delta}"'] & \tilde{D}_p(X) \wedge \tilde{D}_p(Y) \end{tikzcd} \tag{6.13}$$

We define

$$\tilde{c} \;:\; K(n) \wedge K(m) \;\longrightarrow\; K(n + m)$$

as the based map, unique up to homotopy, such that

$$\tilde{c}^*(\iota_{n+m}) \;=\; \iota_n \wedge \iota_m$$

in the group $H^{n+m}(K(n) \wedge K(m); \mathbb{F}_p)$. It induces a based continuous map

$$\tilde{D}_p(\tilde{c}) \;:\; \tilde{D}_p(K(n) \wedge K(m)) \;\longrightarrow\; \tilde{D}_p(K(n + m))$$

on reduced extended powers. We claim that for $X = K(n)$ and $Y = K(m)$, the diagonal (6.12) satisfies

$$\tilde{\Delta}^*(\tilde{\iota}_{n,p} \wedge \tilde{\iota}_{m,p}) \;=\; (\tilde{D}_p(\tilde{c}))^*(\tilde{\iota}_{n+m,p}) \tag{6.14}$$

in the group $H^{(n+m)p}(\tilde{D}_p(K(n) \wedge K(m)); \mathbb{F}_p)$. Because $K(n)$ is $(n-1)$-connected and $K(m)$ is $(m-1)$-connected, the smash product $K(n) \wedge K(m)$ is $(n + m - 1)$-connected. Hence the space $(K(n) \wedge K(m))^{\wedge p}$ is $((n + m)p - 1)$-connected. Proposition 6.3 shows that the map

$$j \;:\; (K(n) \wedge K(m))^{\wedge p} \;\longrightarrow\; \tilde{D}_p(K(n) \wedge K(m))$$

induces an injection on $H^{(n+m)p}(-;\mathbb{F}_p)$. Commutativity of the square (6.13) yields

$$\begin{aligned} j^*(\tilde{\Delta}^*(\tilde{\iota}_{n,p}\wedge\tilde{\iota}_{m,p})) \ &= \ \mathrm{shuffle}^*((j\wedge j)^*(\tilde{\iota}_{n,p}\wedge\tilde{\iota}_{m,p})) \\ &= \ \mathrm{shuffle}^*(j^*(\tilde{\iota}_{n,p})\wedge j^*(\tilde{\iota}_{m,p})) \\ &= \ \mathrm{shuffle}^*((\iota_n\wedge\ldots\wedge\iota_n)\wedge(\iota_m\wedge\ldots\wedge\iota_m)) \\ &= \ (\iota_n\wedge\iota_m)\wedge\ldots\wedge(\iota_n\wedge\iota_m) \\ &= \ \tilde{c}^*(\iota_{n+m})\wedge\ldots\wedge\tilde{c}^*(\iota_{n+m}) \\ &= \ (\tilde{c}\wedge\ldots\wedge\tilde{c})^*(\iota_{n+m}\wedge\ldots\wedge\iota_{n+m}) \\ &= \ (\tilde{c}\wedge\ldots\wedge\tilde{c})^*(j^*(\tilde{\iota}_{n+m,p})) \\ &= \ j^*((\tilde{D}_p(\tilde{c}))^*(\tilde{\iota}_{n+m,p})) \ . \end{aligned}$$

Since j^* is injective in this particular cohomological dimension, this proves relation (6.14).

We we turn the relation (6.14) from a reduced into an unreduced form. We abuse notation and also write

$$\Delta \ : \ D_p(X\times Y) \ \longrightarrow \ D_p(X)\times D_p(Y) \ , \quad [x,y;v] \ \longmapsto \ ([x;v],[y;v])$$

for yet another diagonal map, now for the unreduced extended powers. We write

$$c \ = \ \tilde{c}\circ\Pi \ : \ K(n)\times K(m) \ \longrightarrow \ K(n+m) \ ,$$

which satisfies

$$c^*(\iota_{n+m}) \ = \ \Pi^*(\tilde{c}^*(\iota_{n+m})) \ = \ \Pi^*(\iota_n\wedge\iota_m) \ = \ \iota_n\times\iota_m \ .$$

If X and Y are based, then the following diagram commutes by inspection:

$$\begin{tikzcd} X\times Y\times L(p) \arrow[r, "\Delta"] \arrow[d, "\Delta_{X\times Y}"'] & (X\times L(p))\times(Y\times L(p)) \arrow[d, "\Delta_X\times\Delta_Y"] \\ D_p(X\times Y) \arrow[r, "\Delta"] \arrow[d, "\Pi"'] & D_p(X)\times D_p(Y) \arrow[d, "\Pi"] \\ \tilde{D}_p(X\wedge Y) \arrow[r, "\tilde{\Delta}"'] & \tilde{D}_p(X)\wedge\tilde{D}_p(Y) \end{tikzcd}$$

Then

$$\begin{aligned} (6.15)\qquad \Delta^*(\iota_{n,p}\times\iota_{m,p}) \ &= \ \Delta^*(\Pi^*(\iota_{n,p}\wedge\iota_{m,p})) \ = \ \Pi^*(\tilde{\Delta}^*(\iota_{n,p}\wedge\iota_{m,p})) \\ \overset{(6.14)}{=} \ &\Pi^*(\tilde{D}_p(c)^*(\tilde{\iota}_{n+m,p})) \ = \ D_p(c)^*(\Pi^*(\tilde{\iota}_{n+m,p})) \ = \ D_p(c)^*(\iota_{n+m,p}) \end{aligned}$$

in the group $H^{(n+m)p}(D_p(K(n)\times K(m)))$. Thus

$$\begin{aligned} \mathcal{P}_p(\iota_n\times\iota_m) \ &= \ \mathcal{P}_p((c\circ\Pi)^*(\iota_{n+m})) \\ &= \ \Delta^*_{K(n)\times K(m)}(D_p(c\circ\Pi)^*(\iota_{n+m,p})) \\ &\overset{(6.15)}{=} \Delta^*_{K(n)\times K(m)}(\Delta^*(\iota_{n,p}\times\iota_{m,p})) \\ &= \Delta^*((\Delta_{K(n)}\times\Delta_{K(m)})^*(\iota_{n,p}\times\iota_{m,p})) \\ &= \Delta^*(\Delta^*_{K(n)}(\iota_{n,p})\times\Delta^*_{K(m)}(\iota_{m,p})) \\ &= \Delta^*(\mathcal{P}_p(\iota_n)\times\mathcal{P}_p(\iota_m)) \end{aligned}$$

□

We base $L(2)=\mathbb{R}P^\infty$ at the point $[1,0,0\ldots]$.

Proposition 6.16. *There is a homeomorphism*

$$h \ : \ \tilde{D}_2(S^1) \ \xrightarrow{\cong} \ S^1\wedge\mathbb{R}P^\infty$$

with the property that the composite

$$S^1 \wedge \mathbb{R}P^\infty_+ \xrightarrow{\tilde{\Delta}_{S^1}} \tilde{D}_2(S^1) \xrightarrow[\cong]{h} S^1 \wedge \mathbb{R}P^\infty$$

is homotopic to $S^1 \wedge q\colon S^1 \wedge \mathbb{R}P^\infty_+ \longrightarrow S^1 \wedge \mathbb{R}P^\infty$, where $q\colon \mathbb{R}P^\infty_+ \longrightarrow \mathbb{R}P^\infty$ identifies the external basepoint with the internal basepoint.

Proof. We write $S^1_{\mathrm{sgn}} = \mathbb{R} \cup \{\infty\}$ for the onepoint compactification of $\mathbb{R}$ with the sign involution, sending x to $-x$; the basepoint is the point at infinity. In a first step we exhibit a homeomorphism

$$k \ : \ S^1_{\mathrm{sgn}} \wedge_{C_2} S^\infty_+ \xrightarrow{\cong} \mathbb{R}P^\infty \ . \qquad (6.17)$$

We fix $m \geq 0$ and consider the continuous map

$$\mathbb{R} \times S(\mathbb{R}^m) \longrightarrow \mathbb{R}P^m \ , \quad (x; v_1, v_2, \dots, v_m) \longmapsto [x : v_1 : v_2 : \dots : v_m] \ .$$

For $x \neq 0$, we have

$$[x : v_1 : v_2 : \dots : v_m] = [1 : v_1/x : v_2/x : \dots : v_m/x] \ .$$

So the map extends continuously to

$$S^1 \times S(\mathbb{R}^m) \longrightarrow \mathbb{R}P^m \quad \text{by} \quad (\infty; v_1, v_2, \dots, v_m) \longmapsto [1 : 0 : \dots : 0] \ .$$

Since $\{\infty\} \times S(\mathbb{R}^m)$ is taken to the single point $[1 : 0 : \dots : 0]$, this map factors through a continuous map

$$S^1 \wedge S(\mathbb{R}^m)_+ = (S^1 \times S(\mathbb{R}^m))/(\{\infty\} \times S(\mathbb{R}^m)) \longrightarrow \mathbb{R}P^m \ .$$

This maps is surjective, but not injective: because

$$[-x : -v_1 : -v_2 : \dots : -v_m] = [x : v_1 : v_2 : \dots : v_m] \ ,$$

the pairs (x, v) and $(-x, -v)$ have the same image. So the previous map factors through a continuous surjective map

$$k_m \ : \ S^1_{\mathrm{sgn}} \wedge_{C_2} S(\mathbb{R}^m)_+ \longrightarrow \mathbb{R}P^m$$

on the quotient space. This map is also injective, and hence a continuous bijection from a quasi-compact space to a Hausdorff space. So this map is a homeomorphism. The homeomorphisms k_m are compatible for different values of m, in the sense that the following diagram commutes:

$$\begin{tikzcd} S^1_{\mathrm{sgn}} \wedge_{C_2} S(\mathbb{R}^m)_+ \arrow[r, "{[x,v_1,\dots,v_n] \mapsto [x,v_1,\dots,v_n,0]}"] \arrow[d, "k_m"', "\cong"] & S^1_{\mathrm{sgn}} \wedge_{C_2} S(\mathbb{R}^{m+1})_+ \arrow[d, "\cong"', "k_{m+1}"] \\ \mathbb{R}P^m \arrow[r, "{[y_0:y_1:\dots:y_m] \mapsto [y_0:y_1:\dots:y_m:0]}"'] & \mathbb{R}P^{m+1} \end{tikzcd}$$

So we can pass to the colimit over k in the horizontal directions, and obtain the homeomorphism (6.17).

