Why *abc* is still a conjecture

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In March 2018, the authors spent a week in Kyoto at RIMS of intense and constructive discussions with Prof. Mochizuki and Prof. Hoshi about the suggested proof of the *abc* conjecture. We thank our hosts for their hospitality and generosity which made this week very special.

We, the authors of this note, came to the conclusion that there is no proof. We are going to explain where, in our opinion, the suggested proof has a problem, a problem so severe that in our opinion small modifications will not rescue the proof strategy. We supplement our report by mentioning dissenting views from Prof. Mochizuki and Prof. Hoshi about the issues we raise with the proof and whether it constitutes a gap at all, cf. the report by Mochizuki.

1. With Belyi maps from IUTT to Vojta's inequality and abc

1.1. The goal in terms of Vojta's inequality.

The *abc*-conjecture goes back to Masser and Oesterlé in 1985:

Conjecture 1. *Let $\varepsilon > 0$. Then for all coprime integers a, b, c with $a + b + c = 0$ we have*

$$\log(\max\{|a|, |b|, |c|\}) < (1 + \varepsilon) \cdot \sum_{p|abc} \log(p) + \mathrm{O}(1)$$

where the constant $\mathrm{O}(1)$ depends only on ε.

The *abc*-conjecture is a special case for the projective line $\mathbb{P}^1_{\mathbb{Q}}$ with respect to the divisor $D = 0 + 1 + \infty$ of Vojta's height inequality. The general Vojta inequality is conjectured for the height function $\mathrm{h}_{\omega_{\overline{X}}(D)}(P)$ of any hyperbolic curve $(\overline{X}, D)$, i.e. where $\omega_{\overline{X}}(D)$ is ample.

Definition 2. Let k be a number field, $\mathfrak{o}_k$ its ring of integers, and $\overline{X}/k$ a smooth projective geometrically connected curve. We denote an algebraic closure of k by $\bar{k}$.

(1) The absolute **logarithmic root discriminant** of a point $P \in \overline{X}(\bar{k})$ is

$$\mathfrak{d}(P) := \mathfrak{d}(\kappa(P)) := \frac{1}{[\kappa(P) : \mathbb{Q}]} \log(|\Delta_{\kappa(P)/\mathbb{Q}}|)$$

where $\kappa(P)$ is the residue field of the closed point of $\overline{X}$ supporting P, and $\Delta_{\kappa(P)/\mathbb{Q}} \in \mathbb{Z}$ denotes the discriminant of $\kappa(P)/\mathbb{Q}$.

(2) Let $D \subseteq \overline{X}$ be a reduced (possibly empty) divisor and set $X = \overline{X} \setminus D$. For any closed point $P \in X$ we define the **logarithmic conductor** $n^1_D(P)$ depending on the choice of a proper regular $\mathfrak{o}_k$-model $\overline{\mathscr{X}} \to \mathrm{Spec}(\mathfrak{o}_k)$ as follows.

Let $\mathscr{D}$ be the closure of D in $\overline{\mathscr{X}}$, and let $\kappa(P)$ denote the residue field of P. For the unique extension $\mathscr{P} : \mathrm{Spec}(\mathfrak{o}_{\kappa(P)}) \to \overline{\mathscr{X}}$ of P we consider $\mathscr{P}^{-1}(\mathscr{D})$ as an effective divisor on $\mathrm{Spec}(\mathfrak{o}_{\kappa(P)})$. We then set

$$n^1_D(P) := \deg(\mathscr{P}^{-1}(\mathscr{D})_{\mathrm{red}}) = \frac{1}{[\kappa(P) : \mathbb{Q}]} \sum_{\mathscr{P}(\mathfrak{p}_v) \in \mathscr{D}} \log(N(v)).$$

Here $\mathfrak{p}_v$ is the prime ideal corresponding to the finite place v of k of norm $N(v) = \#\mathfrak{o}_k/\mathfrak{p}_v$. Different choices of models lead to logarithmic conductors that agree up to contributions at a fixed finite set of places, hence up to $\mathrm{O}(1)$. Therefore $n^1_D(-)$ is well defined up to

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a bounded function. It counts, as an Arakelov degree, the number of places where P becomes congruent to a point in the boundary divisor D.

Conjecture 3 (Uniform Vojta inequality for hyperbolic curves)**.** *Let k be a number field and let X/k be a smooth geometrically connected curve with smooth compactification $\overline{X}$ such that $\omega_{\overline{X}}(D)$ is ample, where $D = (\overline{X} \setminus X)_{\mathrm{red}}$. Let $\varepsilon > 0$ and $d \geq 1$. Then for all points $P \in X(\bar{k})$ with $[\kappa(P) : \mathbb{Q}] \leq d$, we have*

$$\mathrm{h}_{\omega_{\overline{X}}(D)}(P) < (1+\varepsilon)\big(\mathfrak{d}(P) + n^1_D(P)\big) + \mathrm{O}_{\varepsilon,d,X}(1) \tag{1.1}$$

Remark 4. The uniform version of Vojta's inequality with respect to the degree of the number field as stated in Conjecture 3 behaves well under branched covers, cf. [GenEll, Proposition 1.7] or [BG06, Theorem 14.416], and indeed is claimed to follow from Mochizuki's proof.

