Geometrically defined cycles on moduli spaces of curves

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1. Moduli spaces of curves and their cohomology

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The moduli space of smooth curves

Definition

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The moduli space of smooth curves

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\[
\mathcal{M}_{g,n} = \left\{(C, p_1, \ldots, p_n): \right\}
\]

\( C \) smooth, compact complex algebraic curve

\( p_1, \ldots, p_n \in C \) distinct points

\( \ast \) alternatively: Riemann surface

complex manifold

of dimension 1
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\[ M_{g,n} \]

\( M_{g,n} \) is smooth, connected space of \( C \)-dimension \( 3g - 3 + n \), but not compact.
The moduli space of smooth curves

Fact

$\mathcal{M}_{g,n}$ is a smooth, connected space of $C^\infty$-dimension $3g - 3 + n$, but not compact.
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$M_{3,2}$
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\( \mathcal{M}_{g,n} \) is smooth, connected space of \( \mathbb{C} \)-dimension \( 3g - 3 + n \), but not compact.
The moduli space of stable curves

Definition (Deligne-Mumford 1969)
Let $g, n \geq 0$ be integers (with $2g - 2 + n > 0$).

$$\overline{M}_{g,n} = \left\{ (C, p_1, \ldots, p_n) : \text{Aut}(C, p_1, \ldots, p_n) \text{ finite} \right\} / \text{iso}$$
Definition (Deligne-Mumford 1969)

Let $g, n \geq 0$ be integers (with $2g - 2 + n > 0$).

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\overline{M}_{g,n} = \left\{ (C, p_1, \ldots, p_n): \begin{array}{c}
C \text{ compact complex algebraic curve of arithmetic genus } g \\
\text{with at worst nodal singularities}
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The moduli space of stable curves

Facts

1. $\overline{M}_{g,n}$ is a smooth, connected, compact space of $\mathbb{C}$-dimension $3g - 3 + n$.

2. The boundary $\partial \overline{M}_{g,n} = \overline{M}_{g,n} \setminus M_{g,n}$ is a closed subset of $\mathbb{C}$-codimension 1 (normal crossing divisor), parametrized by products of smaller-dimensional spaces $\overline{M}_{g_i,n_i}$.
Recursive boundary structure

To \((C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}\) we can associate a stable graph \(\Gamma(C, p_1, \ldots, p_n)\).
Recursive boundary structure

To \((C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}\) we can associate a stable graph \(\Gamma_{(C, p_1, \ldots, p_n)}\).

Conversely, given a stable graph \(\Gamma\) we have a gluing map

\[
\xi_{\Gamma} : \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v), n(v)} = \overline{\mathcal{M}}_{1,3} \times \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_{3,2}
\]
Recursive boundary structure

Proposition

The map $\xi : \Gamma$ is finite with image equal to

$\{(C, p_1, \ldots, p_n) : \Gamma(C, p_1, \ldots, p_n) = \Gamma\}$. 
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The cohomology $H^*(\overline{M}_{g,n})$ of $\overline{M}_{g,n}$
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- $\overline{M}_{g,n}$ compact space $\implies H^*(\overline{M}_{g,n})$ finite-dimensional $\mathbb{Q}$-algebra
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- $\overline{M}_{g,n}$ compact space $\implies H^*(\overline{M}_{g,n})$ finite-dimensional $\mathbb{Q}$-algebra
- (Poincaré duality) For all $0 \leq k \leq \dim = 2(3g - 3 + n)$, the cup product defines a nondegenerate pairing

$$H^k(\overline{M}_{g,n}) \otimes H^{\dim-k}(\overline{M}_{g,n}) \to H^{\dim}(\overline{M}_{g,n}) \cong \mathbb{Q}.$$
The cohomology $H^*(\overline{M}_{g,n})$ of $\overline{M}_{g,n}$