The composite

$$\mathbb{R}P^\infty \xrightarrow{[v] \mapsto [0 \wedge v]} (S^1_{\mathrm{sgn}} \wedge_{C_2} S^\infty_+) \xrightarrow[\cong]{k} \mathbb{R}P^\infty$$

is given by

$$[y_0 : y_1 : \dots] \longmapsto [0 : y_0 : y_1 : \dots] \ .$$

This map is homotopic to the identity, as witnessed by the homotopy

$$\begin{aligned} [0, \pi/2] \times \mathbb{R}P^\infty &\longrightarrow \mathbb{R}P^\infty \\ (t, [y_0 : y_1 : \dots]) &\longmapsto [\sin(t) y_0 : \cos(t) y_0 + \sin(t) y_1 : \cos(t) y_1 + \sin(t) y_2 : \dots] \ . \end{aligned}$$

Now we consider the invertible matrix $A = \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. Since A has positive determinant, the induced homeomorphism on onepoint compactification

$$A \ : \ S^2 \longrightarrow S^2$$

is based homotopic to the identity. This homeomorphism is equivariant for two different involutions on source and target, namely for the twist involution $x \wedge y \mapsto y \wedge x$ on the source, and for the involution

$$S^2 \longrightarrow S^2 \,, \quad x \wedge y \longmapsto (x, -y)$$

on the target. We shall use the suggestive notation $S^1 \wedge S^1_{\mathrm{sgn}}$ for S^2 with this second involution. So A induces another homeomorphism

$$A \wedge_{C_2} S^\infty_+ \;:\; \tilde{D}_2(S^1) = (S^1 \wedge S^1) \wedge_{C_2} S^\infty_+ \;\longrightarrow\; S^1 \wedge (S^1_{\mathrm{sgn}} \wedge_{C_2} S^\infty_+) \,.$$

Since $A \cdot (x, x) = (x, 0)$, the left triangle in the following diagram commutes:

$$\begin{tikzcd}
 & S^1 \wedge \mathbb{R}P^\infty_+ \arrow[dl, bend right, "\tilde{\Delta}_{S^1}"'] \arrow[d, "S^1 \wedge [0,-]"] \arrow[dr, bend left, "S^1 \wedge q"] & \\
\tilde{D}_2(S^1) \arrow[r, "\cong", "A \wedge_{C_2} S^\infty_+"'] & S^1 \wedge (S^1_{\mathrm{sgn}} \wedge_{C_2} S^\infty_+) \arrow[r, "\cong", "S^1 \wedge k"'] & S^1 \wedge \mathbb{R}P^\infty
\end{tikzcd}$$

So $(S^1 \wedge k) \circ (A \wedge_{C_2} S^\infty_+)\colon \tilde{D}_2(S^1) \longrightarrow S^1 \wedge \mathbb{R}P^\infty$ is the desired homeomorphism. $\square$

Let $\iota \in \widetilde{H}^1(S^1; \mathbb{Z})$ be the generator of the first cohomology group of the circle such that $- \wedge \iota$ implements the suspension isomorphism. We use the same name for the image of this class which generates the mod-p cohomology group $H^1(S^1; \mathbb{F}_p)$.

Proposition 6.18. *The relation $\mathcal{P}_2(\iota) = \iota \times u$ holds in the group $H^2(S^1 \times L(2); \mathbb{F}_2)$.*

Proof. We let $g\colon S^1 \longrightarrow \mathbb{R}P^\infty$ be a based map that represents the nontrivial element of $\pi_1(\mathbb{R}P^\infty, *)$. Then

$$g^*(u) \;=\; \iota$$

in $H^1(S^1; \mathbb{F}_2)$. We use the homeomorphism

$$h \;:\; \tilde{D}_2(S^1) \;\xrightarrow{\cong}\; S^1 \wedge \mathbb{R}P^\infty$$

provided by Proposition 6.16. The space $S^1 \wedge \mathbb{R}P^\infty$ is simply connected and has

$$H_2(S^1 \wedge \mathbb{R}P^\infty; \mathbb{Z}) \;\cong\; H_1(\mathbb{R}P^\infty; \mathbb{Z}) \;\cong\; \mathbb{Z}/2 \,.$$

So $\pi_2(S^1 \wedge \mathbb{R}P^\infty, *) \cong \mathbb{Z}/2$ by the Hurewicz theorem. The composite map

$$S^1 \wedge g \;:\; S^1 \wedge S^1 \;\longrightarrow\; S^1 \wedge \mathbb{R}P^\infty$$

is nontrivial on $H^2(-; \mathbb{F}_2)$, and hence not nullhomotopic. The composite

$$S^1 \wedge S^1 \;\xrightarrow{j}\; (S^1 \wedge S^1) \wedge_C S^\infty_+ = \tilde{D}_2(S^1) \;\xrightarrow[\cong]{h}\; S^1 \wedge \mathbb{R}P^\infty$$

is nontrivial on $H^2(-; \mathbb{F}_2)$ by Proposition 6.5, and hence not nullhomotopic. So

$$S^1 \wedge g \sim h \circ j \;:\; S^1 \wedge S^1 \;\longrightarrow\; S^1 \wedge \mathbb{R}P^\infty \,.$$

Hence also

$$j^*(h^*(\iota \wedge u)) \;=\; (h \circ j)^*(\iota \wedge u) \;=\; (S^1 \wedge g)^*(\iota \wedge u) \;=\; \iota \wedge g^*(u) \;=\; \iota \wedge \iota$$

in $H^2(S^1 \wedge S^1; \mathbb{F}_2)$.

The space $\mathbb{R}P^\infty$ is also a $K(\mathbb{F}_2, 1)$, and in this role $u = \iota_1$ is the fundamental class. The class $\tilde{\iota}_{1,2} \in H^2(\tilde{D}_2(\mathbb{R}P^\infty); \mathbb{F}_2)$ was defined in Proposition 6.5 by the property

$$j^*(\tilde{\iota}_{1,2}) \;=\; \iota_1 \wedge \iota_1 \;=\; u \wedge u \,.$$

The following diagram commutes:

$$\begin{tikzcd}
S^1 \wedge S^1 \arrow[r, "g\wedge g"] \arrow[d, "j"'] & \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \arrow[d, "j"] \\
\tilde{D}_2(S^1) \arrow[r, "\tilde{D}_2(g)"'] & \tilde{D}_2(\mathbb{R}P^\infty)
\end{tikzcd}$$

So we obtain the relation

$$j^*(\tilde{D}_2(g)^*(\tilde{\iota}_{1,2})) \;=\; (g\wedge g)^*(j^*(\tilde{\iota}_{1,2})) \;=\; (g\wedge g)^*(u\wedge u) \;=\; \iota\wedge\iota \;.$$

Since $S^1\wedge S^1$ is simply connected, the map

$$j^* \;:\; H^2(\tilde{D}_2(S^1);\mathbb{F}_2) \;\longrightarrow\; H^2(S^1\wedge S^1;\mathbb{F}_2)$$

is injective by Proposition 6.3. So we conclude that

$$\tilde{D}_2(g)^*(\tilde{\iota}_{1,2}) \;=\; h^*(\iota\wedge u)$$

in the group $H^2(\tilde{D}_2(S^1);\mathbb{F}_2)$.

