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REPRESENTATION THEORY OF ALGEBRAS

FOUNDATIONS

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Introduction

Prerequisites. You need a good knowledge of Linear Algebra and some basics from Algebra (rings, ideals, polynomial rings, irreducible polynomials). We will not use any Galois Theory. In fact, there will not even be field extensions. Homological Algebra is also not necessary.

Since I do not want to waste your time: If you know already Homological Algebra, then this lecture course (and its continuation) will be too easy for you and you should maybe try to find a more suitable course.

Topics. This course is on the representation theory of (mostly finite-dimensional) algebras and quivers. By an *algebra* we mean an associative K -algebra with 1, where K is a field.

Content:

- Algebras, modules and homomorphisms between modules (definitions, examples, basic properties).
- Categories and functors.
- Representations of quivers and path algebras.
- Modules of finite length (Theorems of Jordan-Hölder and Krull-Remak-Schmidt, Harada-Sai Lemma).
- Semisimple algebras, semisimple modules, socle and radical of a module, Jacobson radical of an algebra.
- Projective and injective modules.
- Basic algebras, path algebras modulo admissible ideals and Morita equivalence.
- Homological Algebra.
- Homological conjectures for finite-dimensional algebras.
- Auslander-Tachikawa correspondence.
- Auslander-Reiten theory.
- Tilting theory and cluster tilting theory.
- Tame and wild algebras.
- Hereditary algebras.
- Geometric Representation Theory.

Literature. We are not following any particular book. For everyone who wants to read about the topic anyway, there is a list of books at the end of these notes.

However, most of these sources are assuming that you are familiar with Homological Algebra.

Typo hunting and suggestions.

Typos and suggestions for improvement are very welcome. Please send these to me via Email: schroer@math.uni-bonn.de

Many thanks for typo hunting and suggestions for improvement go to:

Acknowledgements. These notes are partially based on a 3 Semester lecture course on Representation Theory held by Claus Michael Ringel at the University of Bielefeld from Autumn 1993. Several parts are directly copied from his old notes whereas others are entirely new. We were planning to convert these notes into a book. That never happened. Instead they became a constantly evolving project. There are major changes each time I'm teaching a 3 or 4 Semester course on Representation Theory in Bonn. I take full responsibility for all inaccuracies, mistakes or other deficiencies. Bonn, 2026, Jan Schröer.

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Zauberberg. The reader might have guessed already that a single semester is not enough to cover all the topics mentioned above. So this course will be continued in SS 27 and SS 27/28 and maybe even beyond that.

The following quote is from the introduction of Thomas Mann's "[Der Zauberberg](#)":

“Wir werden sie ausführlich erzählen, genau und gründlich, – denn wann wäre je die Kurz- oder Langweiligkeit einer Geschichte abhängig gewesen von dem Raum und der Zeit, die sie in Anspruch nahm? Ohne Furcht vor dem Odium der Peinlichkeit, neigen wir vielmehr der Ansicht zu, daß nur das Gründliche wahrhaft unterhaltend sei. Im Handumdrehen also wird der Erzähler mit Hansens Geschichte nicht fertig werden. Die sieben Tage einer Woche werden dazu nicht reichen und auch sieben Monate nicht. Am besten ist es, er macht sich im voraus nicht klar, wieviel Erdenzeit ihm verstreichen wird, während sie ihn umspinnen hält. Es werden, in Gottes Namen, ja nicht geradezu sieben Jahre sein! Und somit fangen wir an.”

Notation and conventions. Throughout let K be a (commutative) field. Set $K^\times = K \setminus \{0\}$. Sometimes we will make additional assumptions on K . (For example, we might assume that K is algebraically closed.)

Let $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ be the natural numbers (including 0). Let $\mathbb{N}_1 := \mathbb{N} \setminus \{0\}$.

For integers $i \leq j$ we define $[i, j] := \{k \in \mathbb{Z} \mid i \leq k \leq j\}$.

All vector spaces will be K -vector spaces, and all linear maps are assumed to be K -linear.

For a subspace U of a vector space V the corresponding factor space is denoted by V/U .

For a vector space V and $n \geq 1$ let V^n be the n -fold direct sum $V \oplus \dots \oplus V$.

For vector spaces V and W let $\text{Hom}_K(V, W)$ be the vector space of linear maps $V \rightarrow W$.

Depending on the situation, by a **basis** of a vector space, we sometimes mean an ordered basis and sometimes an unordered basis. The **standard basis** of K^n is usually denoted by $(e_1, \dots, e_n)$ (ordered) or $\{e_1, \dots, e_n\}$ (unordered).

For a subset X of a vector space V let $\text{Lin}(X)$ be the smallest subspace of V containing X . For $X = \{x_1, \dots, x_n\}$ let $\text{Lin}(x_1, \dots, x_n) := \text{Lin}(X)$.

For a ring R and $m, n \geq 0$ let $M_{m,n}(R)$ be the $(m \times n)$ -matrices with entries in R . Set $M_n(R) := M_{n,n}(R)$.

For elements r and s in a ring R , $[r, s] := rs - sr$ is the *commutator* of r and s .

The *transpose* of a matrix A is denoted by ${}^tA$.

If I is a set, we denote its cardinality by $|I|$. If I' is subset of I we write $I' \subseteq I$. If additionally $I' \neq I$ we also write $I' \subset I$.

For sets M and N let $\text{Abb}(M, N)$ be the set of maps $M \rightarrow N$. By 1_M (or id_M) we denote the map $M \rightarrow M$ defined by $m \mapsto m$ for all $m \in M$.

Given maps $f: L \rightarrow M$ and $g: M \rightarrow N$, we denote the composition by $gf: L \rightarrow N$. Sometimes we also write $g \circ f$ instead of gf .

We use

blue boxes to highlight statements,

green boxes to highlight definitions,

gray boxes for mathematical, personal or historical remarks,

magenta boxes to highlight conjectures and open problems.

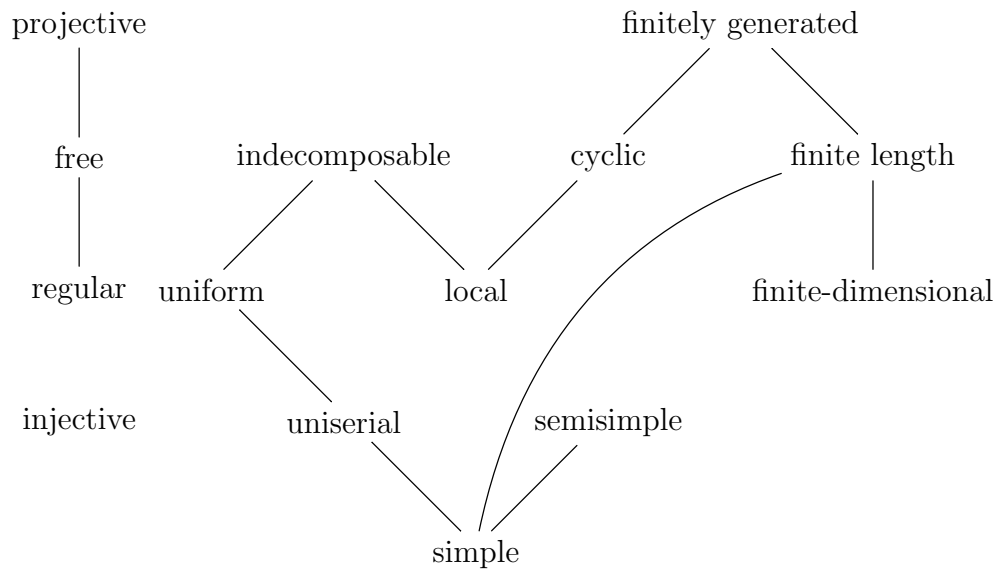
The red sections or paragraphs were not discussed in the lectures. But you should read them anyway.

Classes of algebras. A list of the most important classes of algebras mentioned in this course:

- matrix algebras $M_n(K)$
- polynomial rings $K[T]$
- free algebras $F(J)$
- path algebras KQ
- semisimple algebras
- group algebras KG
- hereditary algebras
- local rings
- selfinjective algebras

Classes of modules. A list of the most important classes of modules mentioned in this course:

- simple and semisimple modules
- indecomposable modules
- uniform modules
- uniserial modules
- cyclic modules
- finitely generated modules
- modules of finite length
- finite-dimensional modules
- local modules
- regular modules ${}_A A$
- free modules
- projective modules
- injective modules
- projective-injective modules



Part 1. Algebras and modules

Summary.

- **Algebras** (Section 1): Definition of an algebra and first examples. Ideals of algebras. Path algebras of quivers.

1. Algebras

1.1. Algebras. Just to be on the safe side, let us spell out what we mean by a *ring* and an *algebra*.

A **ring** is a set R together with two maps $+: R \times R \rightarrow R$, $(a, b) \mapsto a + b$ (the **addition**) and $\cdot: R \times R \rightarrow R$, $(a, b) \mapsto ab$ (the **multiplication**) such that the following hold:

- $(a + b) + c = a + (b + c)$ for all $a, b, c \in R$;
- $a + b = b + a$ for all $a, b \in R$;
- There exists an element $0 \in R$ with $a + 0 = a$ for all $a \in R$;
- For each $a \in R$ there exists an element $-a \in R$ such that $a + (-a) = 0$;
- $(ab)c = a(bc)$ for all $a, b, c \in R$;
- There exists an element $1 \in R$ with $1a = a1 = a$ for all $a \in R$;
- $(a + b)c = ac + bc$ and $a(b + c) = ab + ac$ for all $a, b, c \in R$.