For a thorough discussion of the *abc* conjecture in relation to Voijta's conjecture we refer to [BG06, §12–§14].

1.2. **Frey curves.** Using Belyi maps as in [Belyi, Theorem 2.5] we may reduce Conjecture 3 to the moduli stack of elliptic curves $\mathscr{M}_{ell}$, i.e., to elliptic curves over number fields. In this case a form of the inequality is conjectured as Szpiro's conjecture.

More precisely, in [GenEll, Theorem 2.1 and §3, §4] Mochizuki describes a reduction for all fixed $d \geq 1$ to an inequality for all number fields k of degree $[k : \mathbb{Q}] \leq d$ and all elliptic curves E/k corresponding to points $P \in \mathscr{M}_{ell}(k)$, such that

(1) The local data at places above 2 and ∞ belong to a compact set, hence continuous local invariants at 2 and at ∞ are bounded.

(2) The elliptic curve E/k has split semistable reduction everywhere[1]. Hence, using local Tate uniformization
$$\mathbb{G}_{\mathrm{m}}^{\mathrm{rig}} \to \mathbb{G}_{\mathrm{m}}^{\mathrm{rig}}/q_v^{\mathbb{Z}} \simeq (E \times_k k_v)^{\mathrm{rig}},$$
we have a collection of local Tate parameters $q_v \in k_v$ at finite places v of multiplicative bad reduction, and a finite Arakelov divisor $q_E = \sum_{E \text{ bad red. at } v} v(q_v) \cdot v$ of degree
$$\deg(q_E) = \frac{1}{[k : \mathbb{Q}]} \sum_v v(q_v) \log(N(v)).$$
The corresponding height term $\mathrm{h}(P)$ with respect to log-differentials of $\mathscr{M}_{ell}$ is then thanks and subject to the constraints of (1) that allow us to ignore contributions at infinite places:
$$\mathrm{h}(P) = \frac{1}{6}\deg(q_E) + \mathrm{O}(\mathrm{h}(P)^{1/2} + 1). \tag{1.2}$$
Note that the degree of the boundary $\infty = \bar{\mathscr{M}}_{ell} \setminus \mathscr{M}_{ell}$ is $1/2$ while the degree of the sheaf of log-differentials is $1/12$ explaining the factor of 6.

(3) There is a prime number $\ell = \ell(P)$ such that

(i) there are constants $c_1, c_2 > 0$ depending only on d with
$$\mathrm{h}(P)^{1/2} \leq \ell \leq \mathrm{h}(P)^{1/2} \cdot (c_1 + c_2 \cdot \log(\mathrm{h}(P))), \tag{1.3}$$

(ii) the image of the mod ℓ Galois representation contains $\mathrm{SL}_2(\mathbb{F}_\ell)$:
$$\rho : \mathrm{Gal}_k \to \mathrm{GL}(E[\ell](\bar{k})) \simeq \mathrm{GL}_2(\mathbb{F}_\ell),$$

(iii) locally at places v of bad reduction, the extension arising from the local Tate parametrization
$$0 \to \mu_\ell \to E[\ell] \to \mathbb{Z}/\ell\mathbb{Z} \to 0$$

[1] Arbitrary elliptic curves E_0/k_0 are dealt with after scalar extension such that $E = E_0 \times_{k_0} k$ acquires split semistable reduction. This works because k can be chosen with a uniform bound on $[k : k_0]$, e.g., by adjoining coordinates of all 12-torsion points.

does not split. The extension class is given by $q_v \in k_v^\times/(k_v^\times)^\ell$, so we require q_v not to be an ℓ-th power.

The above set-up misses an exceptional list of elliptic curves (E/k without a "core", with complex multiplication, etc.) that is 'sparse' and can be dealt with separately. We omit the details here.

Remark 5. For fixed $d \geq 1$ and any bound $b > 0$ there are only finitely many E/k corresponding to $P \in \mathscr{M}_{ell}(k)$ as above with $[k:\mathbb{Q}] \leq d$ and $h(P) \leq b$. This follows from Faltings' theorem (Shafarevich conjecture) applied to the Weil restriction $A = \mathrm{R}_{k|\mathbb{Q}}(E)$ which is an abelian variety $A/\mathbb{Q}$ of dimension bounded by d and good reduction outside a fixed finite set of primes depending on the height bound b. There are only finitely many isomorphism classes of such abelian varieties and the map $E/k \mapsto A/\mathbb{Q}$ has finite fibres.