Now we exploit the following homotopy commutative diagram:

$$\begin{tikzcd}
S^1\times\mathbb{R}P^\infty \arrow[r, "\Pi"] \arrow[d, "\Delta_{S^1}"'] & S^1\wedge\mathbb{R}P^\infty_+ \arrow[d, "\tilde{\Delta}_{S^1}"'] \arrow[dr, bend left, "S^1\wedge q"] & \\
D_2(S^1) \arrow[r, "\Pi"'] \arrow[d, "D_2(g)"'] & \tilde{D}_2(S^1) \arrow[r, "\cong", "h"'] \arrow[d, "\tilde{D}_2(g)"'] & S^1\wedge\mathbb{R}P^\infty \\
D_2(\mathbb{R}P^\infty) \arrow[r, "\Pi"'] & \tilde{D}_2(\mathbb{R}P^\infty) &
\end{tikzcd}$$

The commutativity (up to homotopy) of the triangle is part of Proposition 6.16. This yields

$$\begin{aligned}
\mathcal{P}_2(\iota) \;&=\; \Delta^*_{S^1}(D_2(g)^*(\iota_{1,2})) \\
&=\; \Delta^*_{S^1}(D_2(g)^*(\Pi^*(\tilde{\iota}_{1,2}))) \\
&=\; \Pi^*(\tilde{\Delta}^*_{S^1}(\tilde{D}_2(g)^*(\tilde{\iota}_{1,2}))) \\
&=\; \Pi^*(\tilde{\Delta}^*_{S^1}(h^*(\iota\wedge u))) \\
&=\; \Pi^*((S^1\wedge q)^*(\iota\wedge u)) \\
&=\; \Pi^*(\iota\wedge u) \;=\; \iota\times u \;.
\end{aligned}$$

□

The next theorem shows that the total power operation $\mathcal{P}_2$ encodes all the Steenrod operations in one class. The Künneth theorem tells us that the mod-2 cohomology of the product $X\times L(2)$ can be expanded as the tensor product of the cohomology of X and the cohomology of $L(2)$. We recall that the mod-2 cohomology of $L(2)=\mathbb{R}P^\infty$ is a polynomial algebra generated by the non-trivial one-dimensional class (the fundamental class).

Theorem 6.19. *For $p=2$, the total squaring operation $\mathcal{P}_2$ and the operations Sq^i are related by the formula*

$$\mathcal{P}_2(x) \;=\; \sum_{i\geq 0} \mathrm{Sq}^i(x)\times u^{n-i}$$

for $x\in H^n(X;\mathbb{F}_2)$, where $u\in H^1(L(2);\mathbb{F}_2)$ is the generator.

Proof. As an auxiliary notation we let

$$T^i_n \;:\; H^n(X;\mathbb{F}_2) \;\longrightarrow\; H^{n+i}(X\times L(2);\mathbb{F}_2)$$

be the cohomology operations defined by the formula

$$\mathcal{P}_2(x) \;=\; \sum_{i\geq 0} T^i_n(x)\times u^{n-i}\;.$$

We then have to show that $T^i_n = \mathrm{Sq}^i$.

In a first step we note that $T^n_n(x) = x^2$ when x has dimension n. By Lemma 6.11 (i), the restriction of the class $\mathcal{P}_2(x)\in H^{2n}(X\times L(2);\mathbb{F}_2)$ along the inclusion $j\colon X\longrightarrow X\times L(2)$ coincides with the cup-power $x^2\in H^{2n}(X;\mathbb{F}_2)$. Moreover, for $y\in H^*(X;\mathbb{F}_2)$ we have

$$j^*(y\times u^i) \;=\; \begin{cases} y & \text{if } i=0, \text{ and}\\ 0 & \text{if } i\geq 1.\end{cases}$$

So we deduce $x^2 = j^*(\mathcal{P}_2(x)) = T^n_n(x)$, as claimed.

Now we show that the operations T^i_n satisfy the Cartan formula

$$T^i_{k+l}(x\times y) \;=\; \sum_{a+b=i} T^a_k(x)\times T^b_l(y) \tag{6.20}$$

for $x\in H^k(X;\mathbb{F}_2)$ and $y\in H^l(Y;\mathbb{F}_2)$. Indeed, Lemma 6.11 (ii) gives

$$\begin{aligned}\mathcal{P}_2(x\times y) \;&=\; \Delta^*(\mathcal{P}_2(x)\times\mathcal{P}_2(y)) \;=\; \Delta^*\left(\sum_{a,b\geq 0}(T^a_k(x)\times u^{k-a})\times(T^b_l(y)\times u^{l-b})\right)\\ &=\; \sum_{i\geq 0}\left(\sum_{a+b=i} T^a_k(x)\times T^b_l(y)\right)\times u^{k+l-i}\end{aligned}$$

where we used the relation $\Delta^*((\alpha\times u^i)\times(\beta\times u^j)) = \alpha\times\beta\times u^{i+j}$. The Cartan formula (6.20) follows by comparing coefficients of u^{k+l-i}.

Proposition 6.18 provides the relation $\mathcal{P}_2(\iota) = \iota\times u$, where $\iota\in H^1(S^1;\mathbb{F}_2)$ is the generator. This means that $T^0_1(\iota) = \iota$, and $T^i_1(\iota) = 0$ for $i\neq 0$.

Now we verify that the collection of operations $\{T^i_n\}_{n\geq 0}$ form a *stable* cohomology operation. This is actually a formal consequence of the Cartan formula. Indeed, the suspension isomorphism

$$\Sigma \;:\; H^n(X;\mathbb{F}_2) \xrightarrow{\cong} H^{n+1}(\Sigma X;\mathbb{F}_2) = H^{n+1}(X\wedge S^1;\mathbb{F}_2)$$

is given by exterior smash product with the fundamental class $\iota\in\widetilde{H}^1(S^1;\mathbb{F}_2)$. So we get

$$T^i_{n+1}(x\wedge\iota) \;=\; \sum_{j=0}^{i} T^j_n(x)\wedge T^{i-j}_1(\iota) \;=\; T^i_n(x)\wedge\iota\;.$$

The second equation uses that $T^0_1(\iota) = \iota$ and $T^i_1(\iota) = 0$ for $i>0$.

We conclude that $\{T^i_n\}_{n\geq 0}$ form a stable mod-2 cohomology operation such that $T^n_n(x) = x^2$ for all n-dimensional classes x. By the uniqueness property of the squaring operation Sq^i (Theorem 4.1), we thus have $\{T^i_n\}_{n\geq 0} = \mathrm{Sq}^i$. □

The Steenrod squares are usually *defined* by the relation

$$\mathcal{P}_2(x) \;=\; \sum_{i\geq 0}\mathrm{Sq}^i(x)\times u^{n-i}$$

for $x\in H^n(X,\mathbb{F}_2)$, where $u\in H^1(L(2);\mathbb{F}_2)$ is the generator. If this is taken as the definition of the Steenrod squares, then the content of Theorem 6.19 is the proof that $\mathrm{Sq}^i(x) = x^2$ whenever x has degree i, and that the Cartan formula holds.

7. Steenrod operations for odd primes

Theorem 6.19 explains how the Steenrod squares Sq^i can be defined from the total squaring operation $\mathcal{P}_2$. Total power operation encodes all the Steenrod operations in one class. For odd primes p we use the total power operation $\mathcal{P}_p$ to define certain mod-p cohomology operations P^i, called *reduced power operations*. I will only sketch the main steps, but not give full proofs.

Throughout the following discussion, p is an odd prime. We have shown in Proposition 6.2 that

$$H^*(L(p);\mathbb{F}_p) \;=\; \mathbb{F}_p[y]\otimes\Lambda(x) \;=\; \mathbb{F}_p[x,y]/(x^2)\ ,$$

for a specific element $x\in H^1(L(p);\mathbb{F}_p)$, and with $y=\beta(x)$.

We set

$$u = x\cdot y^{p-2}\ \in H^{2p-3}(L(p);\mathbb{F}_p)$$

and

$$v \;=\; y^{p-1}\in\ H^{2p-2}(L(p);\mathbb{F}_p)\ .$$

Then $u^2=0$ and $\beta(u)=v$.

The secret reason for considering the elements u and v is as follows. The multiplicative group $\mathbb{F}_p^\times = \mathbb{F}_p\setminus\{0\}$ acts on the additive group $\mathbb{F}_p$ by multiplication, and this action witnesses $\mathbb{F}_p^\times$ as the automorphism group of $\mathbb{F}_p$. The action of $\mathbb{F}_p^\times$ on $\mathbb{F}_p$ induces an action, up to homotopy, on $L(p)=K(\mathbb{F}_p,1)$: for every $\lambda\in\mathbb{F}_p^\times$, there is a unique homotopy class of based map $t_\lambda\colon L(p)\longrightarrow L(p)$ that induces multiplication by λ on π_1, and hence also on $H_1(-;\mathbb{F}_p)$. Then also

$$t_\lambda^*(x) \;=\; \lambda\cdot x$$

by the universal coefficient theorem, and hence

$$t_\lambda^*(y) \;=\; t_\lambda^*(\beta(x)) \;=\; \beta(t_\lambda^*(x)) \;=\; \beta(\lambda\cdot x) \;=\; \lambda\cdot\beta(x) \;=\; \lambda\cdot y\ .$$

So the classes x and y are *not* invariant under the induced $\mathbb{F}_p^\times$-action on $H^*(L(p);\mathbb{F}_p)$. However,

$$t_\lambda^*(u) \;=\; t_\lambda^*(xy^{p-2}) \;=\; t_\lambda^*(x)\cdot(t_\lambda^*(y))^{p-2} \;=\; (\lambda\cdot x)\cdot(\lambda\cdot y)^{p-2} \;=\; \lambda^{p-1}\cdot xy^{p-2} \;=\; u$$

because $\lambda^{p-1}=1$ for all $\lambda\in\mathbb{F}_p^\times$. Similarly,

$$t_\lambda^*(v) \;=\; t_\lambda^*(y^{p-1}) \;=\; (t_\lambda^*(y))^{p-1} \;=\; \lambda^{p-1}\cdot y^{p-2} \;=\; v\ .$$

In other words: the classes u and v are invariant under the action of $\mathbb{F}_p^\times$. In in fact, they generate the entire subalgebra of $H^*(L(p);\mathbb{F})$ of $\mathbb{F}_p^\times$-invariant classes:

$$(H^*(L(p);\mathbb{F}_p))^{\mathbb{F}_p^\times} \;=\; \mathbb{F}_p[u,v]/(u^2)\ .$$

One can then show:

Proposition 7.1. *For every prime p, every space X and all even numbers $n\geq 0$, the image of the total power operation*

$$\mathcal{P}_p\ :\ H^n(X;\mathbb{F}_p)\ \longrightarrow\ H^{np}(X\times L(p);\mathbb{F}_p)$$

is invariant under the action of the group $\mathbb{F}_p^\times$, induced by the $\mathbb{F}_p^\times$-action on $L(p)$.

