Summer term 2025

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Exercises for **Topology II**
Sheet 2

Exercise 1 (*14 points*)**.** Let (X, x_0) and (Y, y_0) be based CW-complexes (with x_0 and y_0 being 0-cells) of which at least one is locally compact, and let R be a ring. We recall that the *smash product* $X \wedge Y$ is defined as the quotient

$$X \wedge Y = X \times Y/(X \times y_0) \cup (x_0 \times Y).$$

We view $X \wedge Y$ as a based space with basepoint $x_0 \wedge y_0$ the collapsed subspace $(X \times y_0) \cup (x_0 \times Y)$.

Construct an exterior product on cohomology of the form

$$H^n(X, x_0; R) \times H^m(Y, y_0; R) \to H^{n+m}(X \wedge Y, x_0 \wedge y_0; R).$$

Hint. Define a relative version of the exterior product and apply excision.

Remark. The assumption that at least one of X and Y is locally compact is not strictly necessary, but it will make ~~your life~~ solving this exercise slightly easier.

Exercise 2 (*12 points*)**.** Let X and Y be topological spaces. By the universal property of the tensor product, the exterior product maps defined on the last sheet induce additive maps

$$\times\colon H^n(X;\mathbb{Z}) \otimes H^m(Y;\mathbb{Z}) \to H^{n+m}(X \times Y;\mathbb{Z}).$$

Varying m and n, these then assemble into a map of graded abelian groups

$$\times\colon H^*(X;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z}) \to H^*(X \times Y;\mathbb{Z}). \tag{$*$}$$

1. All of $H^*(X;\mathbb{Z})$, $H^*(Y;\mathbb{Z})$, and $H^*(X \times Y;\mathbb{Z})$ are (graded) rings via the cup product. Show that there is a unique ring structure on $H^*(X;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z})$ such the multiplication $\cup$ satisfies

$$(x_1 \otimes y_1) \cup (x_2 \otimes y_2) = (-1)^{n_1 m_2}(x_1 \cup x_2) \otimes (y_1 \cup y_2)$$

   for all $x_i \in H^{m_i}(X;\mathbb{Z})$ and $y_i \in H^{n_i}(Y;\mathbb{Z})$.

2. Show that with respect to this ring structure, the map $(*)$ is a ring homomorphism.

please turn over

Exercise 3 (*14 points*)**.** Let R be a commutative ring and C be a degreewise free chain complex of R-modules whose homology H_*C is also degreewise free and which is concentrated in non-negative degrees, i.e., $C_n = 0$ for $n < 0$. We let HC denote the chain complex given in degree n by

$$(HC)_n = H_n C,$$

with all differentials $d_n\colon (HC)_n \to (HC)_{n-1}$ the zero maps.

1. Show that there exists a chain homotopy-equivalence $f\colon HC \xrightarrow{\sim} C$.

   Hint. First show that there exists a quasi-isomorphism and then use the fact that every quasi-isomorphism between degreewise free chain complexes which are concentrated in non-negative degrees is a chain homotopy-equivalence (as proven in a bonus exercise last term).

2. Let D be another chain complex of R-modules. Show that there exists a chain homotopy-equivalence

$$HC \otimes D \xrightarrow{\sim} C \otimes D.$$

3. Use this to show that the natural map

$$\begin{aligned} \bigoplus_{i=0}^{n} H_i(C) \otimes H_{n-i}(D) &\longrightarrow H_n(C \otimes D) \\ [x] \otimes [y] &\longmapsto [x \otimes y] \end{aligned}$$

   is an isomorphism for all $n \geq 0$.

$*$ **Exercise 4** (*10 bonus points*)**.** We consider $\mathbb{R}\mathrm{P}^2$ with its standard CW-structure $* \subset \mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2$, and we equip $\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2$ with the product CW-structure.

Describe a cellular approximation of the diagonal $\Delta\colon \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2$.

Remark. Let X be a CW-complex with finitely many cells in each dimension. We will see later how one can determine the cup product on the cohomology of X with finitely many cells in each dimension in terms of a cellular approximation of the diagonal $\Delta\colon X \to X \times X$. This will then in particular allow to show that the cup square of the generator of $H^1(\mathbb{R}\mathrm{P}^2, \mathbb{F}_2)$ is non-trivial (hence a generator of H^2).