Claim 6 ([IUTT-4, Theorem 1.10])**.** *Fix a natural number $d \geq 1$.*

There are functions $\alpha_d, \beta_d : \mathbb{N} \to \mathbb{R}$ depending of d, such that $\alpha_d(\ell) \to 0$, for $\ell \to \infty$, and $\beta_d(\ell) = \mathrm{O}_d(1)$ and, for all number fields k of degree $[k:\mathbb{Q}] \leq d$ and elliptic curves E/k as above corresponding to $P \in \mathscr{M}_{ell}(k)$ and $\ell = \ell(P)$,

$$\frac{1}{6}\deg(q_E) \leq (1+\alpha_d(\ell))\big(\mathfrak{d}(P) + n^1_\infty(P)\big) + \beta_d(\ell)\cdot\ell.$$

If $c > 0$ is such that (1.2) says $\mathrm{h}(P) \leq \frac{1}{6}\deg(q_E) + c\cdot(\mathrm{h}(P)^{1/2}+1)$ for all E/k as above, then the inequalities of Claim 6 and (1.3) yield

$$\mathrm{h}(P)\cdot\Big(1 - \frac{c\cdot(1+\mathrm{h}(P)^{-1/2}) + \beta_d(\ell)(c_1 + c_2\cdot\log(\mathrm{h}(P)))}{\mathrm{h}(P)^{1/2}}\Big) \leq (1+\alpha_d(\ell))(\mathfrak{d}(P) + n^1_\infty(P)).$$

Now we choose $\varepsilon > 0$. As soon as for large $\mathrm{h}(P)$ and thus large $\ell = \ell(P) \geq \mathrm{h}(P)^{1/2}$ we have

$$0 < 1 + \alpha_d(\ell) < \Big(1 - \frac{c\cdot(1+\mathrm{h}(P)^{-1/2}) + \beta_d(\ell)(c_1 + c_2\cdot\log(\mathrm{h}(P)))}{\mathrm{h}(P)^{1/2}}\Big)\cdot(1+\varepsilon),$$

we conclude Vojta's inequality

$$\mathrm{h}(P) \leq (1+\varepsilon)(\mathfrak{d}(P) + n^1_\infty(P)) + \mathrm{O}_{\varepsilon,d}(1)$$

by adjusting the constant in $\mathrm{O}_{\varepsilon,d}(1)$ to cover the finitely many cases of small $\mathrm{h}(P)$, cf. Remark 5.

1.3. How IUTT derives the inequality.

For an odd prime number ℓ we set

$$\ell^* := \frac{\ell-1}{2}.$$

Moreover, for a local Tate parameter $q_v \in k_v$ at a finite place v of split multiplicative reduction of E we choose a 2ℓ-th root[2]

$$\underline{\underline{q}}_v \in \bar{k}_v$$

of q_v uniquely defined up to a root of unity. These roots define the Arakelov divisor

$$\underline{\underline{q}}_E = \sum_{E \text{ bad red. at } v} v(\underline{\underline{q}}_v)\cdot v = \frac{1}{2\ell}q_E.$$

Claim 6 is deduced in [IUTT-4, Theorem 1.10] from an inequality that reads

$$-|\log(\underline{\underline{q}})| \leq -|\log(\underline{\underline{\Theta}})| \tag{1.4}$$

and which is the main claim of [IUTT-3, Corollary 3.12], see our comments in Section §2.2. Note that the symbols in (1.4) are not just 'minus the absolute values of the logarithm of something',

[2] These roots play a role later, see Sections §2.1.6-§2.1.9.

but they rather have a distinct involved meaning defined in [IUTT-3, page 16, Corollary 3.12]. In fact, the left side of (1.4) is nothing but

$$-|\log(\underline{\underline{q}})| = -\deg(\underline{\underline{q}}_E) = -\frac{1}{2\ell}\deg(q_E).$$

The right side of (1.4) is in essential approximation ($\mathfrak{d}(P)_v$ is the v-part of the logarithmic root discriminant)

$$\begin{aligned} -|\log(\underline{\underline{\Theta}})| &\approx \frac{1}{\ell^*}\sum_v \sum_{j=1}^{\ell^*}\Big(j\cdot\mathfrak{d}(P)_v - j^2\frac{1}{2\ell[k:\mathbb{Q}]}v(q_v)\log N(v)\Big) \qquad (1.5)\\ &= \frac{1}{\ell^*}\cdot\frac{\ell^*(\ell^*+1)}{2}\mathfrak{d}(P) - \frac{1}{\ell^*}\cdot\frac{\ell^*(\ell^*+1)(2\ell^*+1)}{12\ell}\deg(q_E)\\ &= \frac{\ell^*+1}{2}\Big(\mathfrak{d}(P) - \frac{1}{6}\deg(q_E)\Big). \end{aligned}$$

Substituting this back into (1.4) yields after solving for $\frac{1}{6}\deg(q_E)$ that

$$\frac{1}{6}\deg(q_E) \lessapprox \Big(1 - \frac{12}{\ell(\ell+1)}\Big)^{-1}\cdot\mathfrak{d}(P).$$

The error in the $\lessapprox$ is taken care of with a linear term in ℓ on the right side, so this achieves a proof of Claim 6. We are going to argue that the reasoning for (1.5) in [IUTT-3, Corollary 3.12] rather describes the following inequality (with the factor j^2 missing):

$$\begin{aligned} -|\log(\underline{\underline{\Theta}})| &\approx \frac{1}{\ell^*}\sum_v \sum_{j=1}^{\ell^*}\Big(j\cdot\mathfrak{d}(P)_v - \frac{1}{2\ell[k:\mathbb{Q}]}v(q_v)\log N(v)\Big) \qquad (1.6)\\ &= \frac{\ell^*+1}{2}\cdot\mathfrak{d}(P) - \frac{1}{2\ell}\deg(q_E). \end{aligned}$$

Starting from the corrected inequality we obtain

$$0 \lessapprox \mathfrak{d}(P)$$

which is essentially free of content.