A ring R is **commutative** if $ab = ba$ for all $a, b \in R$.

A **K -algebra** is a K -vector space A together with a map $\cdot: A \times A \rightarrow A$, $(a, b) \mapsto ab$ (the **multiplication**) such that the following hold:

- $(ab)c = a(bc)$ for all $a, b, c \in A$;
- There exists an element $1 \in A$ with $1a = a1 = a$ for all $a \in A$;
- $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in A$;
- Compatibility of multiplication and scalar multiplication: $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $\lambda \in K$ and $a, b \in A$.

The **identity element** 1 is uniquely determined and we often denote it by 1_A .

In other words, a K -algebra is a ring A , which is also a K -vector space such that additionally $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $\lambda \in K$ and $a, b \in A$.

In contrast to the definition of a field, the definitions of a ring and an algebra do not require that the element 0 is different from the element 1 . Thus there is a ring and an algebra which contain just one element, namely $0 = 1$. (This is the only ring or algebra satisfying this equality.)

Examples:

- (i) The set $M_n(K)$ of $(n \times n)$ -matrices is a K -algebra (with respect to the usual addition and multiplication). We will later see that $M_n(K)$ is important, but boring from a representation theoretic point of view. More interesting examples are path algebras of quivers. They will play a prominent role in this course.

- (ii) Let

$$A = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in M_2(K) \mid a, b, c \in K \right\}.$$

This is also a K -algebra (which is less boring than $M_2(K)$).

- (iii) The polynomial ring $K[T]$ in one variable T is a K -algebra.

If A is a K -algebra, then we denote the **opposite algebra** of A by A^{op} . Here we work with the same underlying vector space, but the multiplication map is changed: To avoid confusion, we denote the multiplication of A^{op} by $\star$ which is defined as

$$a_1 \star a_2 = a_2 \cdot a_1 = a_2 a_1$$

for all $a_1, a_2 \in A$ (where $\cdot$ is the multiplication of A). This defines again a K -algebra. Of course we have $(A^{\text{op}})^{\text{op}} = A$.

The **dimension** of a K -algebra A is by definition just the dimension of A as a K -vector space.

Let A and B be K -algebras. A **K -algebra homomorphism** is a K -linear map

$$\eta: A \rightarrow B$$

such that the following hold:

- (i) For all $a_1, a_2 \in A$ we have $\eta(a_1 a_2) = \eta(a_1) \eta(a_2)$;
- (ii) $\eta(1_A) = 1_B$.

In this case, let

$$\text{Ker}(\eta) := \{a \in A \mid \eta(a) = 0\}$$

be the **kernel** of η .

Compositions of K -algebra homomorphisms are again K -algebra homomorphisms.

Let X be a subset of A . Then X is an **algebra generating set** of A if each element in A is finite K -linear combination of finite products $a_1 \cdots a_t$ ($t \geq 0$, repetitions allowed) of elements $a_j \in X$. (By definition such a product is 1 for $t = 0$.) The elements in X are called **algebra generators**. When there is no danger of misunderstanding, we drop the word “algebra” and just use “generating set” and “generators” instead.

A K -algebra A is **finitely generated** if it contains a finite algebra generating set.

Finite-dimensional K -algebras are of course finitely generated. The converse does not hold, e.g. $K[T]$ is infinite dimensional and as a K -algebra generated by T .

1.2. Ideals. Let A be a K -algebra.

A **left ideal** (resp. **right ideal**) in A is a subspace I of A such that $av \in I$ (resp. $va \in I$) for all $a \in A$ and $v \in I$.

A subspace I of A is an **ideal** in A if it is a left and a right ideal. Such an ideal is also called a **two-sided ideal**.

Example: Let $f: A \rightarrow B$ be an algebra homomorphism. Then $\text{Ker}(f)$ is an ideal in A .

For an ideal I let $A/I := \{a + I \mid a \in A\}$ be the **factor algebra** of A modulo I . The addition and scalar multiplication come from the K -vector space A/I . The multiplication is defined by $(a + I) \cdot (b + I) := ab + I$.

If I and J are ideals in A , then the **product** IJ is the subspace of A consisting of all finite sums of elements of the form xy with $x \in I$ and $y \in J$.

One easily checks that IJ is again an ideal.

For $n \geq 0$, and an ideal I in A define $I^0 := A$, $I^1 := I$ and $I^{n+2} := I(I^{n+1})$. We call I **nilpotent** if $I^n = 0$ for some $n \geq 1$.

1.3. Free algebras.

Let J be a set. The **free algebra** $F(J)$ has an algebra generating set $\{x_j \mid j \in J\}$ such that the following hold:

- (i) The monomials $x_{i_1}x_{i_2} \cdots x_{i_t}$ where $i_1, \dots, i_t \in J$ and $t \geq 0$ form a K -basis of $F(J)$. (By convention such a product is the identity 1 of $F(J)$ for $t = 0$.)
- (ii) The product of two basis vectors with $s, t \geq 1$ is defined by

$$(x_{i_1} \cdots x_{i_s}) \cdot (x_{j_1} \cdots x_{j_t}) := x_{i_1} \cdots x_{i_s} x_{j_1} \cdots x_{j_t}.$$

The elements $\{x_j \mid j \in J\}$ are called **free generators** of $F(J)$.

One can see $F(J)$ as a polynomial ring where the x_j are non-commuting variables.

We refrain from defining $F(J)$ formally. One can characterize $F(J)$ by a universal property.

For $J = \{1, \dots, n\}$ let

$$F(n) := K\langle x_1, \dots, x_n \rangle := F(J).$$

Note that $F(1) = K[T]$.

The following proposition is straightforward:

Proposition 1.1. *Let J be a set, and let A be a K -algebra. Then there is a bijection*

$$\eta: \text{Alg}_K(F(J), A) \rightarrow \text{Abb}(J, A)$$

defined by $f \mapsto [j \mapsto f(x_j)]$. Furthermore, $\eta(f)(J)$ is an algebra generating set of A if and only if f is surjective.

Each algebra is isomorphic to a factor algebra of a free algebra:

Corollary 1.2. *Let A be a K -algebra, then there exists a surjective K -algebra homomorphism*

$$f: F(J) \rightarrow A$$

for some set J . With $I := \text{Ker}(f)$ we get an algebra isomorphism $F(J)/I \cong A$.

1.4. Products of algebras. Let A and B be K -algebras. The **product** $A \times B$ of A and B is again a K -algebra with componentwise addition and multiplication, i.e.

$$(a, b) + (a', b') := (a + a', b + b') \quad \text{and} \quad (a, b) \cdot (a', b') := (aa', bb').$$

The scalar multiplication is defined by $\lambda(a, b) := (\lambda a, \lambda b)$. To understand the representation theory of $A \times B$ it is enough to understand the representation theory of A and B . This will be explained in more detail later.

More generally, let $(A_i)_{i \in I}$ be a family of K -algebras. By

$$\prod_{i \in I} A_i$$

we denote the **product** of the algebras A_i . Its elements are the tuples $(a_i)_{i \in I}$ with $a_i \in A_i$ for all i , and the addition and multiplication are defined componentwise, i.e.

$$(a_i)_i + (a'_i)_i := (a_i + a'_i)_i \quad \text{and} \quad (a_i)_i \cdot (a'_i)_i := (a_i a'_i)_i.$$

The scalar multiplication is defined by $\lambda(a_i)_i := (\lambda a_i)_i$.

1.5. Path algebras of quivers. Path algebras of quivers are a central topic in the representation theory of finite-dimensional algebras. They appear everywhere. The necessary definitions and some basic properties can be formulated and proved in terms of Linear Algebra. For a deeper understanding one needs fancier tools like Homological Algebra, Auslander-Reiten Theory, Lie Theory and Algebraic Geometry.

A **quiver** is a quadruple

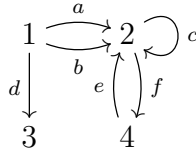
$$Q = (Q_0, Q_1, s, t)$$

where Q_0 and Q_1 are finite sets, and $s, t: Q_1 \rightarrow Q_0$ are maps. The elements in Q_0 are the **vertices** of Q , and the elements in Q_1 are the **arrows** of Q . For an arrow $a \in Q_1$ we call $s(a)$ the **starting vertex** and $t(a)$ the **terminal vertex** of a .

We usually draw an arrow $a \in Q_1$ as

$$s(a) \xrightarrow{a} t(a)$$

Thus we can think of Q as a finite directed graph. But note that multiple arrows and loops (a **loop** is an arrow a with $s(a) = t(a)$) are allowed.



The **opposite quiver** Q^{op} of a quiver Q is obtained from Q by reversing all arrows.

For example, $1 \longrightarrow 2 \longleftarrow 3$ is the opposite quiver of $1 \longleftarrow 2 \longrightarrow 3$.