The standard proof of this proposition uses that the symmetric group Σ_p acts on on X^p by permuting the factors; this action extends the action of the cyclic group C_p, which we identify with the cyclic subgroup of Σ_p generated by the transposition $(1,2,\dots,p)$. Then one exploits the analog of the extended power construction for the action of this larger group.

For $p=3$ there is a direct proof of Proposition 7.1, as follows.

Proof of Proposition 7.1 for $p=3$. We consider the involution

$$\psi\ :\ S^\infty\ \longrightarrow\ S^\infty\ ,\quad \psi(v_1,v_2,\dots)\;=\;(\bar v_1,\bar v_2,\dots)$$

that is complex conjugation in each of the complex coordinates. The third root of unity $\zeta_3=e^{2\pi i/3}$ satisfies $\bar\zeta_3=\zeta_3^2$, so this map satisfies

$$\psi(\zeta_3\cdot v) \;=\; \bar\zeta_3\cdot\psi(v) \;=\; \zeta_3^2\cdot\psi(v)\ .$$

This means that ψ descends to a continuous involution

$$\bar{\psi} \ : \ L(p) = S^\infty/C_p \ \longrightarrow \ S^\infty/C_p = L(p) \ .$$

The restriction of ψ to the 1-skeleton $S(\mathbb{C})$ is complex conjugation, which reverses the orientation of the circle. So the induced homomorphism of fundamental groups

$$\pi_1(\bar{\psi}) \ : \ \pi_1(L(p),*) \ \longrightarrow \ \pi_1(L(p),*)$$

is the inverse map, and thus the unique non-identity automorphism of the cyclic group of order 3. So in the earlier notation, $\bar{\psi} = t_{-1}$, the automorphism associated to $-1 \in \mathbb{F}_3^\times$.

We now define an involution of $D_3(X) = X^3 \times_{C_3} S^\infty$ by

$$\bar{\psi}_X \ : \ X^3 \times_{C_3} S^\infty \ \longrightarrow \ X^3 \times_{C_3} S^\infty \ , \quad [x,y,z;v] \ \longmapsto \ [y,x,z;\psi(v)] \ .$$

Note that the coordinates x and y switch places, which is needed to make the map well-defined on equivalence classes: $(x,y,z;v)$ and $(y,z,x;\zeta_3 v)$ define the same element in $D_3(X)$, and

$$(y,x,z;\psi(v)) \sim (x,z,y;\zeta_3\psi(v)) \sim (z,y,x;\zeta_3^2\psi(v)) = (z,y,x;\psi(\zeta_3 v)) \ .$$

The following square also commutes:

$$\begin{tikzcd}
K(\mathbb{F}_3,n)^{\wedge 3} \arrow[r, "j"] \arrow[d, "\tau\wedge\mathrm{Id}"'] & \tilde{D}_3(K(\mathbb{F}_3,n)) \arrow[d, "\tilde{\psi}_{K(\mathbb{F}_3,n)}"] \\
K(\mathbb{F}_3,n)^{\wedge 3} \arrow[r, "j"'] & \tilde{D}_3(K(\mathbb{F}_3,n))
\end{tikzcd}$$

Here τ switches the first two smash factors of $K(\mathbb{F}_3,n)^{\wedge 3}$. So

$$\begin{aligned}
j^*(\tilde{\psi}^*_{K(\mathbb{F}_3,n)}(\tilde{\iota}_{n,3})) \ &= \ (\tau\wedge\mathrm{Id})^*(j^*(\tilde{\iota}_{n,3})) \\
&= \ (\tau\wedge\mathrm{Id})^*(\iota_n\wedge\iota_n\wedge\iota_n) \\
&= \ \iota_n\wedge\iota_n\wedge\iota_n \ = \ j^*(\tilde{\iota}_{n,3}) \ .
\end{aligned}$$

The third equation uses that n is even, for else a sign would appear. The map $j^*\colon H^{3n}(\tilde{D}_3(K(\mathbb{F}_3,n));\mathbb{F}_3) \longrightarrow H^{3n}(K(\mathbb{F}_3,n)^{\wedge 3};\mathbb{F}_3)$ is injective by Proposition 6.3, so we deduce that

$$\tilde{\psi}^*_{K(\mathbb{F}_3,n)}(\tilde{\iota}_{n,3}) \ = \ \tilde{\iota}_{n,3} \ .$$

The following square commutes:

$$\begin{tikzcd}
D_3(K(\mathbb{F}_3,n)) \arrow[r, "\Pi"] \arrow[d, "\bar{\psi}_{K(\mathbb{F}_3,n)}"'] & \tilde{D}_3(K(\mathbb{F}_3,n)) \arrow[d, "\tilde{\psi}_{K(\mathbb{F}_3,n)}"] \\
D_3(K(\mathbb{F}_3,n)) \arrow[r, "\Pi"'] & \tilde{D}_3(K(\mathbb{F}_3,n))
\end{tikzcd}$$

So

$$\bar{\psi}^*_{K(\mathbb{F}_3,n)}(\iota_{n,3}) \ = \ \bar{\psi}^*_{K(\mathbb{F}_3,n)}(\Pi^*(\tilde{\iota}_{n,3})) \ = \ \Pi^*(\tilde{\psi}^*_{K(\mathbb{F}_3,n)}(\tilde{\iota}_{n,3})) \ = \ \Pi^*(\tilde{\iota}_{n,3}) \ = \ \iota_{n,3} \ .$$

Finally, the following square commutes:

$$\begin{tikzcd}
X\times L(3) \arrow[r, "\Delta_X"] \arrow[d, "X\times\bar{\psi}"'] & D_3(X) \arrow[d, "\bar{\psi}_X"] \\
X\times L(3) \arrow[r, "\Delta_X"'] & D_3(X)
\end{tikzcd}$$

For $X = K(\mathbb{F}_3, n)$, this yields

$$\begin{aligned}(\mathrm{Id} \times \bar{\psi})^*(\mathcal{P}_3(\iota_n)) \ &= \ (\mathrm{Id} \times \bar{\psi})^*(\Delta^*_{K(\mathbb{F}_3,n)}(\iota_{n,3})) \ = \ \Delta^*_{K(\mathbb{F}_3,n)}((\bar{\psi}_{K(\mathbb{F}_3,n)})^*(\iota_{n,3})) \\ &= \ \Delta^*_{K(\mathbb{F}_3,n)}(\iota_{n,3}) \ = \ \mathcal{P}_3(\iota_n) \ .\end{aligned}$$

This is the universal example of the relation we wish to show, so the general case follows by naturality. □

The Künneth theorem says that the exterior product map

$$H^*(X; \mathbb{F}_p) \otimes \mathbb{F}_p[x, y]/(x^2) \ = \ H^*(X; \mathbb{F}_p) \otimes H^*(L(p); \mathbb{F}_p) \ \longrightarrow \ H^*(X \times L(p); \mathbb{F}_p)$$

is an isomorphism of graded-commutative $\mathbb{F}_p$-algebras. So we can expand the total power class $\mathcal{P}_p(x) \in H^{np}(X \times L(p); \mathbb{F}_p)$ in terms of the $\mathbb{F}_p$-basis $\{xy^i, y^i\}$ of $\mathbb{F}_p[x, y]/(x^2)$. By the invariance under the $\mathbb{F}_p^\times$-action, only terms with coefficients $u = xy^{p-2}$ and $v = y^{p-1}$ will show up.

We first work with even-dimensional cohomology classes, and consider the p-th total power operation introduced in (6.10)

$$\mathcal{P}_p \ : \ H^{2k}(X; \mathbb{F}_p) \ \longrightarrow \ H^{2pk}(X \times L(p); \mathbb{F}_p) \ .$$

By the previous discussion, we can expand this operations as

$$\mathcal{P}_p(x) \ = \ \sum_{i=0}^{k} \left(P^i_k(x) \times v^{k-i} \ + \ R^i_k(x) \times uv^{k-i-1}\right) \tag{7.2}$$

for $x \in H^{2k}(X; \mathbb{F}_p)$, for well-defined operations

$$\begin{aligned} P^i_k \ &: \ H^{2k}(X; \mathbb{F}_p) \ \longrightarrow \ H^{2k+2i(p-1)}(X; \mathbb{F}_p) \quad \text{and} \\ R^i_k \ &: \ H^{2k}(X; \mathbb{F}_p) \ \longrightarrow \ H^{2k+2i(p-1)+1}(X; \mathbb{F}_p) \ .\end{aligned}$$

Beware the indexing convention: the subscript of P^i_k and R^i_k is *half* of the cohomology degree of the argument. Since the total power operation is natural in X, the operations P^i_k and R^i_k are also natural in X.