2. Hodge theaters and Frobenioid prime strips

2.1. **Glossary: IUTT-terminology and how we may think of these objects.** The IUTT papers introduce a large amount of terminology. To facilitate the discussion, we will describe (only) the notions that are strictly relevant to explain what we regard as the error. This will involve certain radical simplifications, and it might be argued that such simplifications strip away all the interesting mathematics that forms the core of Mochizuki's proof. Towards this objection we can offer four excuses:

(1) During our discussion in Kyoto, Mochizuki agreed that some of these simplifications are OK, for example regarding the critical notion of $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strips below.
(2) Generally, the discussions in Kyoto were at a level only slightly more sophisticated than what is reflected in the simplifications below, and Mochizuki agreed that this does not result in an essential obfuscation of the ideas. We also discussed the deeper parts of the theory, and Mochizuki agreed that we had a good understanding of the substantial mathematical content.
(3) When it comes to the more drastic simplifications indicated below, such as merely identifying the choice of a Hodge theater with the choice of a curve abstractly isomorphic to X, or simply identifying identical objects along the identity, these are inessential to the point we are making, and Mochizuki was not able to convince us during the week why such a simplification was not allowed.

(4) We are certain that even with all subtleties restored, the issue we are pointing out will prevail, and it is easier to point to the key issue with these surrounding subtleties removed.

2.1.1. *Initial* Θ*-data.* This is essentially an elliptic curve E over a number field k, together with a choice of a prime number ℓ, as above. Often, one considers instead the once-punctured hyperbolic curve $X = E \setminus \{0\}$.

There are many coverings of X that are considered in the IUTT papers, both over k and its localizations. These will play no role for the following discussion, so we omit them here.

2.1.2. *Hodge theater.* These contain data of two types, "étale-like objects" and "Frobenius-like objects".

Roughly, the "étale-like data", often denoted $\mathcal{D}$ or $\mathfrak{D}$, is given by the abstract topological group $\pi_1(X)$, considered as a group up to inner automorphism. Equivalently, as is done in the IUTT papers, we may think of the abstract Galois category of finite étale covers of X, without a choice of base point. At this point, it is useful to recall the following striking result of Mochizuki.

Theorem 7 ([Anab3, Theorem 1.9, Corollary 1.10])**.** *Consider the category whose objects are connected curves X of strictly Belyi type over a field k that is either a number field or a p-adic field, and whose morphisms are finite étale morphisms of schemes. The functor taking X to $\pi_1(X)$ defines a functor to the category whose objects are topological groups, and whose morphisms $G \to H$ are H-conjugacy classes of injective open maps of topological groups. This functor is fully faithful, and one can give an explicit quasi-inverse functor.*

Moreover, once-punctured elliptic curves as considered in the IUTT papers, and their connected finite étale covers, are of strictly Belyi type.[3]

Remark 8. In fact, a slightly stronger statement is true: One can extend the result to cover the morphisms arising from $X_{k_v} \to X$ if X lives over a number field k and k_v is a nonarchimedean localization of k.

Remark 9. Anabelian geometry is supposed to be the key to Mochizuki's proof. However, here we see that in the IUTT papers, we are (for the essential part) in a situation where anabelian geometry holds true in the sense that geometry and group theory are equivalent. We could not find the point where it is essential to work with fundamental groups – there are no additional isomorphisms of fundamental groups that do not come from isomorphisms of schemes, precisely because of Mochizuki's theorem.[4]

The "Frobenius-like picture" is an enhancement of the "étale-like picture", essentially consisting of the topological group $\pi_1(X)$ together with an action on a monoid, e.g. the group $\bar{k}^\times$ of units in $\bar{k}$.[5] Here again, in many situations the forgetful functor from the category of pairs $(\Pi \circlearrowright M)$ of a topological group acting on a topological monoid (and isomorphisms as morphisms) to the category of topological groups Π (and isomorphisms as morphisms) is an equivalence of categories on the essential image of $X \mapsto (\pi_1(X) \circlearrowright \bar{k}^\times)$. This is a consequence of Kummer theory: As U ranges over open subgroups of $\pi_1(X)$, there is an injection

$$\bar{k}^\times \hookrightarrow \varinjlim_{U \subset \pi_1(X)} \mathrm{H}^1(U, \widehat{\mathbb{Z}}(1)) \ ,$$

[3]The final statement follows from the finite étale correspondence $E \setminus \{0\} \xleftarrow{[2]} E \setminus E[2] \to \mathbb{P}^1 \setminus \{0, 1, \lambda, \infty\}$, recalling that strictly Belyi type means that there is such a finite étale correspondence from the curve to a hyperbolic curve of genus 0.

[4]Such isomorphisms exist for Galois groups of p-adic fields, but this did not seem to enter the discussion in a critical way.