Let $Q = (Q_0, Q_1, s, t)$ be a quiver. A sequence

$$a = (a_1, a_2, \dots, a_m)$$

of arrows $a_i \in Q_1$ is a **path** in Q if $s(a_i) = t(a_{i+1})$ for all $1 \leq i \leq m-1$. In this case, $\text{length}(a) := m$ is called the **length** of a . Furthermore set $s(a) = s(a_m)$ and $t(a) = t(a_1)$.

$$t(a) \xleftarrow{a_1} \xleftarrow{a_2} \dots \xleftarrow{a_m} s(a)$$

Instead of $(a_1, a_2, \dots, a_m)$ we often just write $a_1 a_2 \dots a_m$. Additionally there is a path e_i of length 0 for each vertex $i \in Q_0$, and we set $s(e_i) = t(e_i) = i$.

For a paths a we say that a **starts in** $s(a)$ and **ends in** $t(a)$.

A path a of length $m \geq 1$ is an **oriented cycle** in Q if $s(a) = t(a)$.

A quiver Q is **acyclic** if there is no oriented cycle in Q .

In the example above, there are infinitely many paths, e.g. for each $m \geq 1$ there is the path $c^m := (c, \dots, c)$ of length m and the path $(ef)^m := (e, f, e, f, \dots, e, f)$ of length $2m$. Also (f, c, c, e, f, a) is a path, whereas (a, b) and (a, c) are not.

Examples:

- (i) For $n \geq 1$, the **n -loop quiver** $L(n)$ has one vertex 1 and n arrows $1 \rightarrow 1$.

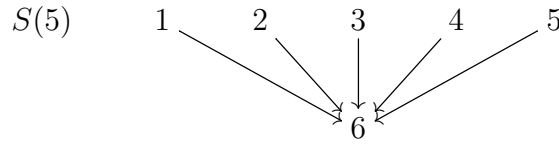
$$L(2) \quad \begin{array}{c} \curvearrowright 1 \curvearrowleft \end{array}$$

- (ii) For $n \geq 1$, the **n -Kronecker quiver** $K(n)$ has two vertices 1 and 2 and n arrows $1 \rightarrow 2$.

$$K(3) \quad 1 \rightrightarrows 2$$

Leopold Kronecker (1823-1891)

- (iii) For $n \geq 1$, the **n -subspace quiver** $S(n)$ has $n + 1$ vertices $1, \dots, n + 1$ and an arrow $i \rightarrow n + 1$ for each $1 \leq i \leq n$.



The **path algebra** KQ of Q over K is the K -algebra with basis the set of all paths in Q . The multiplication of paths a and b is defined as follows: If $a = (a_1, \dots, a_l)$ and $b = (b_1, \dots, b_m)$ are paths in Q with $l, m \geq 1$, then

$$ab := a \cdot b := \begin{cases} (a_1, \dots, a_l, b_1, \dots, b_m) & \text{if } s(a_l) = t(b_1), \\ 0 & \text{otherwise.} \end{cases}$$

If a or b is a path of length 0, then

$$ab := a \cdot b := \begin{cases} a & \text{if } b = e_i \text{ and } s(a) = i, \\ b & \text{if } a = e_i \text{ and } t(b) = i, \\ 0 & \text{otherwise.} \end{cases}$$

These multiplication rules are clearly associative, so extending them K -linearly turns KQ into a K -algebra.

By definition we have

$$e_i e_j = \begin{cases} e_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, observe that

$$1 = \sum_{i \in Q_0} e_i$$

is the identity in KQ .

The set of $Q_0 \cup Q_1$ (i.e. paths of length ≤ 1) is an algebra generating set of KQ .

Examples:

(i) Let Q be the following quiver:

$$\begin{array}{ccccc} 1 & \xleftarrow{a} & 2 & \xrightarrow{b} & 3 \\ & \searrow c & \downarrow d & \downarrow e & \\ & 4 & \xrightarrow{f} & 5 & \end{array}$$

Then the paths in Q are

$$\begin{aligned} & e_1, e_2, e_3, e_4, e_5, \\ & a, b, c, d, e, f, \\ & ca, da, fc, fd, \\ & fda, fca. \end{aligned}$$

Thus, KQ is a 17-dimensional K -algebra. Here are some examples of multiplications of paths:

$$\begin{aligned} e_1 \cdot e_1 &= e_1, & e_3 \cdot e_4 &= 0, & fc \cdot a &= fca, & a \cdot fc &= 0, \\ b \cdot e_2 &= b, & e_2 \cdot b &= 0, & e_3 \cdot b &= b. \end{aligned}$$

(ii) For $m \geq 1$, let $Q = L(m)$ be the m -loop quiver with a single vertex and m loops $a_1, \dots, a_m$. We get a K -algebra isomorphism

$$K\langle x_1, \dots, x_m \rangle \rightarrow KQ$$

defined by $x_i \mapsto a_i$ for $1 \leq i \leq m$. It maps a monomial of the form $x_{i_1} \cdots x_{i_t}$ to the path $(a_{i_1}, \dots, a_{i_t})$.

For $m = 1$, we get $KQ \cong K[T]$, where $K[T]$ is the polynomial ring in one variable T .

1.6. Exercises.

1.6.1. Find examples of non-commutative K -algebras A and B such that $A \cong A^{\text{op}}$ and $B \not\cong B^{\text{op}}$.

1.6.2. Let Q be a quiver. Show that the algebras $(KQ)^{\text{op}}$ and KQ^{op} are isomorphic.

1.6.3. Consider the usual field extension $\mathbb{C}/\mathbb{R}$. This turns $\mathbb{C}$ into an $\mathbb{R}$ -algebra A . Show that A is not isomorphic to a path algebra $\mathbb{R}Q$.

1.6.4. The algebra

$$A := \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{ij} \in K \right\}$$

of upper triangular matrices is isomorphic to the path algebra KQ of the quiver

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Construct this isomorphism. Try to realize the path algebra of

$$1 \xrightarrow{a} 2 \xleftarrow{b} 3$$

as an algebra of matrices.

1.6.5. Let

$$A := \begin{pmatrix} K[T]/(T^2) & K[T]/(T^2) \\ 0 & K \end{pmatrix}.$$

This is a K -algebra via the usual matrix multiplication. Show that A is isomorphic to KQ/I for some quiver Q and some ideal I of KQ .

1.6.6.

The **centre** $Z(R)$ of a ring R is defined as

$$Z(R) := \{c \in R \mid cr = rc \text{ for all } r \in R\}.$$

Let Q be a quiver. Determine the centre $Z(KQ)$.

— — — — — **End of Lecture 1** — — — — —

2. Modules and homomorphisms – basic notions

2.1. Representations of an algebra.

A **representation** of a K -algebra A is a K -algebra homomorphism

$$\eta: A \rightarrow \text{End}_K(V)$$

where V is a K -vector space.

We want to write down explicitly what this means: To every $a \in A$ we associate a map $\eta(a): V \rightarrow V$ such that the following hold:

$$(R1) \quad \eta(a)(v_1 + v_2) = \eta(a)(v_1) + \eta(a)(v_2),$$

$$(R2) \quad \eta(a)(\lambda v) = \lambda(\eta(a)(v)),$$

$$(R3) \quad \eta(a_1 + a_2)(v) = \eta(a_1)(v) + \eta(a_2)(v),$$

$$(R4) \quad \eta(\lambda a)(v) = \lambda(\eta(a)(v)),$$

$$(R5) \quad \eta(a_1 a_2)(v) = \eta(a_1)(\eta(a_2)(v)),$$

$$(R6) \quad \eta(1_A)(v) = v$$

for all $a, a_1, a_2 \in A$, $v, v_1, v_2 \in V$ and $\lambda \in K$. The conditions (R1) and (R2) just mean that for every $a \in A$ the map $\eta(a): V \rightarrow V$ is K -linear. The other rules show that η is a K -algebra homomorphism: (R3) and (R4) say that η is K -linear, (R5) means that η is compatible with the multiplication, and (R6) ensures that the identity element of A is mapped to the identity element of $\text{End}_K(V)$.

We are often a bit sloppy and just say “ V is a representation of A ” without mentioning η .

The following lemma is an easy exercise.

Lemma 2.1. *Let X be an algebra generating set of A . A representation $\eta: A \rightarrow \text{End}_K(V)$ is then uniquely determined by the tuple $(\eta(a))_{a \in X}$.*

Examples:

- (i) Let $F(n) = K\langle x_1, \dots, x_n \rangle$. Then there is a bijection between the set of representations $\eta: F(n) \rightarrow \text{End}_K(V)$ and the set of tuples $(V, f_1, \dots, f_n)$ with $f_i \in \text{End}_K(V)$ for all i . (We have $\eta(x_i) = f_i$ for $1 \leq i \leq n$.)
- (ii) Let $A = K[T]/(T^n)$. Then there is a bijection between the set of representations $\eta: A \rightarrow \text{End}_K(V)$ and the set of pairs (V, f) where $f \in \text{End}_K(V)$ with $f^n = 0$.
- (iii) Let $A = K[x_1, \dots, x_n]$ be the polynomial ring in n commuting variables. Then there is a bijection between the set of representations $\eta: A \rightarrow \text{End}_K(V)$ and the set of tuples $(V, f_1, \dots, f_n)$ with $f_i \in \text{End}_K(V)$ and $f_i f_j = f_j f_i$ for all i, j .
- (iv) Let $f: A \rightarrow B$ be a K -algebra homomorphism. For each representation $\eta: B \rightarrow \text{End}_K(V)$ the composition $\eta f: A \rightarrow \text{End}_K(V)$ is a representation of A .