- $\overline{M}_{g,n}$ compact space $\implies H^*(\overline{M}_{g,n})$ finite-dimensional $\mathbb{Q}$-algebra
- (Poincaré duality) For all $0 \leq k \leq \dim = 2(3g - 3 + n)$, the cup product defines a nondegenerate pairing

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- For $S \subset \overline{M}_{g,n}$ a closed, algebraic subset of $\mathbb{C}$-codimension $d$, there exists a fundamental class

$$[S] \in H_{\dim-2d}(\overline{M}_{g,n}) \xleftarrow{\text{PD}} H^{2d}(\overline{M}_{g,n}).$$
Definition: $\psi$-classes

Let $L_i \to M_{g,n}$ be a complex line bundle, $L_i|_{(C,p_1,...,p_n)} = T^*_{p_i}C$. Then $\psi_i = c_1(L_i) \in H^2(M_{g,n})$.

Definition: $\kappa$-classes

Forgetful morphism $F: M_{g,n+1} \to M_{g,n}$, $(C,p_1,...,p_n,p_{n+1}) \mapsto (C,p_1,...,p_n)$. Then $\kappa_a = F^*(\psi_{n+1})_a \in H^2_a(M_{g,n})$. 
Natural cohomology classes on $\overline{M}_{g,n}$

**Definition: $\psi$-classes**

Let $\mathbb{L}_i \to \overline{M}_{g,n}$ be a complex line bundle, $\mathbb{L}_i|_{(C,p_1,\ldots,p_n)} = T_{p_i}^*C$

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Natural cohomology classes on $\overline{\mathcal{M}}_{g,n}$

**Definition: $ψ$-classes**

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Forgetful morphism $F : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}, (C, p_1, \ldots, p_n, p_{n+1}) \mapsto (C, p_1, \ldots, p_n)$ [C smooth]

$$κ_a = F_*( (ψ_{n+1})^{a+1} ) \in H^{2a}(\overline{\mathcal{M}}_{g,n}).$$
The tautological ring

Definition: the tautological ring

The tautological ring $RH^*(\overline{M}_{g,n}) \subset H^*(\overline{M}_{g,n})$ is spanned as a $\mathbb{Q}$-vector subspace by elements

Example

$[\kappa_1 \kappa_2] = (\xi_\Gamma)^* (\kappa_1 \otimes \psi_h) \in RH^*(\overline{M}_3, 2)$, for $\xi_\Gamma : M_{1,3} \times M_{2,1} \to M_{3,2}$ and $\alpha = \kappa_1 \otimes \psi_h \in H^*(M_{1,3} \times M_{2,1})$.
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$$(\xi\Gamma)^* \left( \prod_{\nu \in V(\Gamma)} \overline{M}_{g(\nu),n(\nu)} \right)$$

Example $\left[ \kappa_{11}^{12} \right] = (\xi\Gamma)^* (\kappa_{1} \otimes \psi_{h}) \in RH^*(\overline{M}_{3,2})$, for $\xi\Gamma : M_{1,3} \times M_{2,1} \to M_{3,2}$ and $\alpha = \kappa_{1} \otimes \psi_{h} \in H^*(M_{1,3} \times M_{2,1})$.
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Example

$$\begin{bmatrix}
1 & 1 \\
2 & 2
\end{bmatrix}$$
The tautological ring

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**Example**

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for $\xi_{\Gamma} : \overline{M}_{1,3} \times \overline{M}_{2,1} \to \overline{M}_{3,2}$
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The tautological ring

Definition: the tautological ring

The tautological ring \( RH^* (\overline{M}_{g,n}) \subset H^* (\overline{M}_{g,n}) \) is spanned as a \( \mathbb{Q} \)-vector subspace by elements

\[
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for \( \xi_\Gamma : \overline{M}_{1,3} \times \overline{M}_{2,1} \to \overline{M}_{3,2} \) and \( \alpha = \kappa_1 \otimes \psi_h \in H^* (\overline{M}_{1,3} \times \overline{M}_{2,1}) \)
Properties of the tautological ring

