Strata of $k$-differentials and double ramification cycles

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1. Two compactifications of loci of $K$-differentials

Let $g,n > 0$ with $2g-2+n > 0$.

$$M_{g,n} = \left\{ (C, p_1, \ldots, p_n) \mid C \text{ smooth genus } g \text{ curve} \right\}_{p_1, \ldots, p_n \in C \text{ distinct points}}$$

→ moduli space of smooth curves

→ smooth orbifold of dim $3g-3+n$

Let $K \geq 0$, $A = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ with $\sum a_i = K(2g-2)$.

$$H^k_g(A) = \left\{ (C, p_1, \ldots, p_n) \mid \omega^k \cong \mathcal{O}_C(\sum a_i + p_i) \right\} \subseteq M_{g,n}$$

iff $3$ meromorphic $K$-differentials $p_i$ on $C$ with $\text{div}(p_i) = \sum a_i p_i$

Example:

$$H^k_{g,k} = \left\{ (E, p, q) \mid E \cong \mathcal{O}_C(p + q) \right\} \subseteq M_{g,n}$$

$$= \left\{ (E, p, q) \mid q \text{ non-trivial } \alpha \text{-torsion pt } \alpha \mid E(p) \right\}$$

Q. Geometry of $H^k_g(A)$, e.g. dimension, smoothness, ...

How to compactify inside the moduli space of stable curves?

Natural cycle class in $CH^k(M_{g,n})$?

A1. Closure $\overline{H^k_g(A)} \subseteq M_{g,n} \rightsquigglyarrow \text{Strata of } K\text{-differentials}$

→ minimal compactification

→ [Bainbridge - Chen - Gendron - Grushevsky - Möller '16, '16]

Characterization of $(C, p_1, \ldots, p_n) \in \overline{H^k_g(A)}$

iff $3$ meromorphic $K$-differentials of $C$, poles & zeros at nodes, K-reduced conditions

→ [BCGGM'19, Constantini - Möller - Zachhuber '19]

Construct smooth, modular compactification

$$\mathcal{P} \mathcal{E}^k_1 \overline{M_{g,n}(A)} \rightsquigglyarrow \overline{H^k_g(A)} \subseteq \overline{M_{g,n}}$$

→ [CMZ'20]

- Express the (orbifold) Euler characteristic of strata $\chi(\overline{H^k_g(A)})$ of differentials in terms of intersection numbers on $\mathcal{P} \mathcal{E}^k_1 \overline{M_{g,n}(A)}$.
- Calculate them in many examples using computers.

Example:

$\chi(\overline{H^1_2}) = -24 \delta$. 
Want formula for \( \widehat{\mathcal{H}}_g(A) \).

**A2. Moduli** \( \widehat{\mathcal{H}}_g(A) \leq \overline{M}_{g,n} \) of twisted K-differentials

\[ \widehat{\mathcal{H}}_g(A) = \{ (C, \rho_1, \ldots, \rho_n) \mid (\ast) \} \leq M_{g,n}, \]

\( \widehat{\mathcal{H}}_g(A) \cap M_{g,n} = \mathcal{H}_g(A) \)

- Normalizing twisted nodes \( N_I \)
- Twisting I on \( T_I \) such that \( \forall I \omega_{\mathbb{C}}^{\omega_k} \equiv \sum_{(\ast)} \zeta_i^{(I_i, q_i)} \cdot C_{\mathbb{C}}(\sum_{\text{heig}_{\text{gen}}} q_i - p_i) \)
- \( \Leftrightarrow \omega_{\mathbb{C}}^{\omega_k} \equiv C_{\mathbb{C}}(\sum_{\text{heig}_{\text{gen}}} (-I_i q_i) + (I_i - k q_i)) \)

**Exa.** \( \widehat{\mathcal{H}}_g^K(a_i - a) \)

- \( C \)
- \( \Delta_i \)
- \( \overline{M}_{g,n} \)
- \( \Delta_i \)

\[ \widehat{\mathcal{H}}_g^K(a_i - a) = \widehat{\mathcal{H}}_g^K(a_i - a) \cup \Delta_i \]

- \( \omega_{\mathbb{C}}^{\omega_k} \equiv C_{\mathbb{C}}(q_i - a_i - (k - k) q_i) \)
- \( \omega_E^{\omega_k} \equiv C_{\mathbb{C}}((k - k) q_i) \)
\( \text{Thom}(FP/5, S^4; \mathbb{Z}) \quad K=1 \)
\( Z = \hat{H}^1_0(A) \text{ component} \)

**Generic T, I**

- Central vertex: \( T \)
- Outlying vertex: \( V_{\text{out}}(I) \)

**Simple star graph**

\( \Rightarrow \) Components of \( \hat{H}^*_{\text{red}}(A) \) supported in boundary of \( \overline{M}_{g,n} \)

are parameterized by products of spaces \( \hat{H}^*_{\text{red}}(A) \)

\( \Rightarrow \) Motivation for definition 2

\[ (C, \mathcal{E}(\tau_{\text{red}})) \xrightarrow{\overline{M}_{g,n}} \text{Pic}_{g,0} = \{ (C, \xi) \} \]
\[ \text{Proj} \text{(BHPSS'20)} \]
\[ \hat{H}^*_{\text{red}}(A) = \sigma_{n,g}^{-1}(\overline{e}) \]

\( \overline{e} \in \text{Pic}_{g,0} \) has pure codim 2; what about \( \hat{H}^*_{\text{red}}(A) \)?