2.2. Modules over an algebra.

Let A be a K -algebra, and let V be a K -vector space. An **A -module structure** on V (or more precisely, a **left A -module structure** on V) is a K -bilinear map

$$\sigma: A \times V \rightarrow V$$

(where we write av instead of $\sigma(a, v)$) such that the following holds:

- (i) For all $a_1, a_2 \in A$ and $v \in V$ we have $(a_1 a_2)v = a_1(a_2 v)$;
- (ii) For all $v \in V$ we have $1_A v = v$.

In other words, we demand that for all $a, a_1, a_2 \in A$, $v, v_1, v_2 \in V$ and $\lambda \in K$ the following hold:

- (M1) $a(v_1 + v_2) = av_1 + av_2$,
- (M2) $a(\lambda v) = \lambda(av)$,
- (M3) $(a_1 + a_2)v = a_1 v + a_2 v$,
- (M4) $(\lambda a)v = \lambda(av)$,
- (M5) $(a_1 a_2)v = a_1(a_2 v)$,
- (M6) $1_A v = v$.

The conditions (M i) with $1 \leq i \leq 4$ mean that σ is K -bilinear. Condition (M5) gives the compatibility with the multiplication, and (M6) ensures that 1_A acts as the identity on V .

An **A -module** (or more precisely a **left A -module**) is a vector space V together with an A -module structure σ on V . Often we just say “ V is an A -module” (or “ V is a module over A ”) without mentioning σ . We sometimes write ${}_A V$ to indicate that V is a left A -module.

The **dimension** of an A -module V is just the dimension of the vector space V .

Occasionally, we will also work with right A -modules.

Let A be a K -algebra, and let V be a K -vector space. A **right A -module structure** on V is a K -bilinear map

$$\rho: V \times A \rightarrow V$$

(where we write va instead of $\rho(v, a)$) such that the following holds:

- (i) For all $a_1, a_2 \in A$ and $v \in V$ we have $v(a_1 a_2) = (va_1)a_2$;
- (ii) For all $v \in V$ we have $v1_A = v$.

A **right A -module** is a vector space V together with a right A -module structure on V . In this case, we sometimes write V_A .

2.3. Modules and representations.

For sets A and V , there is a bijection

$$\text{Abb}(A, \text{Abb}(V, V)) \rightarrow \text{Abb}(A \times V, V).$$

which sends a map $\eta: A \rightarrow \text{Abb}(V, V)$ to the map

$$\begin{aligned} \bar{\eta}: A \times V &\rightarrow V \\ (a, v) &\mapsto \eta(a)(v). \end{aligned}$$

The inverse sends a map $\sigma: A \times V \rightarrow V$ to the map

$$\begin{aligned} \bar{\sigma}: A &\rightarrow \text{Abb}(V, V) \\ a &\mapsto [v \mapsto \sigma(a, v)]. \end{aligned}$$

Now assume that A is a K -algebra and V is a K -vector space. Using the maps above one can show the following:

A map $\eta: A \rightarrow \text{End}_K(V)$ is a representation if and only if $\bar{\eta}: A \times V \rightarrow V$ is an A -module structure on V .

This follows from matching the conditions (Ri) and (Mi) for $1 \leq i \leq 6$.

We often do not distinguish between the two concepts of a module and a representation.

2.4. Homomorphisms between modules.

Let V and W be A -modules. A K -linear map $f: V \rightarrow W$ is a **homomorphism** (or **A -module homomorphism** or **homomorphism of A -modules** or **A -linear map**) if

$$f(av) = af(v)$$

for all $a \in A$ and $v \in V$.

Let V and W be A -modules.

For homomorphisms $f, g: V \rightarrow W$ define

$$f + g: V \rightarrow W$$

by $(f + g)(v) := f(v) + g(v)$.

For $\lambda \in K$ we define

$$\lambda f: V \rightarrow W$$

by $(\lambda f)(v) := \lambda f(v)$.

One easily checks that $f + g$ and λf are also homomorphisms.

Thus the set $\text{Hom}_A(V, W)$ of homomorphisms $V \rightarrow W$ forms a subspace of $\text{Hom}_K(V, W)$.

Sometimes we just write $\text{Hom}(V, W)$ instead of $\text{Hom}_A(V, W)$.

For A -module homomorphisms $f: U \rightarrow V$ and $g: V \rightarrow W$ the composition $gf: U \rightarrow W$ is again a homomorphism.

Namely, for $a \in A$ and $u \in U$ we have

$$gf(au) = g(af(u)) = a(gf(u)).$$

For an A -module V define $\text{End}_A(V) := \text{Hom}_A(V, V)$. We often just write $\text{End}(V)$ instead of $\text{End}_A(V)$. An element $f \in \text{End}(V)$ is called an **endomorphism** of V . One often calls $\text{End}(V)$ the **endomorphism algebra** (or the **endomorphism ring**) of V .

$\text{End}(V)$ is a K -algebra with multiplication given by the composition of endomorphisms.

An injective homomorphism is also called a **monomorphism**, and a surjective homomorphism is an **epimorphism**. A homomorphism which is injective and surjective is an **isomorphism**.

Let $f: V \rightarrow W$ be an A -module isomorphism. Then V and W are **isomorphic** and we write $V \cong W$.

In this case, the inverse $f^{-1}: W \rightarrow V$ is also a homomorphism, and thus also an isomorphism.

Namely, let $a \in A$ and $w \in W$. There is some $v \in V$ with $f(v) = w$. We get

$$f^{-1}(aw) = f^{-1}(af(v)) = f^{-1}(f(av)) = av = af^{-1}(w).$$

Lemma 2.2. *Let V and W be A -modules, and let $\eta: A \rightarrow \text{End}_K(V)$ and $\psi: A \rightarrow \text{End}_K(W)$ be the associated representations. Let X be an algebra generating set of A . For $f \in \text{Hom}_K(V, W)$ the following are equivalent:*

- (i) $f \in \text{Hom}_A(V, W)$;
- (ii) $\psi(a)f = f\eta(a)$ for all $a \in A$;
- (iii) $\psi(a)f = f\eta(a)$ for all $a \in X$.

Proof. Let $f \in \text{Hom}_K(V, W)$. Note that condition (ii) in Lemma 2.2 says that the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \eta(a) \downarrow & & \downarrow \psi(a) \\ V & \xrightarrow{f} & W \end{array}$$

commutes for all $a \in A$.

For $a \in A$ and $v \in V$ we have $af(v) = \psi(a)(f(v))$ and $f(av) = f(\eta(a)(v))$. This implies (i) $\iff$ (ii).

To show that (ii) and (iii) are equivalent, one uses that the diagram

$$\begin{array}{ccccc} & & V & \xrightarrow{f} & W \\ & \nearrow & \downarrow \eta(a_2) & & \downarrow \psi(a_2) \\ & & V & \xrightarrow{f} & W \\ & \searrow & \downarrow \eta(a_1) & & \downarrow \psi(a_1) \\ & & V & \xrightarrow{f} & W \end{array}$$

$\eta(a_1 a_2)$ (left curved arrow from top V to bottom V) $\psi(a_1 a_2)$ (right curved arrow from top W to bottom W)

is commutative for all $a_1, a_2 \in A$. □

Corollary 2.3. *Let V and W be A -modules, and let $\eta: A \rightarrow \text{End}_K(V)$ and $\psi: A \rightarrow \text{End}_K(W)$ be the associated representations. Then the following are equivalent:*

- (i) $V \cong W$;
- (ii) *There exists some vector space isomorphism $f \in \text{Hom}_K(V, W)$ such that $f\eta(a)f^{-1} = \psi(a)$ for all $a \in A$.*

In the situation of Corollary 2.3(ii) we write $\eta \cong \psi$ and say that the representations η and ψ are **isomorphic**.

2.5. Examples of representations and modules.

2.5.1. *Zero module.* Let $V = 0$ be the zero vector space. There is only one K -bilinear map $A \times 0 \rightarrow 0$. This defines a (unique) A -module structure on V .

2.5.2. *Regular module.*

Let $V = A$ be the underlying vector space of A . Then

$$\begin{aligned} A \times V &\rightarrow V \\ (a, v) &\mapsto av \end{aligned}$$

defines an A -module structure on V . (Here av is just the multiplication in A .) One calls V the **regular A -module** (which is often denoted by ${}_A A$) or the **free A -module of rank 1**.

The representation corresponding to ${}_A A$ is

$$\begin{array}{lll} \eta: A \rightarrow \text{End}_K(A) & \text{where} & \lambda_a: A \rightarrow A \\ a \mapsto \lambda_a & & x \mapsto ax \end{array}$$

is the left multiplication with a . We call η the **regular representation of A** .

There is also a **right regular A -module A_A** which is defined in the obvious way.