- explicit, finite list of generators $[\Gamma, \alpha]$ as $\mathbb{Q}$-vector space
- combinatorial description of cup product $[\Gamma, \alpha] \cdot [\Gamma', \alpha']$ (Graber-Pandharipande, 2003)
- list of many linear relations between the generators (Faber-Zagier 2000, Pandharipande-Pixton 2010, Pixton 2012, Pandharipande-Pixton-Zvonkine 2013)
- effective description of isomorphism $RH^{\dim}(\overline{M}_{g,n}) \cong \mathbb{Q}$ (Witten 1991, Kontsevich 1992)
Geometrically defined cycles

Heuristic

For many algebraic-geometric properties $\mathcal{P}$ of smooth pointed curves $(C, p_1, \ldots, p_n)$ (e.g. $\mathcal{P}(C) = \text{"C is hyperelliptic"}$):

$$S_{\mathcal{P}} = \{(C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n} : \mathcal{P}(C, p_1, \ldots, p_n) \text{ is true}\} \subset \mathcal{M}_{g,n}$$
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Goal

Decide if $[S_{\mathcal{P}}] \in H^*(\overline{\mathcal{M}}_{g,n})$ lies in $RH^*(\overline{\mathcal{M}}_{g,n})$. If so, compute formula in terms of generators.
1. Moduli spaces of curves and their cohomology

2. Cycles of twisted $k$-differentials

3. Admissible cover cycles
Meromorphic differential $k$-forms on smooth curves
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\[(T^*_PC)^\otimes_k \]

\[\omega_c^\otimes_k\]
Meromorphic differential $k$-forms on smooth curves
Strata of meromorphic $k$-differentials

**Definition**

Given $g, n, k \geq 0$ and $\mu = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ with $\sum_i m_i = k(2g - 2)$, let

$$\mathcal{H}^k_g(\mu) = \left\{ (C, p_1, \ldots, p_n) : \exists \text{ meromorphic } k\text{-differential } \eta \text{ on } C \text{ with zeros/poles at } p_i \in C \text{ of orders } m_i \right\} \subset \mathcal{M}_{g,n}.$$
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\omega_C^k \cong \mathcal{O}_C(\sum_i m_i p_i) \end{cases} \subset \mathcal{M}_{g,n}$$
Compactifying $H^k_g(\mu)$: twisted differentials

\[ \tilde{H}^k_g(\mu) = \left\{ (C, p_1, \ldots, p_n) : \text{equality of line bundles on partial normalization of } C \right\} \subset M_g, n \]
Compactifying $H^k_g(\mu)$: twisted differentials

Definition (Farkas-Pandharipande 2015)

$\tilde{H}^g_k(\mu) = \{ (C, p_1, \ldots, p_n) \} \subset \overline{M}_{g,n}$
Compactifying $\mathcal{H}_g^k(\mu)$: twisted differentials

$\mathcal{H}_g^k(\mu)$ is the moduli space of twisted differentials for a curve of genus $g$. The compactification is given by

$$\mathcal{H}_g^k(\mu) = \{ (C, p_1, \ldots, p_n) : \text{equality of line bundles on partial normalization of } C \} \subset \overline{\mathcal{M}}_{g,n}.$$
Compactifying $\mathcal{H}_g^k(\mu)$: twisted differentials

$\tilde{\mathcal{H}}_g^k(\mu) = \{ (C, p_1, \ldots, p_n) : \text{equality of line bundles on partial normalization of } C \} \subset \mathcal{M}_g$.

