A compactification of the space of self-maps of $\mathbb{CP}^1$

Johannes Schmitt
ETH Zurich

Abstract

We will describe a parameter space for the morphisms $\mathbb{CP}^1 \to \mathbb{CP}^1$ and compactify it. Then we divide out the simultaneous action of $\text{Aut}(\mathbb{CP}^1)$ on domain and target to obtain compactifications of the moduli space of self-maps of $\mathbb{CP}^1$. Several properties of this new space are given.

The space of morphisms $\mathbb{CP}^1 \to \mathbb{CP}^1$

We identify an algebraic map $\varphi : \mathbb{CP}^1 \to \mathbb{CP}^1$ of degree $d$ with its graph $Γ_\varphi = \{(x, \varphi(x)) : x \in \mathbb{CP}^1\} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$. $Γ_\varphi$ is a complex curve of class $(1, d) \in H_2(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{Z})$. Thus $Γ_\varphi$ is the vanishing set of a global section $s$ of the line bundle $O(1, d)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$, unique up to scaling. This gives a unique element $[s] \in \mathbb{P}(H^0(\mathbb{CP}^1 \times \mathbb{CP}^1, O(1, d))) =: Z_d$.

The set of $[s]$ obtained like this forms an open subset $\text{Rat}_d \subset Z_d$.

The conjugation action of $\text{PGL}_2(\mathbb{C})$

Identifying domain and target of a map $\varphi : \mathbb{CP}^1 \to \mathbb{CP}^1$ to be the same $\mathbb{CP}^1$ we want to study such maps up to simultaneous choice of coordinates on $\mathbb{CP}^1$. A coordinate change then corresponds to the conjugation by an element $ψ \in \text{Aut}(\mathbb{CP}^1) =: \text{PGL}_2(\mathbb{C})$. The induced action $\text{PGL}_2(\mathbb{C}) \times \text{Rat}_d \to \text{Rat}_d, (ψ, ϕ) \mapsto ψ \circ ϕ \circ ψ^{-1}$ extends naturally to $Z_d$ and $\text{Rat}_d$.

Stability conditions

We can then define the quotient of $Z_d$ and $\text{Rat}_d$ by this action using Geometric Invariant Theory (GIT). In order for the quotient to have an algebraic structure, we must restrict ourselves to points $Z_d^ss \subset Z_d$ and $\text{Rat}_d^ss \subset \text{Rat}_d$ that are semistable.

Recursive boundary structure

Consider the boundary divisor $D_{2,k}$ of $M(d, n)$ of maps $f : C \to \mathbb{CP}^1 \times \mathbb{CP}^1$ where one component of $C$ carries the markings $B \subset \{1, \ldots, n\}$ and maps with degree $(0, k)$. By removing this component and putting a new marking at the intersection point with the other component, we obtain the graph of a degree $d-k$ self-map with $n-(|B|+1)$ markings. This leads to a generalization $M(d; d_1, \ldots, d_n)$ of $M(d, n)$ where markings can have a weight playing a role in the definition of semistability in the GIT quotient.

Main definition

$M_d := \text{Rat}_d/\text{PGL}_2(\mathbb{C}) \subset M_u := Z_d^ss/\text{PGL}_2(\mathbb{C})$

$M(d,n) := Y_d^ss/\text{PGL}_2(\mathbb{C})$

First properties

Let $d \geq 2$ even and $n \geq 0$, then we know

- $M(d,n)$ is a normal, projective variety over $\mathbb{C}$
- the map $ϕ : Y_d^ss \to M(d,n)$ is a geometric quotient
- $M(2, 0) \cong \mathbb{CP}^1$; in general $M(d,n)$ is rational
- the action of $\text{PGL}_2(\mathbb{C})$ on $Y_d^ss$ is free away from a locus of codimension $≥ 2$ (unless $(d, n) = (2, 0)$).

Due to this last point, if $M(d,n)$ is $Q$-factorial then the pullback

$ϕ^* : \text{Pic}(M(d,n)) \otimes_{\mathbb{Z}} Q \to \text{Pic}(Y_d^ss) \otimes_{\mathbb{Z}} Q$

is an isomorphism. Using methods adapted from [1] we were able to compute the space on the right.

The Picard group is generated by the boundary divisors of $Y_d^ss$ lying in the semistable locus together with $ev_1^*(O(1, 0))$ for $n = 1, 2$. Intersecting with explicit test curves we showed that all relations among these generators are pullbacks of boundary relations in $M_{2,n}$ under the forgetful map $Y_d^ss \to M_{2,n}$.

Modular interpretation

The space $M_d$ of degree $d$ self maps and its compactification $M_d^u$ have been studied before, see for instance [2]. There it was shown that $M_d$ is a coarse moduli space for families $P \to S$ of degree $d$ self-maps over $S$ up to $S$-automorphisms of $P$. We were able to show that the locus $M_d^u$ of points with trivial $\text{PGL}_2(\mathbb{C})$-isotropy is a fine moduli space parametrizing self maps of flat, projective families $C \to S$ with geometric fibres $\mathbb{CP}^1$ that are not necessarily the trivial family $P_S$. The corresponding universal family is the restriction of the forgetful morphism $M(d, 1) \to M(d, 0)$ to $M_d^u$.

As for the compactification, closed points of $M(d,n)$ are in bijection with stable degree $d$ self-maps of $n$-pointed genus 0 curves (for a suitable definition of `stable self-map'). However, defining a corresponding modular property making $M(d,n)$ a coarse moduli space requires more care and is still work in progress.

Recursion to $M(2, n) \times M(2, n)$

The forgetful map $F$ illustrated above exhibits the divisor $D_{2,k}$ as a fibration above $M(d-k)(0, \ldots, 0, k)$. This recursive structure might be used in exploring the intersection theory on the spaces $M(d; d_1, \ldots, d_n)$.

Contact Information

- Email: johannes.schmitt@math.ethz.ch
- Phone: +41 44 632 3406

References

Intersections of $Q$-divisors on Kontsevich’s moduli space $\overline{M}_{0,n}(\mathbb{P}^1, d)$ and enumerative geometry. Transactions of the American Mathematical Society. 351(4):1481–1505, 1999.