The next step is to show an odd-primary analog of Proposition 6.18, which takes the form

$$\mathcal{P}_p(\iota \wedge \iota) \ = \ (\iota \wedge \iota) \times v \quad \text{in} \quad H^{2p}(S^2 \times L(p); \mathbb{F}_p) \ . \tag{7.3}$$

Here $\iota \wedge \iota \in H^2(S^2; \mathbb{F}_p)$ is the generator that implements the twofold suspension isomorphism. In terms of the recently introduced operations, this means that

$$P^i_1(\iota \wedge \iota) \ = \ \begin{cases} \iota \wedge \iota & \text{if } i = 0, \text{ and} \\ 0 & \text{else.} \end{cases} \tag{7.4}$$

and $R^i_1(\iota \wedge \iota) = 0$ for all $i \geq 0$.

At this stage, we have only defined the operations for even-dimensional classes. Once we have verified that they are compatible with double suspension, we will extend this to odd-dimensional classes in Definition 7.6. A similar argument as in the proof of Theorem 6.19 for $p = 2$ then shows the following result:

Theorem 7.5. *Let p be an odd prime and let P^i_k be the cohomology operation of degree $i(2p-2)$ defined on classes of dimension $2k$ by the formula (7.2).*

- (i) *We have $P^k_k(x) = x^p$ and $P^i_k(x) = 0$ for $i > k$.*
- (ii) *The operations P^i_k satisfy the external Cartan formula*

$$P^i_{k+l}(x \times y) \ = \ \sum_{a+b=i} P^a_k(x) \ \times \ P^b_l(y)$$

for all classes $x \in H^{2k}(X; \mathbb{F}_p)$ and $y \in H^{2l}(Y; \mathbb{F}_p)$.

- (iii) *For fixed i, the operations P^i_k commute with the double suspension isomorphism in the sense that*

$$P^i_{k+1}(x \wedge \iota \wedge \iota) \ = \ P^i_k(x) \wedge \iota \wedge \iota \ \in \ H^{2k+2i(p-1)+2}(X \wedge S^2; \mathbb{F}_p)$$

for every class $x \in H^{2k}(X; \mathbb{F}_p)$.

Proof. (i) As before we let $j\colon H^*(X\times L(p);\mathbb{F}_p)\longrightarrow H^*(X;\mathbb{F}_p)$ be the map induced by the embedding $(-,[1,0,\dots])\colon X\longrightarrow X\times L(p)$. Then for $z\in H^*(X;\mathbb{F}_p)$ we have

$$j^*(z\times x^\varepsilon y^i) \;=\; \begin{cases} z & \text{if } \varepsilon=0 \text{ and } i=0\text{, and}\\ 0 & \text{else.}\end{cases}$$

The restriction of $\mathcal{P}_p(x)$ is the cup power x^p by Lemma 6.11 (i); so restriction of defining formula (7.2) gives

$$x^p \;=\; j^*(\mathcal{P}_p(x)) \;=\; P_k^k(x)\ .$$

(ii) Lemma 6.11 (ii) gives

$$\begin{aligned}
\mathcal{P}_p(x\times y) &= \Delta^*(\mathcal{P}_p(x)\times\mathcal{P}_p(y))\\
&= \Delta^*\left(\sum_{a,b\geq 0}\left(P_k^a(x)\times v^{k-a} \;+\; R_k^a(x)\times uv^{k-a-1}\right)\times\left(P_l^b(y)\times v^{l-b}\;+\;R_l^b(y)\times uv^{l-b-1}\right)\right)\\
&= \sum_{i\geq 0}\sum_{a+b=i}\Big(\,P_k^a(x)\times P_l^b(y)\times v^{k+l-i} \;+\; (P_k^a(x)\times R_l^b(y)+R_k^a(x)\times P_l^b(y))\times uv^{k+l-i-1}\Big)
\end{aligned}$$

where we used the relations $\Delta^*((\alpha\times u^\varepsilon v^i)\times(\beta\times u^{\varepsilon'}v^j))=\alpha\times\beta\times u^{\varepsilon+\varepsilon'}v^{i+j}$ and $u^2=0$. We obtain the external Cartan formula by comparing coefficients of v^{k+l-i}.

(iii) As in Theorem 6.19, this part is a formal consequence of the Cartan formula. The suspension isomorphism

$$\Sigma \;:\; H^n(X;\mathbb{F}_p)\;\xrightarrow{\cong}\; H^{n+1}(X\wedge S^1;\mathbb{F}_p)$$

is given by exterior smash product with the fundamental class $\iota\in\widetilde{H}^1(S^1;\mathbb{F}_p)$. So we get

$$P^i_{k+1}\,(x\wedge\iota\wedge\iota) \;=\; \sum_{j=0}^{i} P_k^{i-j}(x)\wedge P_1^j(\iota\wedge\iota) \;=\; P_k^i(x)\wedge\iota\wedge\iota\ .$$

The second equation holds since P_1^0 is the identity operation on the fundamental class $\iota\wedge\iota$, and all other operations vanish on this class, see (7.4). □

Definition 7.6. Let p be an odd prime, let X be a space, and let $n\geq 0$. We define the stable mod-p cohomology operation

$$P^i \;:\; H^n(X;\mathbb{F}_p)\;\longrightarrow\; H^{n+2i(p-1)}(X;\mathbb{F}_p)$$

of type $(\mathbb{F}_p, 2i(p-1),\mathbb{F}_p)$ by the formula

$$P^i(x) \;=\; \begin{cases} P^i_{n/2}(x) & \text{for even } n\text{, and}\\ \Sigma^{-1}P^i_{(n+1)/2}(\Sigma(x)) & \text{for odd } n.\end{cases}$$

By part (iii) of Theorem 7.5, this definition indeed satisfies the stability condition.

We summarize the main properties of the reduced power operations in the next theorem.

Theorem 7.7. *Let p be an odd prime and let P^i be the stable mod-p cohomology operations of degree $2i(p-1)$ defined in 7.6.*

(i) *The operation P^0 is the identity operation.*
(ii) *(Unstability condition) We have $P^i(x)=x^p$ if $|x|=2i$ and $P^i(x)=0$ if $|x|<2i$.*
(iii) *(Cartan formulas) Let X and Y be spaces and let $x\in H^*(X;\mathbb{F}_p)$ and $y\in H^*(Y;\mathbb{F}_p)$ be cohomology classes. Then we have*

$$P^i(x\cup y) \;=\; \sum_{a+b=i} P^a(x)\cup P^b(y) \quad\text{and}\quad P^i(x\times y)\;=\;\sum_{a+b=i}P^a(x)\times P^b(y)\ .$$

8. Adem relations

The *Adem relations* are relations that express the composite of two Steenrod operations as a sum of composites of other operations. The Adem relations generate all relations between the Steenrod operations. In (6.10) we have defined the total power operations

$$\mathcal{P}_p \ : \ H^n(X;\mathbb{F}_p) \ \longrightarrow \ H^{np}(X\times L(p);\mathbb{F}_k) \ .$$

The Adem relations ultimately arise from a symmetry of the iterated total power operations $\mathcal{P}_p\circ\mathcal{P}_p$. We use the following result without proof.

Theorem 8.1. *Let p be any prime, and let n be a number which is even in case p is odd. Then image of the iterated total power operation*

$$\mathcal{P}_p\circ\mathcal{P}_p \ : \ H^n(X;\mathbb{F}_p) \ \longrightarrow \ H^{np^2}(X\times L(p)\times L(p);\mathbb{F}_p)$$

is invariant under the involution induced by the automorphism of $X\times L(p)\times L(p)$ which interchanges the two factors of $L(p)$.

To deduce the Adem relations for $p=2$ from the symmetry property in Theorem 8.1, we follow the elegant method of Bullet and Macdonald [BM]. We denote by $P(t)\in\mathcal{A}_2[\![t]\!]$ the formal power series with coefficients in the mod-2 Steenrod algebra $\mathcal{A}_2$ given by

$$P(t) \ = \ \sum_{i=0}^{\infty} \mathrm{Sq}^i\cdot t^i \ .$$

Proposition 8.2. *The formal power series $P(t)$ with coefficients in $\mathcal{A}_2$ satisfies the identity*

$$P(1+t)\cdot P(t^2) \ = \ P(t+t^2)\cdot P(1) \ .$$

Proof. Since the cohomology algebra $H^*(L(2);\mathbb{F}_2)$ is polynomial on the 1-dimensional class u, the Künneth isomorphism

$$\times \ : \ H^*(X;\mathbb{F}_2)\otimes H^*(L(2);\mathbb{F}_2) \ \cong \ H^*(X\times L(2);\mathbb{F}_2)$$

gives an identification

$$H^*(X\times L(2);\mathbb{F}_2) \ = \ H^*(X;\mathbb{F}_2)[v] \ ,$$

where $v=1\times u$. In the same fashion we identify the cohomology algebra $H^*(X\times L(2)\times L(2);\mathbb{F}_2)$ with $H^*(X;\mathbb{F}_2)[s,t]$ where $s=1\times 1\times u$ and $t=1\times u\times 1$ are polynomial generators of $H^*(L(2)\times L(2);\mathbb{F}_2)$.