Diagram:
- $\overline{\mathcal{M}}_{3,2}$
- $\mathcal{H}_3^1(2,2)$
- $\mathcal{H}_1^1(2,2,4)$
- $\mathcal{H}_2^1(2)$
- normalization
Compactifying $\mathcal{H}_g^k(\mu)$: twisted differentials

**Definition (Farkas-Pandharipande 2015)**

$$\tilde{\mathcal{H}}_g^k(\mu) = \left\{ (C, p_1, \ldots, p_n) : \left( \text{equality of line bundles on partial normalization of } C \right) \right\} \subset \overline{M}_{g,n}$$
Definition (Farkas-Pandharipande 2015)

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Compactifying $\mathcal{H}_g^k(\mu)$: twisted differentials

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Dimension of moduli space of twisted $k$-differentials

Theorem ($k = 1$: Farkas-Pandharipande 2015, $k > 1$: S. 2016, Bainbridge-Chen-Gendron-Grushevsky-Möller 2016, Mondello)
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For $k \geq 1$, all components of $\tilde{H}_g^k(\mu)$ are of codimension $g$ in $\overline{M}_{g,n}$
For $k \geq 1$, all components of $\tilde{H}_g^k(\mu)$ are of codimension $g$ in $\overline{\mathcal{M}}_{g,n}$, except if $\mu = k \cdot \mu'$ for some $\mu' \geq 0$. In this case, the sublocus

$$\overline{H}_g^1(\mu') \subset \tilde{H}_g^k(\mu)$$

is a union of components of codimension $g - 1$. 

Theorem ($k = 1$: Farkas-Pandharipande 2015, $k > 1$: S. 2016, Bainbridge-Chen-Gendron-Grushevsky-Möller 2016, Mondello)
For $k \geq 1$, all components of $\tilde{H}_g^k(\mu)$ are of codimension $g$ in $\overline{M}_{g,n}$, except if $\mu = k \cdot \mu'$ for some $\mu' \geq 0$. In this case, the sublocus

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is a union of components of codimension $g - 1$.

Note

We have $\mathcal{H}_g^1(\mu') \subset \mathcal{H}_g^k(\mu)$ since

$$\omega_C \cong \mathcal{O}_C(\sum_i \frac{m_i}{k} p_i) \quad \implies \quad \omega_C^\otimes k \cong \mathcal{O}_C(\sum_i m_i p_i).$$
Conjectural relation to Pixton’s cycle

Let $k \geq 1$ and assume $\mu$ is not of the form $\mu = k \mu'$ for $\mu' \geq 0$, so $\tilde{H}_k g(\mu)$ has pure codimension $g$. Then we have

$$\sum_{Z \text{comp. of } \tilde{H}_k g(\mu)} \left[ Z \right] = 2 - g P_{g, k} g(\tilde{\mu}) \in H_{2g}(M_g, \mathbb{A})$$

for $\tilde{\mu} = (m_1 + k, \ldots, m_n + k)$. Note Pixton’s cycle $P_{g, k} g(\tilde{\mu})$ is explicit sum of generators of $RH_{2g}(M_g, \mathbb{A})$. Explicit list of components $\left[ Z \right]$, each parametrized by products of $H_{k_j} g_j(\mu_j)$.
Conjectural relation to Pixton’s cycle

Conjecture ($k = 1$ Janda-Pandharipande-Pixton-Zvonkine [FP-Appendix], $k \geq 1$ S.)
Conjectural relation to Pixton’s cycle

**Conjecture (k = 1 Janda-Pandharipande-Pixton-Zvonkine [FP-Appendix], k ≥ 1 S.)**

Let $k \geq 1$ and assume $\mu$ is not of the form $\mu = k\mu'$ for $\mu' \geq 0$, so $\tilde{H}_g^k(\mu)$ has pure codimension $g$. 

Note Pixton’s cycle $P_g^k(\tilde{\mu})$ is explicit sum of generators of $R^g H_{2g}(M_g, n)$ explicit list of components $[Z]$, each parametrized by products of $H_{k_j}^g(\mu_j)$. 
Conjectural relation to Pixton’s cycle

Conjecture ($k = 1$ Janda-Pandharipande-Pixton-Zvonkine [FP-Appendix], $k \geq 1$ S.)