2.5.3. *Modules over K .* Let $A = K$. The A -modules are nothing else than K -vector spaces. Two K -modules V and W are isomorphic as modules if and only if V and W are isomorphic as vector spaces. As we learned in Linear Algebra this is the case if and only if $\dim(V) = \dim(W)$.

2.5.4. *A module over $M_n(K)$.* Let $A = M_n(K)$ and $V = K^n$. Then

$$\begin{aligned} A \times V &\rightarrow V \\ (a, v) &\mapsto av \end{aligned}$$

defines an A -module structure on V . (Here av is just the matrix multiplication. Note that we treat the elements of K^n as column vectors.)

2.5.5. *Matrix representations.* Recall that we can identify $\text{End}_K(K^n)$ and $M_n(K)$.

A representation of the form $A \rightarrow \text{End}_K(K^n)$ is called a **matrix representation** of A .

Each finite-dimensional representation is isomorphic to a matrix representation.

Namely, let $\eta: A \rightarrow \text{End}_K(V)$ be an n -dimensional representation. Choosing a K -basis $B = (b_1, \dots, b_n)$ of V we get the coordinate isomorphism $\mathbf{c}_B: V \rightarrow K^n$. Then

$$\begin{aligned} \psi: A &\rightarrow \text{End}_K(K^n) \\ a &\mapsto \mathbf{c}_B \eta(a) \mathbf{c}_B^{-1} \end{aligned}$$

is a representation. Note that $\mathbf{c}_B: V \rightarrow K^n$ becomes an A -module isomorphism: For $a \in A$ and $v \in V$ we have

$$\mathbf{c}_B(av) = \mathbf{c}_B(\eta(a)(v)) = \mathbf{c}_B \eta(a) (\mathbf{c}_B^{-1} \mathbf{c}_B(v)) = \psi(a) \mathbf{c}_B(v) = a \mathbf{c}_B(v).$$

Thus we have $\eta \cong \psi$.

2.5.6. *Representations of $K[T]$.* Let $A = K[T]$, and let $\eta: A \rightarrow \text{End}_K(V)$ and $\psi: A \rightarrow \text{End}_K(W)$ be representations. These are isomorphic if and only if there exists some isomorphism $f: V \rightarrow W$ of K -vector spaces such that

$$f \eta(T) f^{-1} = \psi(T).$$

Now assume that V and W are finite-dimensional. Without loss of generality we can assume that η and ψ are matrix representations. Thus $\eta \cong \psi$ if and only if the matrices $\eta(T)$ and $\psi(T)$ are conjugate. We solved this classification problem in Linear Algebra (Jordan Normal Form for K algebraically closed and Frobenius Normal Form for general K).

2.6. How to compute homomorphism spaces. For simplicity we restrict to finitely generated algebras. Let A be a finitely generated K -algebra, and let $X = \{a_1, \dots, a_n\}$ be an algebra generating set of A .

Let $\eta: A \rightarrow \text{End}_K(V)$ be a representation. Thus V is an A -module. Consider the tuple $(V, f_1, \dots, f_n)$ where $f_i := \eta(a_i)$ for $1 \leq i \leq n$. (For $A = F(n)$ each tuple $(V, f_1, \dots, f_n)$ with $f_i \in \text{End}_K(V)$ for $1 \leq i \leq n$ occurs in this way. In general, this is not the case.)

Let $(W, g_1, \dots, g_n)$ be also an A -module.

Then

$$\text{Hom}_A(V, W) = \{h \in \text{Hom}_K(V, W) \mid g_i h = h f_i \text{ for all } 1 \leq i \leq n\}.$$

If the f_i and g_i are given explicitly as matrices, then $\text{Hom}_A(V, W)$ can be computed by solving a system of linear equations.

Example: Let $A = K[T]$. Let V be the A -module corresponding to the representation $\eta: A \rightarrow \text{End}_K(K^3)$ defined by $T \mapsto \eta(T)$ where

$$\eta(T) := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We get

$$\text{End}_A(V) = \{f \in M_3(K) \mid f\eta(T) = \eta(T)f\} = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in K \right\}.$$

This follows from the calculation

$$\eta(T)f = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$f\eta(T) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} \end{pmatrix}.$$

Let W be the A -module corresponding to the representation $\psi: A \rightarrow \text{End}_K(K^2)$ defined by $T \mapsto \psi(T)$ where

$$\psi(T) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

With V as before, we get

$$\text{Hom}_A(V, W) = \{f \in M_{2,3}(K) \mid \psi(T)f = f\eta(T)\} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \end{pmatrix} \mid a, b \in K \right\}.$$

— — — — — **End of Lecture 2** — — — — —

2.7. Posets and lattices.

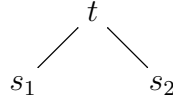
A **partially ordered set** (or **poset**) is given by $(S, \leq)$ where S is a set and $\leq$ is a relation on S which is **transitive** ($s_1 \leq s_2 \leq s_3$ implies $s_1 \leq s_3$), **reflexive** ($s_1 \leq s_1$) and **anti-symmetric** ($s_1 \leq s_2$ and $s_2 \leq s_1$ implies $s_1 = s_2$).

One can try to visualize a partially ordered set $(S, \leq)$ using its **Hasse diagram** $\mathcal{H}(S)$: This is an oriented graph with vertices the elements of S , and one draws an arrow $s \rightarrow t$ if $s < t$ and if $s \leq m \leq t$ implies $s = m$ or $m = t$.

Usually one tries to draw the diagram with arrows pointing upwards and then one forgets the orientation of the arrows and just uses unoriented edges.

For example, the following Hasse diagram describes the partially ordered set $(S, \leq)$ with three elements s_1, s_2, t with $s_i < t$ for $i = 1, 2$, and s_1 and s_2 are not comparable

in $(S, \leq)$.



[Helmut Hasse \(1898-1979\)](#) was a member of the NSDAP and an active supporter of the NS regime. Despite being quite a fascist, Hasse was very fond of [Emmy Noether \(1882-1935\)](#) and had an extensive correspondence with her, even after her emigration (being Jewish, she lost her position in 1933 and emigrated to the USA). It took till 1948 before Hasse was more or less acquitted from his NS sins. Hasse and Krull also had an extensive correspondence between 1928 - 1970. Both collections of letters can be easily found online.

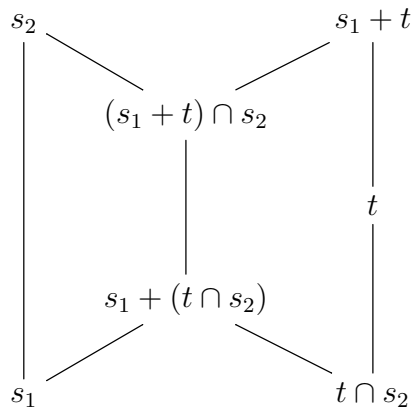
For a subset $T \subseteq S$ an **upper bound** for T is some $s \in S$ such that $t \leq s$ for all $t \in T$. A **supremum** of T is a smallest upper bound s_0 for T , i.e. s_0 is an upper bound and if s is an upper bound for T , then $s_0 \leq s$.

Similarly, define a **lower bound** and an **infimum** of T .

The poset $(S, \leq)$ is a **lattice** if for any two elements $s, t \in S$ there is a supremum and an infimum of $T = \{s, t\}$. In this case, we write $s + t$ (or $s \cup t$) for the supremum and $s \cap t$ for the infimum. One calls $(S, \leq)$ a **complete lattice** if there is a supremum and infimum for every subset of S .

Let s_1, s_2, t be elements in a lattice $(S, \leq)$ with $s_1 \leq s_2$. Then

$$s_1 + (t \cap s_2) \leq (s_1 + t) \cap s_2.$$



A lattice $(S, \leq)$ is called **modular** if

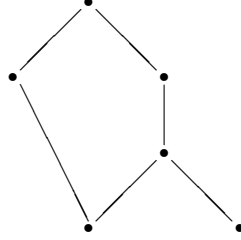
$$s_1 + (t \cap s_2) = (s_1 + t) \cap s_2$$

for all elements $s_1, s_2, t \in S$ with $s_1 \leq s_2$.

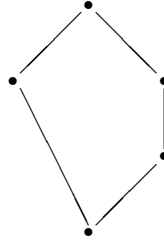
Examples:

(i) The natural numbers $\mathbb{N}$ together with the usual ordering form a modular lattice, but this lattice is not complete. For example, the subset $\mathbb{N}$ itself does not have a supremum in $\mathbb{N}$.

(ii) This is not a lattice:



(iii) This is a complete lattice, but it is not modular:



2.8. Submodules.

Let V be an A -module. A subspace U of V is a **submodule** of V if $au \in U$ for all $a \in A$ and $u \in U$.

Then U is again an A -module in the obvious way.

How to compute submodules: For simplicity we restrict again to finitely generated algebras. Let A be a finitely generated K -algebra, and let $X = \{a_1, \dots, a_n\}$ be an algebra generating set of A . Let $\eta: A \rightarrow \text{End}_K(V)$ be a representation. Thus V is an A -module. As before, we consider the tuple $(V, f_1, \dots, f_n)$ where $f_i := \eta(a_i)$ for $1 \leq i \leq n$.

