Summer term 2025 Prof. Dr. Stefan Schwede
Dr. Tobias Lenz

Exercises for **Topology II**
Sheet 1

Exercise 1 (*10 points*)**.** Let $p\colon \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2/\mathbb{R}\mathrm{P}^1$ be the collapse map, and recall that the target is homeomorphic to the 2-sphere S^2 (for example, because $\mathbb{R}\mathrm{P}^2$ admits a 2-dimensional CW-structure with 1-skeleton $\mathbb{R}\mathrm{P}^1$ and precisely one 2-cell).

1. Show that the induced map on homology $H_n(p;\mathbb{Z})\colon H_n(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \to H_n(\mathbb{R}\mathrm{P}^2/\mathbb{R}\mathrm{P}^1;\mathbb{Z})$ is trivial for all $n > 0$.
2. Show that the induced map on second cohomology $H^2(p;\mathbb{Z})\colon H^2(\mathbb{R}\mathrm{P}^2/\mathbb{R}\mathrm{P}^1;\mathbb{Z}) \to H^2(\mathbb{R}\mathrm{P}^2;\mathbb{Z})$ is nontrivial.
3. Explain why this implies that there cannot be a section of the surjection

   $$\Phi\colon H^2(X;\mathbb{Z}) \to \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z})$$

   which is natural in all topological spaces X.

   Remark. Similar examples show that there does not exist a natural section of

   $$\Phi\colon H^n(X;\mathbb{Z}) \to \mathrm{Hom}(H_n(X;\mathbb{Z}),\mathbb{Z})$$

   for any $n \geq 2$; on the other hand, Φ is even an isomorphism for $n = 0, 1$.

Exercise 2 (*15 points*)**.** Let X and Y be simplicial sets and R a ring. We define an *exterior product*

$$-\times -\colon H^n(X;R) \times H^m(Y;R) \to H^{n+m}(X \times Y;R)$$

via

$$x \times y \coloneqq p_X^*(x) \cup p_Y^*(y),$$

where $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$ are the projections. Show:

1. The exterior product is associative, i.e., the relation

   $$(x \times y) \times z = x \times (y \times z)$$

   in $H^{n+m+k}(X \times Y \times Z;R)$ holds for all $x \in H^n(X;R)$, $y \in H^m(Y;R)$ and $z \in H^k(Z;R)$.
2. If R is commutative, then the exterior product is commutative in the following sense: for $x \in H^n(X;R)$ and $y \in H^m(Y;R)$ we have

   $$y \times x = (-1)^{nm} \cdot \tau^*(x \times y)$$

   in $H^{n+m}(Y \times X;R)$, where $\tau\colon X \times Y \to Y \times X$ is the homeomorphism swapping the two factors.
3. If x and y are classes in the cohomology of the same space X (not necessarily in the same degree), we have

   $$x \cup y = \Delta^*(x \times y)$$

   where $\Delta\colon X \to X \times X$ is the diagonal map. In other words: The cup-product and the exterior product determine each other.

please turn over

Exercise 3 (*15 points*)**.** Let X be a space, $A, B \subset X$ open subsets and R a ring. Recall from the lecture that the Alexander–Whitney map induces a bilinear map

$$H^n(X, A; R) \times H^m(X, B; R) \;\to\; H^{n+m}(X, A \cup B; R)$$

on relative cohomology, called the *relative cup-product.*

1. Show that the relative cup-product is associative.
2. We assume that $X = A \cup B$ and both A and B are contractible. Show that the cup-product of two classes of degree > 0 in $H^*(X; R)$ is always trivial.
3. Recall that the (unreduced) suspension ΣX of a space X is defined as the quotient space of $X \times [0, 1]$ by the equivalence relation generated by the identifications $(x_1, 0) \sim (x_2, 0)$ and $(x_1, 1) \sim (x_2, 1)$ for all $x_1, x_2 \in X$. Show that the cup-product of any two cohomology classes in $H^*(\Sigma X; \mathbb{Z})$ of positive dimension is trivial.

$*$ **Exercise 4** (*10 bonus points*)**.** Fix a generator $\tau \in H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}$. For every topological space X, we define a map

$$\beta\colon [X, S^1] \to H^1(X; \mathbb{Z})$$

from the set of unbased homotopy classes of maps $X \to S^1$ via $[f] \mapsto f^*\tau$.

1. Show that β is well-defined and a natural transformation of contravariant functors **Top** $\to$ **Set** (where we make $[X, S^1]$ contravariantly functorial in X via precomposition).
2. Equip X with an arbitrary basepoint. Show that the forgetful map $[X, S^1]_* \to [X, S^1]$ from based to unbased homotopy classes is bijective.

   Hint. Use the group structure on S^1 coming from the multiplication on $\mathbb{C}$.
3. Assume now that X admits a CW-structure. Show that β is bijective.

   Hint. Reduce to the case where X has a single 0-cell and no cells above dimension 2.

Remark. The set $[X, S^1]$ carries an abelian group structure via the multiplication on S^1, and one can show that β is actually an isomorphism of groups. More generally, one can find for any $n \geq 0$ and any abelian group A a topological abelian group $K(A, n)$ together with natural group isomorphisms $[X, K(A, n)] \cong H^n(X; A)$ for all CW-complexes X.