Let $k \geq 1$ and assume $\mu$ is not of the form $\mu = k\mu'$ for $\mu' \geq 0$, so $\tilde{\mathcal{H}}^k_g(\mu)$ has pure codimension $g$. Then we have

$$
\sum_{Z \text{ comp. of } \tilde{\mathcal{H}}^k_g(\mu)} [Z] = \tilde{\mathcal{H}}^k_g(\mu) \in H^{2g}(\overline{M}_{g,n}),
$$

Note: Pixton’s cycle $P^g_{g}(\tilde{\mu})$ is explicit sum of generators of $\mathcal{R}H^{2g}(M_{g,n})$.
Conjectural relation to Pixton’s cycle

**Conjecture (k = 1 Janda-Pandharipande-Pixton-Zvonkine [FP-Appendix], k ≥ 1 S.)**

Let \( k \geq 1 \) and assume \( \mu \) is not of the form \( \mu = k\mu' \) for \( \mu' \geq 0 \), so \( \tilde{H}_g^k(\mu) \) has pure codimension \( g \). Then we have

\[
\sum_{Z \text{ comp. of } \tilde{H}_g^k(\mu)} \left( \begin{array}{c} \text{combinatorial factor} \\ \end{array} \right) [Z] = \quad \in H^{2g}(\overline{M}_{g,n}),
\]

Note Pixton's cycle \( P_g, k_g(\tilde{\mu}) \) is an explicit sum of generators of \( RH^{2g}(M_{g,n}) \).
Conjectural relation to Pixton’s cycle

Conjecture \((k = 1 \text{ Janda-Pandharipande-Pixton-Zvonkine [FP-Appendix], } k \geq 1 \text{ S.})\)

Let \(k \geq 1\) and assume \(\mu\) is not of the form \(\mu = k\mu'\) for \(\mu' \geq 0\), so \(\tilde{H}_{g}^{k}(\mu)\) has pure codimension \(g\). Then we have

\[
\sum_{Z \text{ comp. of } \tilde{H}_{g}^{k}(\mu)} \left( \begin{array}{c} \text{combinatorial factor} \\ \end{array} \right) [Z] = 2^{-g} P_{g}^{k}(\tilde{\mu}) \in H^{2g}(\overline{M}_{g,n}),
\]

for \(\tilde{\mu} = (m_{1} + k, \ldots, m_{n} + k)\).
Conjectural relation to Pixton’s cycle

Conjecture \((k = 1 \text{ Janda-Pandharipande-Pixton-Zvonkine [FP-Appendix], } k \geq 1 \text{ S.})\)

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\[
\sum_{Z \text{ comp. of } \tilde{H}_g^k(\mu)} \left( \text{combinatorial factor} \right) [Z] = 2^{-g} P_g^{g,k}(\tilde{\mu}) \in H^{2g}(\overline{M}_{g,n}),
\]

for \(\tilde{\mu} = (m_1 + k, \ldots, m_n + k)\).

Note

- Pixton’s cycle \(P_g^{g,k}(\tilde{\mu})\) is explicit sum of generators of \(RH^{2g}(\overline{M}_{g,n})\)
Conjectural relation to Pixton’s cycle

Conjecture ($k = 1$ Janda-Pandharipande-Pixton-Zvonkine [FP-Appendix], $k \geq 1$ S.)

Let $k \geq 1$ and assume $\mu$ is not of the form $\mu = k\mu'$ for $\mu' \geq 0$, so $\tilde{H}_g^k(\mu)$ has pure codimension $g$. Then we have

$$\sum_{Z \text{ comp. of } \tilde{H}_g^k(\mu)} \left( \begin{array}{c} \text{combinatorial factor} \\ \end{array} \right) [Z] = 2^{-g} P_{g}^{g,k}(\widetilde{\mu}) \in H^{2g}(\overline{M}_{g,n}),$$

for $\widetilde{\mu} = (m_1 + k, \ldots, m_n + k)$.