Under this identification Theorem 6.19 yields

$$\mathcal{P}_2(x) \ = \ \sum_{i=0}^{\infty}\mathrm{Sq}^i(x)\cdot v^{n-i} \ \in \ H^{2n}(X\times L(2);\mathbb{F}_2)$$

for every n-dimensional cohomology class x. The Cartan formula shows that the total squaring operation is multiplicative, i.e., we have $\mathrm{Sq}(xy)=\mathrm{Sq}(x)\,\mathrm{Sq}(y)$ for $x,y\in H^*(X;\mathbb{F}_2)$. Moreover, the value of the total square on $v=1\times u\in H^1(X\times L(2);\mathbb{F}_2)$ is given by

$$\mathcal{P}_2(v) \ = \ \sum_{i\geq 0}\mathrm{Sq}^i(1\times u)\times u^{1-i} = (1\times u\times u)+(1\times u^2\times 1) \ = \ ts+t^2$$

in $H^2(X\times L(2)\times L(2);\mathbb{F}_2)$. For the iterated total squaring operation we thus get the formula

$$\begin{aligned}
\mathcal{P}_2(\mathcal{P}_2(x)) \ &= \ \mathcal{P}_2\left(\sum_{j\geq 0}\mathrm{Sq}^j(x)\cdot v^{n-j}\right) \ = \ \sum_{j\geq 0}\mathcal{P}_2(\mathrm{Sq}^j(x))\cdot(\mathcal{P}_2(v))^{n-j}\\
&= \ \sum_{j\geq 0}(\sum_{i\geq 0}\mathrm{Sq}^i(\mathrm{Sq}^j(x))\cdot s^{n+j-i})\cdot(ts+t^2)^{n-j}\\
&= \ s^n(s+t)^n t^n\sum_{i,j\geq 0}\mathrm{Sq}^i(\mathrm{Sq}^j(x))s^{-i}(t+t^2s^{-1})^{-j} \ .
\end{aligned}$$

The second equation is the fact that $\mathcal{P}_2$ is additive and multiplicative, the latter by Lemma 6.11. By Theorem 8.1, this expression is symmetric in s and t, hence so is the expression

$$\sum_{i,j\geq 0} \mathrm{Sq}^i(\mathrm{Sq}^j(x)) \cdot s^{-i}(t+t^2s^{-1})^{-j} \;=\; P(s^{-1})P((t+t^2s^{-1})^{-1})(x) \ .$$

Since this holds for all spaces X and for all cohomology classes x, we get the equality of (formal Laurent power series of) stable cohomology operations, i.e., we have

$$P(s^{-1})P((t+t^2s^{-1})^{-1}) \;=\; P(t^{-1})P((s+s^2t^{-1})^{-1}) \ . \tag{8.3}$$

If we substitute $s=(1+v)^{-1}$ and $t=(v+v^2)^{-1}$, then

$$t+s^{-1}t^2 \;=\; \frac{1}{v+v^2}+\frac{1+v}{(v+v^2)^2} \;=\; \frac{1+v^2}{(v+v^2)^2} \;=\; \frac{1}{v^2} \qquad \text{and} \qquad s+t^{-1}s^2=1$$

where we exploited characteristic 2. So substituting into (8.3) gives $P(1+v)P(v^2)=P(v+v^2)P(1)$. $\square$

Example 8.4. We expand the relation of Proposition 8.2 modulo t^3 to obtain the Adem relations for $\mathrm{Sq}^1\,\mathrm{Sq}^j$ and for $\mathrm{Sq}^2\,\mathrm{Sq}^j$:

$$\begin{aligned}
P(1+t)\cdot P(t^2) \;&=\; (\sum_{i\geq 0}\mathrm{Sq}^i \cdot (1+t)^i)\cdot(\sum_{j\geq 0}\mathrm{Sq}^j \cdot t^{2j}) \\
&\equiv\; (\sum_{i\geq 0}\mathrm{Sq}^i \cdot (1+it+\binom{i}{2}t^2))\cdot(1+\mathrm{Sq}^1 \cdot t^2) \\
&\equiv\; (\sum_{i\geq 0}\mathrm{Sq}^i) + (\sum_{i\geq 0}\mathrm{Sq}^i \cdot i)\cdot t + (\sum_{i\geq 0}\mathrm{Sq}^i\cdot\binom{i}{2} + \mathrm{Sq}^i\,\mathrm{Sq}^1)\cdot t^2
\end{aligned}$$

$$\begin{aligned}
P(t+t^2)\cdot P(1) \;&=\; (\sum_{i\geq 0}\mathrm{Sq}^i \cdot (t+t^2)^i)\cdot(\sum_{j\geq 0}\mathrm{Sq}^j) \\
&\equiv\; (1+\mathrm{Sq}^1 \cdot (t+t^2)+\mathrm{Sq}^2 \cdot t^2)\cdot(\sum_{j\geq 0}\mathrm{Sq}^j) \\
&=\; (\sum_{j\geq 0}\mathrm{Sq}^j) + (\sum_{j\geq 0}\mathrm{Sq}^1\,\mathrm{Sq}^j \cdot (t+t^2)) + (\sum_{j\geq 0}\mathrm{Sq}^2\,\mathrm{Sq}^j \cdot t^2) \\
&=\; (\sum_{j\geq 0}\mathrm{Sq}^j) + (\sum_{j\geq 0}\mathrm{Sq}^1\,\mathrm{Sq}^j)\cdot t + (\sum_{j\geq 0}\mathrm{Sq}^1\,\mathrm{Sq}^j + \mathrm{Sq}^2\,\mathrm{Sq}^j)\cdot t^2
\end{aligned}$$

Comparing coefficients of t yields

$$\sum_{i\geq 0}\mathrm{Sq}^i \cdot i \;=\; \sum_{j\geq 0}\mathrm{Sq}^1\,\mathrm{Sq}^j \ ;$$

in cohomology degree $j+1$ this yields the relation

$$\mathrm{Sq}^1\,\mathrm{Sq}^j \;=\; \mathrm{Sq}^{j+1}\cdot(j+1) \;=\; \begin{cases} \mathrm{Sq}^{j+1} & \text{for } j \text{ even, and} \\ 0 & \text{for } j \text{ odd.} \end{cases}$$

For example $\mathrm{Sq}^1\,\mathrm{Sq}^1=0$, which we already knew because the composite of two Bockstein operations is zero. And $\mathrm{Sq}^1\,\mathrm{Sq}^2=\mathrm{Sq}^3$, $\mathrm{Sq}^1\,\mathrm{Sq}^3=0$, etc.

Comparing coefficients of t^2 yields

$$\sum_{i\geq 0}\mathrm{Sq}^i\cdot\binom{i}{2} + \mathrm{Sq}^i\,\mathrm{Sq}^1 \;=\; \sum_{j\geq 0}\mathrm{Sq}^1\,\mathrm{Sq}^j + \mathrm{Sq}^2\,\mathrm{Sq}^j \ ;$$

in cohomology degree $j+2$ this yields the relation

$$\binom{j+2}{2}\operatorname{Sq}^{j+2}+\operatorname{Sq}^{j+1}\operatorname{Sq}^1 \;=\; \operatorname{Sq}^1\operatorname{Sq}^{j+1}+\operatorname{Sq}^2\operatorname{Sq}^j \ ,$$

or equivalently

$$\begin{aligned}
\operatorname{Sq}^2\operatorname{Sq}^j \;&=\; \binom{j+2}{2}\operatorname{Sq}^{j+2} \;+\; \operatorname{Sq}^{j+1}\operatorname{Sq}^1 \;+\; \operatorname{Sq}^1\operatorname{Sq}^{j+1}\\
&=\; \begin{cases}
\operatorname{Sq}^{j+2} \;+\; \operatorname{Sq}^{j+1}\operatorname{Sq}^1 & \text{for } j\equiv 0 \text{ modulo } 4\\
\operatorname{Sq}^{j+2} \;+\; \operatorname{Sq}^{j+1}\operatorname{Sq}^1 \;+\; \operatorname{Sq}^{j+2} & \text{for } j\equiv 1 \text{ modulo } 4\\
\qquad\quad\ \operatorname{Sq}^{j+1}\operatorname{Sq}^1 & \text{for } j\equiv 2 \text{ modulo } 4\\
\qquad\quad\ \operatorname{Sq}^{j+1}\operatorname{Sq}^1 \;+\; \operatorname{Sq}^{j+2} & \text{for } j\equiv 3 \text{ modulo } 4
\end{cases}\\
&=\; \begin{cases}
\operatorname{Sq}^{j+2} \;+\; \operatorname{Sq}^{j+1}\operatorname{Sq}^1 & \text{for } j\equiv 0,3 \text{ modulo } 4\\
\qquad\quad\ \operatorname{Sq}^{j+1}\operatorname{Sq}^1 & \text{for } j\equiv 1,2 \text{ modulo } 4
\end{cases}
\end{aligned}$$