[5]There is one notable more interesting case related to the monoid of divisors on tempered coverings of X at places of bad reduction, by means of which Mochizuki encodes the Θ-function.

and there are almost no compatible automorphisms of $\bar{k}^\times$ and its associated

$$\widehat{\mathbb{Z}}(1) = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \bar{k}^\times)$$

that commute with this injection: The automorphisms of the latter are just $\widehat{\mathbb{Z}}^\times$, and except for ± 1, these do not respect the subset $\bar{k}^\times$ along the above inclusion[6]. Thus, in this case the assertion is only true up to sign, but for example if k_v is a nonarchimedean localization, it becomes true if one replaces $\bar{k}^\times$ by

$$\mathfrak{o}^{\triangleright}_{\bar{k}_v} := \mathfrak{o}_{\bar{k}_v} \cap \bar{k}_v^\times .$$

This modification is often made in the IUTT papers.

We note that the monoid $\mathfrak{o}^\times_{\bar{k}_v}$ does not have this rigidity property: In this case, the pair $(\pi_1(X) \circlearrowright \mathfrak{o}^\times_{\bar{k}_v})$ admits extra automorphisms given by the action of $\widehat{\mathbb{Z}}^\times$ on $\mathfrak{o}^\times_{\bar{k}_v}$. This is the "non-rigidity of the unit group portion".

Thus, up to equivalence of categories, the "Frobenius-like picture" is often equivalent to a category of certain hyperbolic curves, cf. e.g. [IUTT-1, Corollary 5.3, Corollary 6.12 (i)], but not always.

In any case, a ($\Theta^{\pm \mathrm{ell}}NF$-)Hodge theater is a certain amount of data that abstractly comes from the fixed once-punctured elliptic curve X. The natural functor from the category whose only object is X and whose morphisms are the automorphisms of X (which has one object with two automorphisms, the 'identity' and 'negation') to the category of $\Theta^{\pm \mathrm{ell}}NF$-Hodge theaters is an equivalence of categories, as follows by combining [IUTT-1, Corollary 6.12 (i), Proposition 6.6 (iii), Corollary 5.6 (ii), Proposition 4.8 (ii), Definition 6.13]. In other words, up to equivalence of categories, choosing a Hodge theater is equivalent to choosing a once-punctured elliptic curve abstractly isomorphic to X, and this equivalence of categories is constructive in the sense that one can give an explicit functor that takes a Hodge theater and produces a once-punctured elliptic curve. Of course, the category of elliptic curves abstractly isomorphic to X (and isomorphisms of curves) is equivalent to the category whose only object is X that we started with.

2.1.3. log-*links.* The log-link introduced in [IUTT-3] is based on the following construction for an algebraic closure K of a nonarchimedean local field of characteristic 0, cf. [IUTT-3, Definition 1.1 (i)]. The logarithm map defines a well-defined surjective[7] map

$$\log : \mathfrak{o}_K^\times \to K$$

that turns multiplication into addition. The kernel of the map is given by the roots of unity $(\mathbb{Q}/\mathbb{Z})(1) \cong \mathfrak{o}_K^\mu \subset \mathfrak{o}_K^\times$, and thus defining $\mathfrak{o}_K^{\times\mu} := \mathfrak{o}_K^\times / \mathfrak{o}_K^\mu$, the logarithm induces a bijection

$$\log : \mathfrak{o}_K^{\times\mu} \xrightarrow{\sim} K \ .$$

Using this observation, Mochizuki defines an endofunctor log on the category of topological fields isomorphic to algebraic closures of nonarchimedean local fields as follows. For any such field K, the underlying topological space of the new topological field $\log(K)$ will be $\mathfrak{o}_K^{\times\mu}$. The addition on $\log(K)$ is given by the multiplication on $\mathfrak{o}_K^{\times\mu}$, and the multiplication on $\log(K)$ is defined via transport of structure along the bijection

$$\log : \log(K) = \mathfrak{o}_K^{\times\mu} \xrightarrow{\sim} K.$$

It follows that the endofunctor log is naturally equivalent to the identity, with the natural equivalence to the identity given by $\log : \log(K) \cong K$.

[6]In fact, any $\varepsilon \in \hat{\mathbb{Z}}^\times$ that respects $\bar{k}^\times$ must also respect $k^\times = \bar{k}^\times \cap \varprojlim_n k^\times/(k^\times)^n$ sitting via Kummer theory inside $\varinjlim_{U \subset \pi_1(X)} \mathrm{H}^1(U, \widehat{\mathbb{Z}}(1))$. But $k^\times$ is the direct sum of a finite torsion group with a free abelian group of countable rank, hence multiplication by ε must preserve $\mathbb{Z}$ inside $\hat{\mathbb{Z}}$. This shows that necessarily $\varepsilon = \pm 1$.

[7]Note that K is algebraically closed and thus the image of log is divisible rather than contained in the maximal ideal.