**2. Dimension theory & weighted fundamental class**

\( \text{Thom}(FP/15; \mathbb{Z}_k; S^4/6; \mathbb{Z}_k) \)

For \( k \geq 1 \), \( \hat{H}^*_{\text{red}}(A) \) has pure codim \( g \) in \( \overline{M}_{g,n} \),
except if \( A = A' \) for \( A' \in \mathbb{Z}^n_{\geq 1} \) in which case

\( \hat{H}^*_{\text{red}}(A) \subset \hat{H}^*_{\text{red}}(A') \)

in a union of copy of codim \( g-1 \).

Idea of Pf: (over \( \overline{M}_{g,n} \)) \( \sigma_{n,g} \) and \( e \) meet transversally \( \Rightarrow \hat{H}^*_{\text{red}}(A) = \overline{M}_{g,n} \) smooth

in \( \Theta \overline{M}_{g,n} \) : recursive argument \( \square \)

\( \Rightarrow \) What about cycle theory ?

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Conjecture A \((\text{Janda-Pandharipande-Pixton-Zvonkine}^{15}(k,n))\)

Let \(k \geq 1\) and \(A \neq kA^1\) for \(A \in \mathbb{Z}_{\geq 0}^n\). \(\widetilde{\Theta}_g^k(A)\) pull cographic

\[
\sum_{Z \text{ component of } \widetilde{\Theta}_g^k(A)} m_Z \cdot [Z] = \sum_{\text{expt. } \tilde{A}} \tilde{P}_g^{\tilde{A}}(\tilde{A}) \in \text{Ch}^g(\mathcal{M}_{3,n}) \quad (**)
\]

\(\text{ Proof } ([\text{HS'15}, \text{BHPSS'20}])\)

Recall

\[
\sigma_{\tilde{A}} = \sigma_{\tilde{A}^1} \quad \gamma \Downarrow \quad \text{Pixton's formula}
\]

\[
\mathcal{M}_{3,n} \longrightarrow \mathcal{M}_{3,1}
\]

\[
\implies [\text{HS'15}] \quad \text{Intersection multiplicity of } \sigma_{\tilde{A}} \text{ and } \gamma \text{ along } Z \text{ in } \widetilde{\Theta}_g^k(A)
\]

\[
\implies \text{LHS of } (**); \sigma_{\tilde{A}}(\tilde{A}) \text{ along } [\tilde{e}]
\]

\[
\implies \text{RHS of } (**); \tilde{P}_g^{\tilde{A}}(\tilde{A}) \text{ along } [\tilde{e}]
\]

\(\implies [\tilde{e}] \in \text{Ch}^g(\mathcal{P}_{g,0}) \text{ is the universal twisted DR cycle}\)

\(\approx \text{ all classical DR cycles are pullbacks under } \sigma_{\tilde{A}}: \mathcal{M}_{3,n} \longrightarrow \mathcal{P}_{g,0}\)

\(\approx \text{ What about cycles } [\tilde{\Theta}_g^k(A)]?\)

Corollary \([\tilde{\Theta}_g^k(A)] + \text{ (boundary cycle)} = \text{ (explicit formula)}\)

\[
\text{parametrized by small-dim'lspace } \mathcal{P}_{g,0}^k(\tilde{A})
\]

\(\Rightarrow \text{ we can set up recursion}\)

\(\Rightarrow \text{ recursive formula for } [\tilde{\Theta}_g^k(A)]\).

\(\approx \text{ discussion of taut. classes & Pixton's formula}\)

\(\approx \text{ proof that } [\tilde{e}] = \tilde{P}_g^{\tilde{A}} \in \text{Ch}^g(\mathcal{P}_{g,0})\)
3. Chow group of $Pic_{g_0}$

- Use operational / bivariant / Chow scheme approach (Fulton, chap 17)
- Let $S$ be finite type scheme
  $\xymatrix{
  S \ar[r] \ar@/=1pc/[rr] & Pic_{g_0} \ar[r] & C_{d,g} \ar[d]_s \\
  S \ar[r] & \text{family of curves} & + \text{line bundle}
  }$

**An operator class** $\alpha \in CH^*(Pic_{g_0})$ in data of:

$$
(\alpha(y) : CH^*(S) \to CH^*_{c}(S)) \quad \beta \quad (y^* \cap \beta) \quad \text{all such maps}
$$

Compatible with prop. pushf.,
flat pullback,
Gysin pullback.