The first Adem relations explicitly look as follows:

$$\begin{aligned}
\operatorname{Sq}^1\operatorname{Sq}^1 \;&=\; 0\\
\operatorname{Sq}^1\operatorname{Sq}^2 \;&=\; \operatorname{Sq}^3\\
\operatorname{Sq}^2\operatorname{Sq}^2 \;&=\; \operatorname{Sq}^3\operatorname{Sq}^1\\
\operatorname{Sq}^1\operatorname{Sq}^3 \;&=\; 0\\
\operatorname{Sq}^2\operatorname{Sq}^3 \;&=\; \operatorname{Sq}^5+\operatorname{Sq}^4\operatorname{Sq}^1
\end{aligned}$$

From these one can deduce some other relations, for example

$$\operatorname{Sq}^3\operatorname{Sq}^2 \;=\; \operatorname{Sq}^1\operatorname{Sq}^2\operatorname{Sq}^2 \;=\; \operatorname{Sq}^1\operatorname{Sq}^3\operatorname{Sq}^2 \;=\; 0\ .$$

Now we derive all the Adem relations in their usual form, still following Bullet and Macdonald [BM]. We will use the *residue calculus* which we briefly recall. Let R be any ring, and denote by $R(\!(t)\!)$ the ring of Laurent power series over R, i.e., the collection of formal sums

$$f(t) \;=\; \sum_{i>>-\infty}^{\infty} f_i\cdot t^i$$

where $f_i \in R$ and $f_i = 0$ for almost all negative values of i. The *residue* of a Laurent power series $f(t)\in R(\!(t)\!)$ is defined as the coefficient of t^{-1},

$$\operatorname{Res}(f) \;=\; f_{-1} \;\in\; R\ .$$

The notation $\operatorname{Res} f(t)\,dt$ is also used. Now suppose that $\tau(t)\in\mathbb{Z}[t]$ is a *polynomial* with integer coefficients without constant term and with linear term equal to t, i.e. $\tau(t)\equiv t \mod (t^2)$. Then τ is invertible in the ring $R(\!(t)\!)$, and hence we can substitute τ into any Laurent power series $f(t)$ to get a new Laurent power series $f(\tau(t))$. We are interested in this for $\tau(t)=t+t^2$. In this situation we have the following 'residue formula':

Proposition 8.5. *Let R be a ring, and let $f(t)\in R(\!(t)\!)$ be a Laurent power series over R. Then*

$$\operatorname{Res}(f) \;=\; \operatorname{Res}\left[f(t+t^2)\cdot(1+2t)\right]\ .$$

Proof. Both sides of the equation are R-linear in the Laurent power series f. Moreover, if f is a power series (as opposed to Laurent power series), i.e., if the coefficients f_i of f are trivial for negative values of i, then both sides of the equation are trivial. A general Laurent power series has only finitely many non-zero coefficients of negative powers of the indeterminate, so by R-linearity it is enough to check the desired equation for $R=\mathbb{Z}$ and $f(t)=t^{-j}$ with $j\geq 1$. In other words, we must show that

$$\operatorname{Res}\left[(t+t^2)^{-j}\cdot(1+2t)\right] \;=\; \begin{cases} 1 & \text{for } j=1, \text{ and}\\ 0 & \text{for } j\geq 2.\end{cases}$$

For $j = 1$ we have

$$\begin{aligned}(t+t^2)^{-1} &= t^{-1}(1+t)^{-1} = t^{-1}\cdot(1-t+t^2-t^3+\dots) \\ &= t^{-1}-1+t-t^2+t^3+\dots\ ,\end{aligned}$$

so indeed $\mathrm{Res}\left[(t+t^2)^{-1}\right] = 1$.

The *formal derivative* of a Laurent power series $f(t) = \sum a_i \cdot t^i \in \mathbb{Z}(\!(t)\!)$ is

$$f'(t) \;=\; \frac{df}{dt} \;=\; \sum_i i\cdot a_i\cdot t^{i-1}\ .$$

For all $m \in \mathbb{Z}$ we have

$$\frac{d}{dt}(f^m) \;=\; m\cdot\frac{df}{dt}\cdot f^{m-1}\ .$$

So

$$\frac{d}{dt}((t+t^2)^{1-j}) \;=\; (1-j)\cdot(1+2t)\cdot(t+t^2)^{-j}\ .$$

Because the residue of every formal derivative is trivial, we conclude that

$$0 \;=\; \mathrm{Res}\left[\frac{d}{dt}((t+t^2)^{1-j})\right] \;=\; (1-j)\cdot\mathrm{Res}\left[(1+2t)\cdot(t+t^2)^{-j}\right]\ .$$

Because $\mathbb{Z}$ is torsion free, for $j \neq 1$ we can conclude

$$\mathrm{Res}\left[(1+2t)\cdot(t+t^2)^{-j}\right] \;=\; 0\ . \qquad \square$$

Theorem 8.6. (Adem relations) *The Steenrod squaring operations satisfy the following relations*

$$\mathrm{Sq}^a\,\mathrm{Sq}^b \;=\; \sum_{i=0}^{[a/2]} \binom{b-i-1}{a-2i}\ \mathrm{Sq}^{a+b-i}\,\mathrm{Sq}^i$$

for all $a, b \geq 0$.

Proof. We fix non-negative integers a and b. Then we have

$$\begin{aligned}
\mathrm{Sq}^a\,\mathrm{Sq}^b &= \mathrm{Coeff}_{t^a}\left\{\sum_{j=0}^{a+b}\mathrm{Sq}^j\,\mathrm{Sq}^{a+b-j}\,t^j\right\}\\
&= \mathrm{Res}\left\{\sum_{j=0}^{a+b}\mathrm{Sq}^j\,\mathrm{Sq}^{a+b-j}\,t^{j-a-1}\right\}\\
&= \mathrm{Res}\left\{\sum_{j=0}^{a+b}\mathrm{Sq}^j\,\mathrm{Sq}^{a+b-j}\cdot(t+t^2)^{j-a-1}\right\} \qquad (8.7)\\
&= \mathrm{Res}\left\{\sum_{j=0}^{a+b}\mathrm{Sq}^j\,\mathrm{Sq}^{a+b-j}\cdot(t+t^2)^{j}\cdot(t+t^2)^{-a-1}\right\}\\
&= \mathrm{Res}\left\{\sum_{j=0}^{a+b}\mathrm{Sq}^j\,\mathrm{Sq}^{a+b-j}(1+t)^{j}\cdot t^{2(a+b-j)}\cdot(t+t^2)^{-a-1}\right\} \qquad (8.8)\\
\underset{i=a+b-j}{=}\ & \mathrm{Res}\left\{\sum_{i=0}^{a+b}\mathrm{Sq}^{a+b-i}\,\mathrm{Sq}^{i}(1+t)^{a+b-i}\cdot t^{2i}\cdot(t+t^2)^{-a-1}\right\}\\
&= \mathrm{Res}\left\{\sum_{i=0}^{a+b}\mathrm{Sq}^{a+b-i}\,\mathrm{Sq}^{i}(1+t)^{b-i-1}\cdot t^{2i-a-1}\right\}\\
&= \mathrm{Coeff}_{t^a}\left\{\sum_{i=0}^{a+b}\mathrm{Sq}^{a+b-i}\,\mathrm{Sq}^{i}(1+t)^{b-i-1}t^{2i}\right\}\\
&= \sum_{i=0}^{a+b}\binom{b-i-1}{a-2i}\mathrm{Sq}^{a+b-i}\,\mathrm{Sq}^{i}\ .
\end{aligned}$$

Equation (8.7) is the residue formula (8.5), which simplifies to $\mathrm{Res}(f) = \mathrm{Res}\left[f(t+t^2)\right]$ because $2 = 0$ in the mod-2 Steenrod algebra. Equation (8.8) uses the power series identity $P(t + t^2)P(1) = P(1 + t)P(t^2)$ of Theorem 8.2, namely the part which has dimension $a + b$ with respect to the grading coming from the Steenrod algebra $\mathcal{A}_2$. The Adem relation as stated in the theorem now follow since the binomial coefficient $\binom{b-j-1}{a-2j}$ vanishes for $j > a/2$. □

Remark 8.9. With respect to binomial coefficients $\binom{n}{m}$ for integers n and m, possibly negative, we recall that

$$\binom{n}{m} = \begin{cases} \frac{n\cdot(n-1)\cdots(n-m+1)}{m\cdot(m-1)\cdots 1} & \text{if } m \geq 1,\\ 1 & \text{if } m = 0,\\ 0 & \text{if } m < 0.\end{cases}$$

With these conventions, the formula

$$(1+t)^n = \sum_{i=0}^{\infty}\binom{n}{i}\cdot t^i$$

holds for all integers n, positive or negative, in the power series ring $\mathbb{Z}[[t]]$.