The links in IUTT are all of the following form. Given two Hodge theaters HT_1 and HT_2, one extracts certain coarse data of the same categorical type on both sides, and then chooses an isomorphism between them. In the case of the log-link, the data is essentially that of $\pi_1(X)$ acting on $\overline{k}_v$ (for v ranging over the finite places; there is some similar story at the archimedean places that we will ignore). Letting $\Pi_i \circlearrowright K_i$ denoting these pairs deduced from HT_i for $i = 1, 2$, the log-link consists of an isomorphism $\Pi_1 \cong \Pi_2$ and an isomorphism

$$\mathfrak{o}^{\triangleright}_{\log(K_1)} \cong \mathfrak{o}^{\triangleright}_{K_2}$$

of topological monoids that is equivariant for the $\Pi_1 \cong \Pi_2$-actions.[8]

2.1.4. *Global realified Frobenioids.* Mochizuki introduces the difficult notion of a Frobenioid in his papers [Frd1], [Frd2]. However, the notion of a global realified Frobenioid is very elementary, cf. [IUTT-1, Example 3.5]. In this case, it simply amounts to a collection of ordered 1-dimensional $\mathbb{R}$-vector spaces $\mathbb{R}_v$ parametrized by the places v of k, together with a subspace

$$D_0 \subset \bigoplus_v \mathbb{R}_v$$

of codimension 1 such that there is an ordered isomorphism $\mathbb{R}_v \cong \mathbb{R}$ under which D_0 is the kernel of the map

$$\bigoplus_v \mathbb{R}_v \cong \bigoplus_v \mathbb{R} \to \mathbb{R}$$

given by sending 1 to $\log(N(v))$ at finite places, and 2π at infinite places.

In other words, one can define another ordered 1-dimensional $\mathbb{R}$-vector space

$$\mathbb{R}_{\odot} := (\bigoplus_v \mathbb{R}_v)/D_0,$$

and then the natural maps $\mathbb{R}_v \to \mathbb{R}_{\odot}$ are ordered isomorphisms for all v. These considerations show that the category of global realified Frobenioids is equivalent to the category of ordered 1-dimensional $\mathbb{R}$-vector spaces. Moreover, if one fixes isomorphisms $\mathbb{R}_v \cong \mathbb{R}$ as above and defines $\gamma_v \in \mathbb{R}_v$ as the preimage of 1, then the image of $\gamma_v/\log(N(v))$ under the map $\mathbb{R}_v \to \mathbb{R}_{\odot}$ (resp. the image of $\gamma_v/2\pi$ for infinite places v) is an element

$$\gamma_{\mathrm{can}} \in \mathbb{R}_{\odot}$$

independent of v (but depending on the trivialization; cf. however the discussion of $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strips, where this trivialization is canonical).

2.1.5. *$\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strips.* Another kind of coarse data that can be extracted from Hodge theaters is the data of a $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strip, cf. [IUTT-2, Definition 4.9]. In general, prime strips denote data that is parametrized by all places of the number field k. In the case of a $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strip, this data is, at nonarchimedean places, given by the pair

$$G_v \circlearrowright \mathfrak{o}^{\times\mu}_{\bar{k}_v} \times \mathfrak{o}^{\blacktriangleright}_{\bar{k}_v}$$

where $\mathfrak{o}^{\blacktriangleright}_{\bar{k}_v} \simeq \mathbb{N}$ with trivial $G_v = \mathrm{Gal}_{k_v}$-action.[9] There is additional global data in the form of a global realified Frobenioid. At each finite place, one defines $\mathbb{R}_v$ as the 1-dimensional $\mathbb{R}$-vector

[8] Technically, Mochizuki does not choose any such isomorphism, but works with what he calls the full poly-isomorphism, i.e. the set of all such isomorphisms. Although we spent a lot of time discussing the necessity of introducing full poly-isomorphisms over choosing one such isomorphism, Mochizuki was not able to explain this convincingly in our opinion. On a related note, if one remembers that all Hodge theaters really come from our fixed curve X, there is a completely natural isomorphism $\Pi_1 \cong \Pi_2$ given by $\Pi_1 = \pi_1(X) = \Pi_2$ and $K_1 \cong K_2$ as $K_1 = \bar{k} = K_2$, and thus a natural $\Pi_1 \cong \Pi_2$-equivariant isomorphism $\log(K_1) \cong K_1 \cong K_2$ (and thus the same on $\mathfrak{o}^{\triangleright}$). Choosing these "obvious" isomorphisms did not result in any problem that would be solved by allowing some other (possibly indeterminate) isomorphism.

[9] There is some extra structure called a $\times\mu$-Kummer structure on $\mathfrak{o}^{\times\mu}_{\bar{k}_v}$, given as the images of $(\mathfrak{o}^{\times}_{\bar{k}_v})^H \to \mathfrak{o}^{\times\mu}_{\bar{k}_v}$ for all open subgroups H of G_v. As this plays no role for us, we will ignore it.

space with basis element γ_v given by the generator of $\mathfrak{o}^{\blacktriangleright}_{\bar{k}_V}$, and one defines $D_0 \subset \bigoplus_v \mathbb{R}_v$ as the usual divisors of arithmetic degree 0. In other words, the forgetful functor from $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strips to global realified Frobenioids factors over the category of trivialized global realified Frobenioids, and the category of $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strips is equivalent to the product of the categories of pairs of topological groups acting on topological monoids abstractly isomorphic to $G_v \circlearrowright \mathfrak{o}^{\times\mu}_{\bar{k}_v} \times \mathbb{N}$, over all places v (where one has to modify this at archimedean places).