Note

- Pixton’s cycle $P_{g}^{g,k}(\widetilde{\mu})$ is explicit sum of generators of $RH^{2g}(\overline{M}_{g,n})$
- explicit list of components $[Z]$,
Conjectural relation to Pixton’s cycle

**Conjecture \((k = 1 \text{ Janda-Pandharipande-Pixton-Zvonkine [FP-Appendix], } k \geq 1 \text{ S.})\)**

Let \(k \geq 1\) and assume \(\mu\) is not of the form \(\mu = k \mu'\) for \(\mu' \geq 0\), so \(\tilde{H}^k_g(\mu)\) has pure codimension \(g\). Then we have

\[
\sum_{\text{combinatorial factor}} [Z] = 2^{-g} P^g,k(\tilde{\mu}) \in H^{2g}(\overline{M}_{g,n}),
\]

for \(\tilde{\mu} = (m_1 + k, \ldots, m_n + k)\).

**Note**

- Pixton’s cycle \(P^g,k(\tilde{\mu})\) is **explicit** sum of generators of \(RH^{2g}(\overline{M}_{g,n})\)
- **explicit** list of components \([Z]\), each parametrized by products of \(\overline{H}^{k_j}_{g_j}(\mu_j)\)
Applications and Evidence

Application : Recursion for $[\mathcal{H}_g^k(\mu)]$

The conjecture effectively determines the classes $[\mathcal{H}_g^k(\mu)]$. 
Applications and Evidence

Application: Recursion for $[\mathcal{H}_g^k(\mu)]$

The conjecture effectively determines the classes $[\mathcal{H}_g^k(\mu)]$.

Evidence

Conjecture is true for

- $g = 0$ trivial ($1 = 1$)
Applications and Evidence

Application: Recursion for $\overline{H}_g^k(\mu)$

The conjecture effectively determines the classes $\overline{H}_g^k(\mu)$.

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- $g = 0$ trivial ($1 = 1$)
- $g = 1$ (FP-Appendix)
Applications and Evidence

**Application : Recursion for** \( [\mathcal{H}_g^k(\mu)] \)

The conjecture effectively determines the classes \( [\mathcal{H}_g^k(\mu)] \).

**Evidence**

Conjecture is true for
- \( g = 0 \) trivial \((1 = 1)\)
- \( g = 1 \) (FP-Appendix)
- \( g = 2 \)
Applications and Evidence

Application : Recursion for $\left[ H^k_g(\mu) \right]$

The conjecture effectively determines the classes $\left[ H^k_g(\mu) \right]$.

Evidence

Conjecture is true for

- $g = 0$ trivial ($1 = 1$)
- $g = 1$ (FP-Appendix)
- $g = 2$
  - $k = 1$ and $\mu = (3, -1), (2, 1, -1)$ (FP-Appendix)
Applications and Evidence

Application: Recursion for $\overline{H}_g^k(\mu)$

The conjecture effectively determines the classes $\overline{H}_g^k(\mu)$.

Evidence

Conjecture is true for

- $g = 0$ trivial ($1 = 1$)
- $g = 1$ (FP-Appendix)
- $g = 2$
  - $k = 1$ and $\mu = (3, -1), (2, 1, -1)$ (FP-Appendix)
  - $k = 2$ and $\mu = (3, 1), (2, 1, 1)$ (S)
1. Moduli spaces of curves and their cohomology