A subspace U of V is then a submodule of V if and only if

$$f_i(U) \subseteq U$$

for all $1 \leq i \leq n$.

A submodule U of V is a **proper submodule** if $U \subset V$, i.e. $U \neq V$.

A module V is **simple** (or **irreducible**) if $V \neq 0$ and the only submodules are 0 and V .

Examples:

(i) Let V be an A -module. Then 0 and V are submodules of V .

(ii) Let $\eta: K[T] \rightarrow \text{End}_K(K^2)$ be the representation defined by

$$T \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and let $V = K^2$ be the corresponding module. Then V has exactly one non-zero proper submodule, namely $\text{Lin}(e_1)$. (Why?)

(iii) The submodules of the regular module ${}_A A$ are exactly the left ideals in A .

The set of submodules of a module V becomes a poset by defining $U_1 \leq U_2$ whenever $U_1 \subseteq U_2$.

The following lemma is straightforward:

Lemma 2.4. *Let $(U_i)_{i \in I}$ be a family of submodules of a module. Then the sum $\sum_{i \in I} U_i$ and the intersection $\bigcap_{i \in I} U_i$ are again submodules.*

Lemma 2.5. *Let V be a module. Then the set of all submodules of V is a complete lattice.*

Proof. The supremum of a set $\{U_i \mid i \in I\}$ of submodules is $\sum_{i \in I} U_i$, and the infimum is $\bigcap_{i \in I} U_i$. $\square$

Lemma 2.6 (Dedekind Lemma). *Let U_1, U_2, W be submodules of a module V such that $U_1 \subseteq U_2$. Then we have*

$$U_1 + (W \cap U_2) = (U_1 + W) \cap U_2.$$

Richard Dedekind (1831-1906)

Proof. It is sufficient to prove the statement for subspaces of vector spaces.

Let $u_1 \in U_1$ and $u_2 \in W \cap U_2$. Then $u_1 + u_2$ belongs to $U_1 + W$ and (since $U_1 \subseteq U_2$) also to U_2 .

For the other inclusion let $u \in U_1$, $w \in W$ and assume $u + w \in U_2$. Then $w = (u + w) - u$ belongs to U_2 and thus to $W \cap U_2$. It follows that $u + w \in U_1 + (W \cap U_2)$. $\square$

Corollary 2.7. *The poset of submodules of a module is a complete modular lattice.*

2.9. Factor modules. Let U be a submodule of V . Recall that

$$V/U := \{v + U \mid v \in V\}$$

and $v + U = v' + U$ if and only if $v - v' \in U$.

The **factor module** V/U corresponding to U is defined by

$$\begin{aligned} A \times V/U &\rightarrow V/U \\ (a, v + U) &\mapsto av + U. \end{aligned}$$

This is well defined since U is a submodule: Given $v + U = v' + U$, we have $av + U = av' + U$ if and only if $a(v - v') \in U$. This is the case, since $v - v' \in U$ and U is a submodule.

Lemma 2.8. *Let V be a module, and let U be a submodule of V . Then*

$$W \mapsto W/U$$

defines an inclusion preserving bijection between the set of submodules W of V with $U \subseteq W$ and the set of submodules of V/U .

Proof. We know from Linear Algebra (K -modules) that $W \mapsto W/U$ defines a bijection between the set of subspaces W of V with $U \subseteq W$ and the set of subspaces of V/U .

Let W be a submodule of V with $U \subseteq W$. Then W/U is obviously a submodule of V/U .

Vice versa, let W/U be a submodule of V/U . Thus

$$a(w + U) = aw + U \in W/U$$

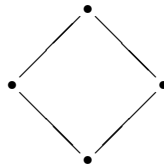
for all $a \in A$ and $w \in W$. This implies $aw \in W$. Thus W is a submodule of V . $\square$

2.10. Examples of submodule lattices.

2.10.1. Consider the $K[T]$ -module (K^2, f_1) where

$$f_1 = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}.$$

If $c_1 \neq c_2$, then (K^2, f_1) has exactly two non-zero proper submodules, namely $\text{Lin}(e_1)$ and $\text{Lin}(e_2)$. Here is the submodule lattice of (K^2, f_1) :



If $c_1 = c_2$, then each subspace of K^2 is a submodule.

2.10.2. Consider the $F(2)$ -module (K^2, f_1, f_2) where

$$f_1 = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $c_1 \neq c_2$. The module (K^2, f_1, f_2) is simple. Here is its submodule lattice:



2.10.3. Let V be the $F(2)$ -module (K^2, f_1, f_2) where

$$f_1 = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

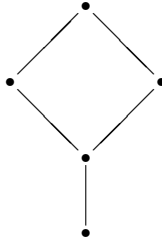
Then (K^2, f_1, f_2) has exactly one non-zero proper submodule, namely $U = \text{Lin}(e_1)$. We have $U \cong (K, c_1, 0)$, and $V/U \cong (K, c_2, 0)$. In particular, U and V/U are not isomorphic if and only if $c_1 \neq c_2$. Here is the submodule lattice of (K^2, f_1, f_2) :



2.10.4. Let (K^3, f_1, f_2) be the $F(2)$ -module defined by

$$f_1 = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

with pairwise different c_i . The non-zero proper submodules are $\text{Lin}(e_3)$, $\text{Lin}(e_1, e_3)$ and $\text{Lin}(e_2, e_3)$. The submodule lattice of (K^3, f_1, f_2) looks like this:



2.10.5. Let $A = F(2)$, and let $V = (K^n, f_1, f_2)$ be the A -module defined by

$$f_1 = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ 1 & & & 0 \end{pmatrix}.$$

(The empty spots in the matrices are meant to be 0 entries.) Then V is a simple A -module. To prove this, one first shows that the only submodules of the $K[T]$ -module (K^n, f_1) are $U_i := \text{Lin}(e_1, e_2, \dots, e_i)$ for $i \geq 0$. (By convention, $U_0 := 0$.)

2.10.6. Let $A = K[T]$, and let f be an irreducible polynomial in $K[T]$. Then the factor module $V = K[T]/(f)$ is a simple A -module with $\dim(V) = \deg(f)$. For example, for $K = \mathbb{Q}$ and $n \geq 1$, the polynomial $f = X^n - 2$ is irreducible in $K[T]$. Thus, in this case, we have an n -dimensional simple A -module for each $n \geq 1$.

2.11. Matrix version of submodules and factor modules. Let V be a finite-dimensional A -module, and let U be a submodule of V . Let $\eta: A \rightarrow \text{End}_K(V)$, $\eta_U: A \rightarrow \text{End}_K(U)$ and $\eta_{V/U}: A \rightarrow \text{End}_K(V/U)$ be the associated representations.

Choose a basis $B = (b_1, \dots, b_n)$ of V such that $B_1 = (b_1, \dots, b_s)$ is a basis of U . Then $B_2 = (b_{s+1} + U, \dots, b_n + U)$ is a basis of V/U . The coordinate isomorphism $\mathbf{c}_B: V \rightarrow K^n$ gives a representation $\eta_B: A \rightarrow \text{End}_K(K^n)$ defined by $a \mapsto \mathbf{c}_B \eta(a) \mathbf{c}_B^{-1}$.

For $a \in A$ we get

$$\eta_B(a) = \begin{pmatrix} A_a & B_a \\ 0 & C_a \end{pmatrix}$$

where $A_a = \mathbf{c}_{B_1} \eta_U(a) \mathbf{c}_{B_1}^{-1}$ and $C_a = \mathbf{c}_{B_2} \eta_{V/U}(a) \mathbf{c}_{B_2}^{-1}$. In other words, η_U and $\eta_{V/U}$ are isomorphic to the matrix representations $a \mapsto A_a$ and $a \mapsto C_a$, respectively.

— — — — — **End of Lecture 3** — — — — —

2.12. Kernel, image and cokernel.

If $f: V \rightarrow W$ is a homomorphism, then

$$\text{Ker}(f) := \{v \in V \mid f(v) = 0\}$$

is the **kernel** of f ,

$$\text{Im}(f) := \{f(v) \mid v \in V\}$$

is the **image** of f , and

$$\text{Cok}(f) := W/\text{Im}(f)$$

is the **cokernel** of f .

$\text{Ker}(f)$ is a submodule of V :

For $a \in A$ and $v \in \text{Ker}(f)$, we have $f(av) = af(v) = a0 = 0$.

$\text{Im}(f)$ is a submodule of W :

For $a \in A$ and $v \in V$, we have $af(v) = f(av) \in \text{Im}(f)$.

Here are two easy examples of homomorphisms:

Let V be an A -module, and let U be a submodule of V .

Let

$$\begin{aligned}\iota_U: U &\rightarrow V \\ u &\mapsto u\end{aligned}$$

be the **(canonical) inclusion**, and let

$$\begin{aligned}\pi_U: V &\rightarrow V/U \\ v &\mapsto v + U\end{aligned}$$

be the **(canonical) projection**.

The maps ι_U and π_U are homomorphisms.

Note that $\text{Cok}(\iota_U) = V/U$ and $\text{Ker}(\pi_U) = U$.