The Adem relations 8.6 are often stated under the condition that $a < 2b$. In that case $i \leq a/2$ implies that $b - i - 1$ is non-negative. However, the relations hold for all non-negative a and b, just that some of the binomial coefficients which arise when $a \geq 2b$ may have negative numerators.

The Adem relations allow us to write some of the Steenrod squares as sum of products of Steenrod square of smaller degrees:

$$\begin{aligned}
\mathrm{Sq}^3 &= \mathrm{Sq}^1\,\mathrm{Sq}^2 \\
\mathrm{Sq}^5 &= \mathrm{Sq}^1\,\mathrm{Sq}^4 \;=\; \mathrm{Sq}^2\,\mathrm{Sq}^3 + \mathrm{Sq}^4\,\mathrm{Sq}^1 \\
\mathrm{Sq}^6 &= \mathrm{Sq}^2\,\mathrm{Sq}^4 + \mathrm{Sq}^5\,\mathrm{Sq}^1 \\
\mathrm{Sq}^7 &= \mathrm{Sq}^1\,\mathrm{Sq}^6 \ .
\end{aligned}$$

These particular Steenrod operations are thus *decomposable* in the Steenrod algebra $\mathcal{A}_2$. The next corollary shows how this fits into a general pattern. The operations Sq^{2^i} for $i \geq 0$ are indecomposable, i.e., they cannot be written as sums of products of cohomology operations of smaller degrees.

Corollary 8.10. *Let n be a positive integer which is not a power of 2. Then the Steenrod operation Sq^n is decomposable in the Steenrod algebra $\mathcal{A}_2$, i.e., in the square of the ideal generated by the positive dimensional elements of $\mathcal{A}_2$.*

Proof. By hypothesis we can write $n = 2^i(2k+1)$ with $i \geq 0$ and $k \geq 1$. We have the Adem relation

$$\mathrm{Sq}^{2^i}\,\mathrm{Sq}^{n-2^i} \;=\; \sum_{j=0}^{2^{i-1}} \binom{n-2^i-j-1}{2^i-2j}\,\mathrm{Sq}^{n-j}\,\mathrm{Sq}^j \ .$$

The summand indexed by $j=0$ contributes the term $\binom{n-2^i-1}{2^i}\mathrm{Sq}^n$. We claim that the binomial coefficient $\binom{n-2^i-1}{2^i} = \binom{2^{i+1}k-1}{2^i}$ is odd. This implies that

$$\mathrm{Sq}^n \;=\; \mathrm{Sq}^{2^i}\,\mathrm{Sq}^{n-2^i} \;+\; \sum_{j=1}^{2^{i-1}} \binom{n-2^i-j-1}{2^i-2j}\,\mathrm{Sq}^{n-j}\,\mathrm{Sq}^j \ ,$$

so we conclude that Sq^n is decomposable in the mod-2 Steenrod algebra if n is not a power of 2. To evaluate the binomial coefficient we use that $\binom{2^{i+1}k-1}{2^i}$ is the coefficient of t^{2^i} in the polynomial $(1+t)^{2^{i+1}k-1}$. In characteristic 2, that polynomial evaluates to

$$(1+t)^{2^{i+1}k-1} \;=\; ((1+t)^{2^{i+1}})^k \cdot (1+t)^{-1} \;=\; (1+t^{2^{i+1}})^k \cdot (1+t+t^2+\cdots) \ .$$

Since $(1+t^{2^{i+1}})^k$ is congruent to 1 modulo t^{2^i+1}, the coefficient of t^{2^i} in $(1+t)^{2^{i+1}k-1}$ is indeed congruent to 1 mod 2. $\square$

Construction 8.11. The Adem relations can be used to show that certain composite of Hopf maps are stably essential. We need the following observation: suppose that

$$\alpha\colon S^m \longrightarrow S^k \quad \text{and} \quad \beta\colon S^n \longrightarrow S^m$$

are continous pointed maps between spheres. Suppose that the composite $\alpha\beta\colon S^n \longrightarrow S^k$ is null-homotopic, and let

$$H\colon S^n \times [0,1] \;\longrightarrow\; S^k$$

be a pointed homotopy from $\alpha\beta$ to the constant map. Since H ends in the constant map, it factors over a map

$$\bar{H} \;:\; CS^n = (S^n \times [0,1])/(S^n \times \{1\}) \;\longrightarrow\; S^k$$

from the cone of S^n. The maps $\bar{H}$ and α glue together to give a map

$$\alpha \cup \bar{H} \;:\; C(\beta) = S^m \cup_\beta CS^n \longrightarrow S^k$$

from the mapping cone of β. We let $C(\alpha,\beta,H)$ be the mapping cone of $\alpha \cup \bar{H}\colon C(\beta) \longrightarrow S^k$. This space has a CW-structure with 4 cells in dimensions 0, k, $m+1$ and $n+2$. Moreover, it contains the mapping cone of α as its $(m+1)$-skeleton, and the quotient of $C(\alpha,\beta,H)$ by its k-skeleton (which is the sphere S^k) is homeomorphic to a certain suspension of the mapping cone of β.

Example 8.12. Now we show how Construction 8.11 and the Adem relation $\mathrm{Sq}^2\,\mathrm{Sq}^2 = \mathrm{Sq}^3\,\mathrm{Sq}^1$ can be used to show that the composite η^2 is stably essential. Suppose that for some n the composite

$$S^{n+2} \xrightarrow{\eta} S^{n+1} \xrightarrow{\eta} S^n$$

is null-homotopic. After choosing a null-homotopy H we can form the space $C(\eta,\eta,H)$ with cells in dimension 0, n, $n+2$ and $n+4$. The reduced mod-2 cohomology of this space is one-dimensional in dimensions n, $n+2$ and $n+4$, and trivial in all other dimensions. Since the $(n+2)$-cell is attached to the n-cell by η, the Steenrod operation Sq^2 is an isomorphism from $H^n(C(\eta,\eta,H);\mathbb{F}_2)$ to $H^{n+2}(C(\eta,\eta,H);\mathbb{F}_2)$, and similarly from there to $H^{n+4}(C(\eta,\eta,H);\mathbb{F}_2)$. But since the group $H^{n+1}(C(\eta,\eta,H);\mathbb{F}_2)$ vanishes, we get that

$$\mathrm{Sq}^2\,\mathrm{Sq}^2 = \mathrm{Sq}^3\,\mathrm{Sq}^1 \ : \ H^n(C(\eta,\eta,H);\mathbb{F}_2) \ \longrightarrow \ H^{n+4}(C(\eta,\eta,H);\mathbb{F}_2)$$

is trivial, a contradiction. Hence no suspension of η^2 is ever null-homotopic.

The same kind of reasoning yields other non-triviality results for certain composites of Hopf maps, using that 2ι, η, ν and σ are detected in mod-2 cohomology by the Steenrod operations Sq^1, Sq^2, Sq^4 and Sq^8, respectively. In the following table we list some Adem relations and the composite which are non-trivial by the above argument.

relation	stably essential product
$\mathrm{Sq}^1\,\mathrm{Sq}^4 = \mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^2\,\mathrm{Sq}^3$	2ν
$\mathrm{Sq}^1\,\mathrm{Sq}^8 = \mathrm{Sq}^8\,\mathrm{Sq}^1 + \mathrm{Sq}^2\,\mathrm{Sq}^7$	2σ
$\mathrm{Sq}^2\,\mathrm{Sq}^2 = \mathrm{Sq}^3\,\mathrm{Sq}^1$	$\eta\eta$
$\mathrm{Sq}^2\,\mathrm{Sq}^8 = \mathrm{Sq}^9\,\mathrm{Sq}^1 + \mathrm{Sq}^8\,\mathrm{Sq}^2 + \mathrm{Sq}^4\,\mathrm{Sq}^6$	$\eta\sigma$
$\mathrm{Sq}^4\,\mathrm{Sq}^4 = \mathrm{Sq}^7\,\mathrm{Sq}^1 + \mathrm{Sq}^6\,\mathrm{Sq}^2$	$\nu\nu$
$\mathrm{Sq}^8\,\mathrm{Sq}^8 = \mathrm{Sq}^{15}\,\mathrm{Sq}^1 + \mathrm{Sq}^{14}\,\mathrm{Sq}^2 + \mathrm{Sq}^{12}\,\mathrm{Sq}^4$	$\sigma\sigma$

Some products of Hopf maps do not occur in the table: 2η, $\eta\nu$ and $\nu\sigma$. These products are in fact stably null-homotopic.

We conclude by stating, without proof, the Adem relations for the odd-primary Steenrod algebra.

Theorem 8.13. (Odd-primary Adem relations) *Let p be an odd prime. Then the power operations P^i satisfy the relations*

$$P^a P^b \ = \ \sum_{j=0}^{[a/p]} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^j$$

for all $0<a<pb$. Moreover, the power operations and the mod-p Bockstein β satisfy the relations

$$\begin{aligned} P^a \beta P^b \ = \ & \sum_{j=0}^{[a/p]} (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta P^{a+b-j} P^j \\ + \ & \sum_{j=0}^{[(a-1)/p]} (-1)^{a+j-1} \binom{(p-1)(b-j)-1}{a-pj-1} P^{a+b-j} \beta P^j \end{aligned}$$

for all $0<a\leq pb$.

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