2.1.6. *Abstract pilot objects.* Pilot objects occur in [IUTT-3, Definition 3.8]. They essentially consist of generators of the $\mathfrak{o}^{\blacktriangleright}_{\bar{k}_v}$-portions of a $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strip. One can define a "pilot element"

$$\gamma_{\text{pilot}} \in \mathbb{R}_{\odot}$$

as the image of $(v(\underline{\underline{q}}_v)\gamma_v)_v \in \bigoplus_v \mathbb{R}_v$ (where we declare this entry to be 0 outside the places of bad reduction) under the projection to $\mathbb{R}_{\odot} = (\bigoplus_v \mathbb{R}_v)/D_0$ associated to the global realified Frobenioid of the $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strip. Recall $\mathbb{R}_{\odot}$ is actually canonically trivialized by the normalization that maps γ_{can} to 1. Under this normalization, the pilot element becomes

$$\mathbb{R}_{\odot} \ni \gamma_{\text{pilot}} \mapsto \frac{1}{2l}\deg(q_E) \in \mathbb{R}.$$

In many definitions below, the interesting things happen at the bad places, and we will only discuss this part in detail.

2.1.7. *Concrete q-pilot object.* On the other hand, Mochizuki defines certain concrete[10] pilot objects: A q-pilot object and a Θ-pilot object. These are only defined internally within a Hodge theater. The concrete q-pilot object is the collection of $\underline{\underline{q}}_v \in \mathfrak{o}^{\triangleright}_{\bar{k}_v}$ at bad places (and certain placeholder elements at other places). This is encoded in the $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strip $\mathscr{F}_q^{\Vdash\blacktriangleright\times\mu}$ given by the collection of

$$G_v \circlearrowright \mathfrak{o}^{\times\mu}_{\bar{k}_v} \times \underline{\underline{q}}_v^{\mathbb{N}}.$$

Note that the $\underline{\underline{q}}_v$, when regarded as actual elements of $\mathfrak{o}_{\bar{k}_v}$, naturally define an Arakelov divisor, whose Arakelov degree agrees with the number given by the abstract γ_{pilot} of $\mathscr{F}_q^{\Vdash\blacktriangleright\times\mu}$ when one chooses the natural isomorphism $\mathbb{R}_{\odot,q} \cong \mathbb{R}$.

2.1.8. *Concrete Θ-pilot object.* The concrete Θ-pilot object is the collection of

$$\underline{\underline{q}}_v^{j^2} \in \mathfrak{o}^{\triangleright}_{\bar{k}_v}$$

for $j = 1, \ldots, \ell^*$ (at bad places). Up to some 2ℓ-th roots of unity, these arise naturally as the values of a Θ-function at certain 2ℓ-torsion points, and Mochizuki devises an ingenious algorithm to recover this data very directly from the data of $\pi_1(X)$ acting on a certain monoid of divisors on tempered coverings of X.[11]

The (j-th) concrete Θ-pilot object defines a natural Arakelov divisor whose Arakelov degree equals j^2 times the Arakelov degree of the q-pilot object.

On the other hand, one can form a $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strip $\mathscr{F}_\Theta^{\Vdash\blacktriangleright\times\mu}$ given by the pair

$$G_v \circlearrowright \mathfrak{o}^{\times\mu}_{\bar{k}_v} \times ((\underline{\underline{q}}_v^{j^2})_{j=1,\ldots,\ell^*})^{\mathbb{N}}$$

that is generated by $\mathfrak{o}^{\times\mu}_{\bar{k}_v}$ and (a formal symbol) $(\underline{\underline{q}}_v^{j^2})_{j=1,\ldots,\ell^*}$; these are all abstractly isomorphic to the standard $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strip. In particular, the associated abstract pilot object (still called the Θ-pilot object by Mochizuki) is an element of $\mathbb{R}_{\odot,\Theta}$ that encodes the arithmetic

[10] The emphasis on abstract vs. concrete pilot objects is ours; Mochizuki does not properly distinguish them, which is part of our main concern.

[11] There is an issue here that this is not one object but ℓ^* many. This can be resolved in a number of ways, e.g. by passing to a "diagonal" copy; or more concretely by forming averages when one extracts numbers such as arithmetic degrees.

degree of the j-th concrete Θ-pilot object only when the identification $\mathbb{R}_{\odot,\Theta} \cong \mathbb{R}$ is scaled by j^2; the necessity of this scaling is critical and will play a key role below.

2.1.9. Θ-*link.* The Θ-link between two Hodge theaters HT_1 and HT_2 is the (full poly-)isomorphism between the $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strip $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}_{\Theta,1}$ constructed from the Θ-pilot object in HT_1 and the $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strip $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}_{q,2}$ constructed from the q-pilot object in HT_2. In particular, the choice of such an isomorphism identifies the corresponding abstract pilot objects. However, the Θ-link forgets about the concrete embeddings of $\underline{\underline{q}}_v$ into $\mathfrak{o}_{\bar{k}_v}$ and $\Theta_v \sim (\underline{\underline{q}}_v^{j^2})_{j=1,\ldots,\ell^*}$ into $\mathfrak{o}_{\bar{k}_v}$.