2.13. Homomorphism Theorems.

Theorem 2.9 (Homomorphism Theorem). *Let $f: V \rightarrow W$ be a homomorphism of modules. Then f induces an isomorphism*

$$\bar{f}: V/\text{Ker}(f) \rightarrow \text{Im}(f)$$

defined by $\bar{f}(v + \text{Ker}(f)) = f(v)$.

Proof. We know from Linear Algebra that $\bar{f}$ is a well defined isomorphism of vector spaces. Next one easily checks that $\bar{f}$ is a module homomorphism: Let V and W be A -modules. Then

$$\bar{f}(a(v + \text{Ker}(f))) = \bar{f}(av + \text{Ker}(f)) = f(av) = af(v) = a\bar{f}(v + \text{Ker}(f))$$

for all $a \in A$ and $v \in V$. □

We derive some consequences of Theorem 2.9:

Corollary 2.10 (First Isomorphism Theorem). *If $U_1 \subseteq U_2$ are submodules of a module V , then*

$$V/U_2 \cong (V/U_1)/(U_2/U_1).$$

Proof. Note that U_2/U_1 is a submodule of V/U_1 . Thus we can build the factor module $(V/U_1)/(U_2/U_1)$. Let

$$V \rightarrow V/U_1 \rightarrow (V/U_1)/(U_2/U_1)$$

be the composition of the two canonical projections. This homomorphism is obviously surjective and its kernel is U_2 . Now we use Theorem 2.9. □

$$\begin{array}{c}
 V \\
 | \\
 U_2 \\
 | \\
 U_1
 \end{array}$$

Corollary 2.11 (Second Isomorphism Theorem). *If U_1 and U_2 are submodules of a module V , then*

$$U_1/(U_1 \cap U_2) \cong (U_1 + U_2)/U_2.$$

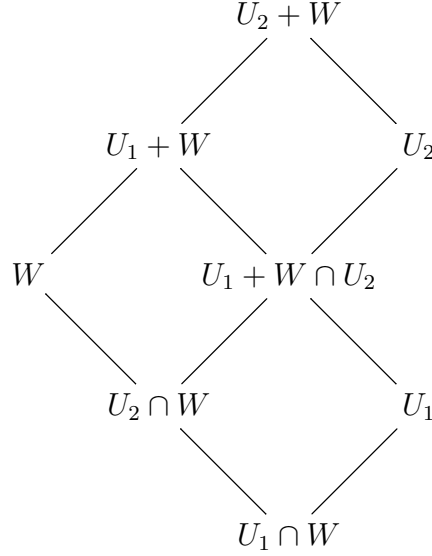
Proof. Let

$$U_1 \rightarrow U_1 + U_2 \rightarrow (U_1 + U_2)/U_2$$

be the composition of the inclusion $U_1 \rightarrow U_1 + U_2$ and the projection $U_1 + U_2 \rightarrow (U_1 + U_2)/U_2$. This homomorphism is surjective: If $u_1 \in U_1$ and $u_2 \in U_2$, then $u_1 + u_2 + U_2 = u_1 + U_2$ is the image of u_1 . Furthermore, its kernel is $U_1 \cap U_2$: An element $u_1 \in U_1$ is mapped to 0 if and only if $u_1 + U_2 = U_2$, thus if and only if $u_1 \in U_1 \cap U_2$. Now we use Theorem 2.9. $\square$

$$\begin{array}{ccc}
 & U_1 + U_2 & \\
 & / \quad \backslash & \\
 U_1 & & U_2 \\
 & \backslash \quad / & \\
 & U_1 \cap U_2 &
 \end{array}$$

If $U_1 \subseteq U_2$ and W are submodules of a module V , then we get the following picture:



(Note that by the Dedekind Lemma 2.6, we do not need to add brackets in the expression $U_1 + W \cap U_2$.) Now Corollary 2.11 yields the isomorphisms

$$\begin{aligned} (U_2 \cap W)/(U_1 \cap W) &\cong (U_1 + W \cap U_2)/U_1, \\ U_2/(U_1 + W \cap U_2) &\cong (U_2 + W)/(U_1 + W). \end{aligned}$$

2.14. Exact sequences. Let I be an **interval** in $\mathbb{Z}$, i.e. I is of one of the following forms:

$$\begin{aligned} [a, b] &:= \{m \in \mathbb{Z} \mid a \leq m \leq b\}, \\ (-\infty, b] &:= \{m \in \mathbb{Z} \mid m \leq b\}, \\ [a, \infty) &:= \{m \in \mathbb{Z} \mid a \leq m\}, \\ (-\infty, \infty) &:= \mathbb{Z} \end{aligned}$$

where $a, b \in \mathbb{Z}$ with $a \leq b$.

A tuple $(f_i)_{i \in I}$ of A -module homomorphisms is an **exact sequence** provided the composition $f_{i+1} \circ f_i$ is defined and $\text{Im}(f_i) = \text{Ker}(f_{i+1})$ for all $i, i+1 \subseteq I$.

$$\cdots \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \xrightarrow{f_{i+1}} V_{i+2} \xrightarrow{f_{i+2}} \cdots$$

A diagram

$$0 \rightarrow U \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0$$

of homomorphisms is a **short exact sequence** if $(0, f, g, 0)$ is an exact sequence.

Thus

$$0 \rightarrow U \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0$$

is a short exact sequence if and only if f is a monomorphism, g is an epimorphism, and $\text{Im}(f) = \text{Ker}(g)$. We say that this sequence **starts in** U and **ends in** W . Its **middle term** is V and its **end terms** are U and W . For such a short exact sequence we often write (f, g) instead of $(0, f, g, 0)$.

Examples: Let $f: V \rightarrow W$ be a homomorphism.

- (i) Let $\iota: \text{Ker}(f) \rightarrow V$ be the inclusion, and let $\pi: W \rightarrow \text{Cok}(f)$ be the projection. Then the sequence

$$0 \rightarrow \text{Ker}(f) \xrightarrow{\iota} V \xrightarrow{f} W \xrightarrow{\pi} \text{Cok}(f) \rightarrow 0$$

is exact.

- (ii) The exact sequence in (i) is obtained by “glueing” the short exact sequences

$$0 \rightarrow \text{Ker}(f) \xrightarrow{\iota} V \rightarrow \text{Im}(f) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{Im}(f) \rightarrow W \xrightarrow{\pi} \text{Cok}(f) \rightarrow 0.$$

2.15. Schur’s Lemma.

A **skew field** is a non-zero ring D such that every non-zero element in D has a multiplicative inverse. Such a skew field is called a **K -skew field** if D is a K -algebra.

Lemma 2.12 (Schur’s Lemma). *Let S be a simple A -module. Then $\text{End}_A(S)$ is a K -skew field.*

Issai Schur (1875-1941)

Proof. We know that $\text{End}_A(S)$ is a K -algebra. Let $f \in \text{End}_A(S)$. It follows that $\text{Im}(f)$ and $\text{Ker}(f)$ are submodules of S . Since S is simple we get either $\text{Ker}(f) = 0$ and $\text{Im}(f) = S$, or we get $\text{Ker}(f) = S$ and $\text{Im}(f) = 0$. In the first case, f is an isomorphism, and in the second case $f = 0$. Thus every non-zero element in $\text{End}_A(S)$ is invertible. $\square$

In other words, if S is a simple module, then every endomorphism of S is either 0 or an isomorphism.

Example: Let $A = F(2)$, and let V be the A -module (K^2, f_1, f_2) discussed in Section 2.10.3. If $c_1 \neq c_2$, then $\text{End}_A(V) \cong K$ (thus $\text{End}_A(V)$ is a K -skew field but V is not simple). If $c_1 = c_2$, then $\text{End}_A(V) \cong K[T]/(T^2)$.

2.16. Decompositions and direct sums of modules.

Let V be a module, and let U_1 and U_2 be submodules of V . If $U_1 \cap U_2 = 0$ and $U_1 + U_2 = V$, then we write

$$V = U_1 \oplus U_2$$

and call this a **direct decomposition** of V . We say that V is the **(internal) direct sum** of the submodules U_1 and U_2 .

A submodule U of V is a **direct summand** of V if there exists a submodule U' such that

$$U \oplus U' = V.$$

In this case, we say that U' is a **direct complement** of U in V .

Lemma 2.13. *Let $U_1 \subseteq U_2$ be submodules of a module V .*

- (i) *If U_1 is a direct summand of V , then U_1 is a direct summand of U_2 .*
- (ii) *If U_2 is a direct summand of V , then U_2/U_1 is a direct summand of V/U_1 .*

Proof. (i): If $U_1 \oplus W = V$ for some submodule W , then

$$U_1 \oplus (W \cap U_2) = (U_1 \oplus W) \cap U_2 = V \cap U_2 = U_2.$$

(ii): If $U_2 \oplus W = V$ for some submodule W , then

$$U_2/U_1 \oplus (W + U_1)/U_1 = V/U_1.$$

□

Let V and W be A -modules. Then

$$V \oplus W := V \times W := \{(v, w) \mid v \in V, w \in W\}$$

carries an A -module structure defined by $(a, (v, w)) \mapsto (av, aw)$. The module $V \oplus W$ is called the **(external) direct sum** of V and W .

Note that an external direct sum $V \oplus W$ is the internal direct sum of its submodules $V \times 0$ and $0 \times W$.