With some work:

$$
\overline{c} \in Pic_{g_0} \to [\overline{c}] \in CH^*_h(Pic_{g_0})
$$

"Poincaré dual of zero class"

4. Tautological classes on $Pic_{g_0}$

- Define $R^*(Pic_{g_0}) \subseteq CH^*_h(Pic_{g_0})$
- Express $[\overline{c}]$ as elem. in $R^*(Pic_{g_0})$

- $c \in L \in \text{universal line bundle!}$

- $Pic_{g_0} \to \eta := F_*(c_1(s^2)) \in CH^*_h(Pic_{g_0})$

- Boundary strata of $Pic_{g_0}$

- Probable graphs $T \xymatrix{T^* \ar[d]_s \ar[r] & \eta \ar@{..>}[l]_s}$

- $\Sigma S_i = 0$
Given \( T_{2g} \) have glueing morphisms

\[
\pi_{T_2} : \text{Pic}_{T_2} \to \text{Pic}_{g,0}
\]

\[
\text{Pic}_{T_2} = \left\{ (0,0,0,0) \right\}
\]

\[
\text{Pic}_{g,0,\text{mod, str}} = \sum (C, \mu_1, \ldots, \mu_n, \xi) \]

\[
\xi_i \to \text{Pic}_{g,0,\text{mod}} \text{ line bundle, } L_i |_{(C, \mu_1, \ldots, \mu_n)} = T_{\mu_i} C
\]

\[
\Rightarrow \psi_i = c_i(L_i) \in CH^1(\text{Pic}_{g,0,\text{mod}}) \xrightarrow{\text{via } \pi_{T_2}^*} \text{ in } CH^1(\text{Pic}_{T_2})
\]

Pixton's formula (shape)

\[
P_{g,0}^\psi = \sum_{T_2, w, c} h^c(\pi_{T_2}^*) \text{ (polynomial in } \psi\text{-classes) on } \text{Pic}_{T_2}
\]

[BPSS 20]

\[
\bar{M}_{g,n,B}(X) = \sum (C, \mu_1, \ldots, \mu_n, f \to X) \text{ stable maps of degree } B
\]

\[
\text{every cycle of } C \text{ not connected by } f \text{ is stable.}
\]

\[
\text{Given } B \in H_2(X, \mathbb{Z}) : \\
\text{every cycle of } C \text{ not connected by } f \text{ is stable.}
\]

\[
\text{Given } A = (a_1, \ldots, a_n) \in \mathbb{Z}^n, w \leq a_i \Rightarrow spg(A) \text{, the paper}
\]

[BPZ] defines a DR cycle \( DR_{g, A, B}(X, L) \) on \( \bar{M}_{g,n,B}(X) \)

Compactifying the condition

\[
f^* L = C \cdot (\Sigma a_i p_i)
\]

They show a Pixton-style formula \( P_{g, A, B}^\psi(X,L) \) for \( DR_{g, A, B}(X, L) \) by localization on \( P(\mathbb{H}) \)
What we can show:

\[ Y : \overline{M}_{g,n,\beta}(X) \rightarrow \text{Pic}_{g,0} \]
\[ ((C_{\beta_1+\cdots+\beta_n}, X)) \mapsto (\gamma_1(\alpha)(-\Sigma_{\alpha})) \]

\[ \Rightarrow Y^*([E_J]) \cap \overline{M}_{g,n,\beta}(X)^{[\text{div}]} = DR_{g,\alpha,\beta}(X_{|\mathcal{C}^g}) \]
\[ Y^*(P_g^s) \cap \cdots = P_{g,\alpha,\beta}^s (X_{|\mathcal{C}^g}) \]

Idea of Proof of Main Theorem

For \( X = \mathbb{P}^n \), \( \beta = \alpha \) \[ \text{[LL]} \] we can use the maps \( Y \) above as "charts" of \( \text{Pic}_{g,0} \)

\[ \Rightarrow \text{known equal from } \text{[PPPZ]} \Rightarrow \text{[LL]} \]
\[ P_g^s \text{ act in same way on } E_{-J}^{\text{div}} \text{ via } Y \]

\[ \Rightarrow \text{ for } n, d \gg 0 \text{, there is large open subs. of } \overline{M}_{g,n,\beta}(X) \]
\[ \text{on which Witt. fund. class = usual fund. class} \]

\[ \Rightarrow \text{verify that knowing act. of } [E_J, P_g^s \text{ on there } \]
\[ \text{is enough to show equality in } \text{Ch}^*_0(\text{Pic}_{g,0}). \]