The attentive reader will realize that there is in fact a canonical choice for the Θ-link: The $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strips are given by data of the form $G_v \circlearrowright \mathfrak{o}^{\times\mu}_{\bar{k}_v} \times \mathbb{N}$ on both sides (at finite places), and this data is canonically the same on both sides. It is simply the name of the generator of the monoid $\mathbb{N}$ that appears that is called Θ respectively q.

2.2. **Proof of** [IUTT-3, Corollary 3.12]**.** Now let us try to unravel what happens in the critical step in the series of papers, namely towards the end of Step (xi) in the proof of [IUTT-3, Corollary 3.12]: "If one interprets the above discussion in terms of the notation introduced in the statement of Corollary 3.12, then one concludes [...] that $-|\log(\underline{\underline{q}})| \leq -|\log(\underline{\underline{\Theta}})| \in \mathbb{R}$."

An extremely rough outline of what happens in this step is the following. One considers two Hodge theaters HT_1 and HT_2 linked by a Θ-link, so in particular the abstract Θ-pilot object from HT_1 is mapped to the abstract q-pilot object belonging to HT_2. As we indicated earlier, there is no clear distinction between abstract and concrete pilot objects in Mochizuki's work, so it is argued in [IUTT-3, Corollary 3.12] that the multiradial algorithm [IUTT-3, Theorem 3.11][12] implies that up to certain indeterminacies, e.g. (Ind 1,2,3) (without which the conclusion would be obviously false), this becomes an identification of concrete Θ-pilot objects and concrete q-pilot objects (encoded via their action on processions of tensor packets of log-shells), and then the inequality follows directly.

As we are interested in comparing real numbers, and we have seen that various copies of ordered 1-dimensional $\mathbb{R}$-vector spaces arise, it is critical to spell out all identifications of copies of real numbers that are in place. In particular, in order to say that the abstract Θ-pilot object encodes the arithmetic degree of the (j-th) concrete Θ-pilot object, we saw that it was necessary to change the isomorphism $\mathbb{R} \cong \mathbb{R}_{\odot,\Theta}$ by the scalar j^2, $j = 1, \ldots, \ell^*$ (or their average, if one is interested in the averaged degree). Trying to unravel exactly what is going on, we were drawing the following diagram in Kyoto. There are several ordered 1-dimensional $\mathbb{R}$-vector spaces appearing:

1. The ones $\mathbb{R}_{\odot,\Theta}$, $\mathbb{R}_{\odot,q}$ where the pilot elements γ_{pilot} of the abstract pilot objects live.
2. The ones $\mathbb{R}_{\odot_c,\Theta_j}$, $j = 1, \ldots, \ell^*$, and $\mathbb{R}_{\odot_c,q}$ where the pilot elements γ_{pilot} of the concrete pilot objects live (of which there really are ℓ^* many in the case of Θ).
3. The standard real numbers $\mathbb{R}$ where arithmetic degrees live. For clarity, we take two copies of them $\mathbb{R}_\Theta$ and $\mathbb{R}_q$, one for each Hodge theater.

In order for a meaningful inequality to be concluded, one must consistently identify all of these. For this purpose, we were drawing the following diagram (the left half arises from HT_1,

[12] We pause to observe that with the simplifications outlined above, such as identifying identical copies of objects along the identity, the critical [IUTT-3, Theorem 3.11] does not become false, but trivial.

the Θ-side, and the right half arises from HT_2, the q-side):

$$\begin{array}{ccccc} & \mathbb{R}_{\odot,\Theta} & \xrightarrow[\cong]{\Theta\text{-link}} & \mathbb{R}_{\odot,q} & \\ \swarrow & & & & \searrow \\ (\mathbb{R}_{\odot_c,\Theta_j})_{j=1,\ldots,\ell^*} & & & & \mathbb{R}_{\odot_c,q} \\ \searrow & & & & \swarrow \\ & \mathbb{R}_{\Theta} & \xrightarrow{=} & \mathbb{R}_q & \end{array}$$

There is one consistent choice of isomorphisms given by using the natural isomorphisms $\mathbb{R}_{\odot,\Theta} \cong \mathbb{R}_{\odot,q} \cong \mathbb{R}$ coming from the observation that the global realified Frobenioids coming from $\mathscr{F}^{\Vdash\blacktriangleright\times\mu}$-prime strips are always canonically trivial using the various γ_{can}. However, we saw that with these isomorphisms, the abstract Θ-pilot object does not encode the arithmetic degree of the Θ-divisor. Thus, Mochizuki wanted to introduce scalars of j^2 somewhere on the left part of this diagram (which strictly speaking leads to inconsistencies, i.e. monodromy, on the left part of the diagram alone, which arguably can be overcome by using averages). However, it is clear that this will result in the whole diagram having monodromy j^2, i.e. being inconsistent.

The conclusion of this discussion is that with consistent identifications of copies of real numbers, one must in (1.5) omit the scalars j^2 that appear, which leads to an empty inequality.

We voiced these concerns in this form at the end of the fourth day of discussions. On the fifth and final day, Mochizuki tried to explain to us why this is not a problem after all. In particular, he claimed that up to the "blurring" given by certain indeterminacies the diagram does commute; it seems to us that this statement means that the blurring must be by a factor of at least $\mathrm{O}(\ell^2)$ rendering the inequality thus obtained useless.

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