In case the distinction is necessary, it will always be clear from the context if a direct sum is internal or external.

Matrix version: Assume that V is finite-dimensional, and let $V = U_1 \oplus U_2$ be a direct decomposition.

Let $\eta: A \rightarrow \text{End}_K(V)$ and $\eta_{U_i}: A \rightarrow \text{End}_K(U_i)$ ($i = 1, 2$) be the associated representations.

Choose a basis $B = (b_1, \dots, b_n)$ of V such that $B_1 = (b_1, \dots, b_s)$ is a basis of U_1 and $B_2 = (b_{s+1}, \dots, b_n)$ is a basis of U_2 . The coordinate isomorphism $\mathbf{c}_B: V \rightarrow K^n$

gives a representation $\eta_B: A \rightarrow \text{End}_K(K^n)$ defined by $a \mapsto \mathbf{c}_B \eta(a) \mathbf{c}_B^{-1}$. We know that $\eta \cong \eta_B$.

For $a \in A$ we get

$$\eta_B(a) = \begin{pmatrix} A_a & 0 \\ 0 & C_a \end{pmatrix}$$

where $A_a = \mathbf{c}_{B_1} \eta_{U_1}(a) \mathbf{c}_{B_1}^{-1}$ and $C_a = \mathbf{c}_{B_2} \eta_{U_2}(a) \mathbf{c}_{B_2}^{-1}$. In other words, η_{U_1} and η_{U_2} are isomorphic to the matrix representations $a \mapsto A_a$ and $a \mapsto C_a$, respectively.

— — — — — **End of Lecture 4** — — — — —

References

BOOKS ON REPRESENTATIONS THEORY OF ALGEBRAS

- [ASS06] I. Assem, D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory*. London Mathematical Society Student Texts, 65. Cambridge University Press, Cambridge, 2006. x+458 pp.
- [ARS95] M. Auslander, I. Reiten, S. Smalø, *Representation theory of Artin algebras*. Corrected reprint of the 1995 original. Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge, 1997. xiv+425 pp.
- [Ba15] M. Barot, *Introduction to the representation theory of algebras*. Springer, Cham, 2015. x+179 pp.
- [Ba98] D. J. Benson, *Representations and cohomology. I. Basic representation theory of finite groups and associative algebras*. Second edition. Cambridge Studies in Advanced Mathematics, 30. Cambridge University Press, Cambridge, 1998. xii+246 pp.
- [Bb98] D. J. Benson, *Representations and cohomology. II. Cohomology of groups and modules*. Second edition. Cambridge Studies in Advanced Mathematics, 31. Cambridge University Press, Cambridge, 1998. xii+279 pp.
- [DW17] H. Derksen, J. Weyman, *An introduction to quiver representations*. Graduate Studies in Mathematics, 184. American Mathematical Society, Providence, RI, 2017. x+334 pp.
- [DK80] Y. A. Drozd, V. Kirichenko, *Finite-dimensional algebras*. Translated from the 1980 Russian original and with an appendix by Vlastimil Dlab. Springer-Verlag, Berlin, 1994. xiv+249 pp.
- [EH18] K. Erdmann, T. Holm, *Algebras and representation theory*. Springer Undergraduate Mathematics Series. Springer, Cham, 2018. ix+298 pp.
- [GR92] P. Gabriel, A. V. Roiter, *Representations of finite-dimensional algebras*. Translated from the Russian. With a chapter by B. Keller. Reprint of the 1992 English translation. Springer-Verlag, Berlin, 1997. iv+177 pp.
- [H88] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*. London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988. x+208 pp.
- [KZ98] S. König, A. Zimmermann, *Derived equivalences for group rings*. With contributions by Bernhard Keller, Markus Linckelmann, Jeremy Rickard and Raphaël Rouquier. Lecture Notes in Mathematics, 1685. Springer-Verlag, Berlin, 1998. x+246 pp.
- [P82] R. S. Pierce, *Associative algebras*. Graduate Texts in Mathematics, 88. Studies in the History of Modern Science, 9. Springer-Verlag, New York-Berlin, 1982. xii+436 pp.
- [Ri84] C. M. Ringel, *Tame algebras and integral quadratic forms*. Lecture Notes in Mathematics, 1099. Springer-Verlag, Berlin, 1984. xiii+376 pp.
- [SSa07] D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 2. Tubes and concealed algebras of Euclidean type*. London Mathematical Society Student Texts, 71. Cambridge University Press, Cambridge, 2007. xii+308 pp.
- [SSb07] D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 3. Representation-infinite tilted algebras*. London Mathematical Society Student Texts, 72. Cambridge University Press, Cambridge, 2007. xii+456 pp.
- [SY11] A. Skowroński, K. Yamagata, *Frobenius algebras. I. Basic representation theory*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2011. xii+650 pp.
- [SY17] A. Skowroński, K. Yamagata, *Frobenius algebras. II. Tilted and Hochschild extension algebras*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2017. x+619 pp.
- [Z14] A. Zimmermann, *Representation theory. A homological algebra point of view*. Algebra and Applications, 19. Springer, Cham, 2014. xx+707 pp.

BOOKS ON REPRESENTATION THEORY OF RINGS

- [L99] T.Y. Lam, *Lectures on Modules and Rings*. Graduate Texts in Mathematics, 189. Springer-Verlag, New York, 1999. xxiv+557 pp.

BOOKS ON CATEGORY THEORY

- [A10] S. Awodey, *Category Theory*. Second edition. Oxford Logic Guides, 52. Oxford University Press, Oxford, 2010. xvi+311 pp.
- [B94] F. Borceux, *Handbook of categorical algebra. 1. Basic category theory*. Encyclopedia of Mathematics and its Applications, 50. Cambridge University Press, Cambridge, 1994. xvi+345 pp.
- [L14] T. Leinster, *Basic category theory*. Cambridge Studies in Advanced Mathematics, 143. Cambridge University Press, Cambridge, 2014. viii+183 pp.
- [ML98] S. Mac Lane, *Categories for the working mathematician*. Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998. xii+314 pp.

BOOKS ON HOMOLOGICAL ALGEBRA

- [CE56] H. Cartan, S. Eilenberg, *Homological algebra*. With an appendix by David A. Buchsbaum. Reprint of the 1956 original. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1999. xvi+390 pp.
- [GM03] S. Gelfand, Y. Manin, *Methods of homological algebra*. Second edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. xx+372 pp.
- [HSt97] P. J. Hilton, U. Stammach, *A course in homological algebra*. Second edition. Graduate Texts in Mathematics, 4. Springer-Verlag, New York, 1997. xii+364 pp.
- [ML75] S. Mac Lane, *Homology*. Reprint of the 1975 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. x+422 pp.
- [N60] D. Northcott, *An introduction to homological algebra*. Reprint of the 1960 original. Cambridge University Press, Cambridge, 2008. xii+282 pp.
- [R09] J. Rotman, *An introduction to homological algebra*. Second edition. Universitext. Springer, New York, 2009. xiv+709 pp.
- [W94] C. Weibel, *An introduction to homological algebra*. Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994. xiv+450 pp.

HOMOLOGICAL DIMENSIONS AND HOMOLOGICAL CONJECTURES

- [AR75] M. Auslander, I. Reiten, *On a generalized version of the Nakayama conjecture*. Proc. Amer. Math. Soc. 52 (1975), 69–74.
- [AR91] M. Auslander, I. Reiten, *Applications of contravariantly finite subcategories*. Adv. Math. 86 (1991), no. 1, 111–152.
- [Ba60] H. Bass, *Finitistic dimension and a homological generalization of semiprimary rings*. Trans. Amer. Math. Soc. 95 (1960), 466–488.
- [H] D. Happel, *Homological conjectures in representation theory of finite-dimensional algebras*. Unpublished note.
- [N58] T. Nakayama, *On algebras with complete homology*. Abh. Math. Sem. Univ. Hamburg 22 (1958), 300–307.
- [T64] H. Tachikawa, *On dominant dimension of QF-3 algebras*. Trans. Amer. Math. Soc. 112 (1964), 249–266.
- [ZH95] B. Zimmermann-Huisgen, *The finitistic dimension conjectures—a tale of 3.5 decades*. Abelian groups and modules (Padova, 1994), 501–517, Math. Appl., 343, Kluwer Acad. Publ., Dordrecht, 1995.

REPRESENTATION THEORY OF QUIVERS

- [BGP82] I.N. Bernstein, I.M. Gelfand, V.A. Ponomarev, *Coxeter functors, and Gabriel's theorem*. In: Representation theory. Including selected papers translated from the Russian., 157–172. London Mathematical Society Lecture Note Series, 69. Cambridge University Press, Cambridge-New York, 1982. i+272 pp.
- [CB92] W. Crawley-Boevey, *Lectures on Representations of Quivers*. Lecture Notes. This is not published but can be found on Crawley-Boevey's webpage.
- [G72] P. Gabriel, *Unzerlegbare Darstellungen. I.* (German. English summary), Manuscripta Math. 6 (1972), 71–103; correction, *ibid.* 6 (1972), 309.
- [Kr08] H. Krause, *Representations of quivers via reflection functors*. Lecture notes (2008), 42 pp., arXiv:0804.1428

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