2. Cycles of twisted $k$-differentials

3. Admissible cover cycles
Ramified covers of smooth curves: hyperelliptic case
Ramified covers of smooth curves: hyperelliptic case

\[ \Phi \downarrow \mathbb{P}^1 \]

\[ C \]
Ramified covers of smooth curves: hyperelliptic case
Ramified covers of smooth curves: hyperelliptic case

\[ \varphi \downarrow \mathbb{P}^1 \]

\[ C \]

\[ \mathbb{P}^1 \]
Ramified covers of smooth curves: hyperelliptic case
Ramified covers of smooth curves: hyperelliptic case

\[ \mathcal{C} \xrightarrow{\varphi} \mathbb{P}^1 \]

Points of ramification:

Points of branch:

\[ \mathbb{P}^1 \]
Ramified covers of smooth curves: hyperelliptic case

\[ \varphi : 2:1 \]

\[ \mathbb{P}^1 \cdot \cdot \cdot \cdot \cdot \]

\[ \text{ramification Points} \]

\[ \text{branch points} \]
Ramified covers of smooth curves: hyperelliptic case

Diagram:

- **C**: Smooth curve
- **$\mathbb{P}^1$**: Projective line
- **$\gamma$**: Ramified cover
- **Ramification Points**: Points where the covering ramifies
- **Conjugate Pair**: Points in the conjugate pair
- **Branch Points**: Additional points on the projective line

The diagram illustrates a 2:1 ramified cover $\gamma$ from a smooth curve $C$ to the projective line $\mathbb{P}^1$. The ramification points and conjugate pair are indicated on the curve, while the branch points are shown on the projective line.
Loci of hyperelliptic and bielliptic curves

Definition

Let $g, n, m \geq 0$ be integers with $0 \leq n \leq 2g + 2$. Define

$$
\text{Hyp}_{g, n, 2m} = \left\{ (C, (p_i)_i^{n}, (q_j, q'_j)_j^{m}) : \begin{array}{l}
C \text{ hyperelliptic} \\
\text{ram. points } p_i, \\
\text{conj. pairs } q_j, q'_j
\end{array} \right\} \subset \mathcal{M}_{g, n+2m}.
$$
Loci of hyperelliptic and bielliptic curves

**Definition**

Let $g, n, m \geq 0$ be integers with $0 \leq n \leq 2g + 2$. Define

\[ \text{Hyp}_{g,n,2m} = \left\{ (C, (p_i)_{i=1}^n, (q_j, q'_j)_{j=1}^m) : \begin{array}{l} \text{C hyperelliptic} \\ \text{ram. points } p_i, \\ \text{conj. pairs } q_j, q'_j \end{array} \right\} \subset \mathcal{M}_{g,n+2m}. \]

**Definition**

Let $g, n, m \geq 0$ be integers with $0 \leq n \leq 2g + 2$. Define

\[ \text{B}_{g,n,2m} = \left\{ (C, (p_i)_{i=1}^n, (q_j, q'_j)_{j=1}^m) : \begin{array}{l} \text{C bielliptic} \\ \text{ram. points } p_i, \\ \text{conj. pairs } q_j, q'_j \end{array} \right\} \subset \mathcal{M}_{g,n+2m}. \]
Compactification via admissible covers

Goal

Study admissible cover cycles like $Hyp_{g,n,2m}$ and $Bhyp_{g,n,2m} \in H^\ast(M_{g,n+2m})$. 

$\overline{M}_{3,1}$
Compactification via admissible covers

Goal

Study admissible cover cycles like \([\text{Hyp}, n, 2m] \) and \([\text{Bhyp}, n, 2m] \) \(\in H^*(\overline{M}_{g,n} + 2m)\).
Compactification via admissible covers

**Goal**

Study admissible cover cycles like $Hyp_{3,1}$ and $B_{3,1}$, $2m \in H^*(\bar{M}_{3,1}, \mathbb{Z})$. 

[Diagram showing $\bar{M}_{3,1}$ and an admissible cover with a 2:1 map]
Compactification via admissible covers

Goal

Study admissible cover cycles like $\text{Hyp}_{g,n,2m}$ and $\text{Bhyp}_{g,n,2m} \in H^\ast(M_g, n + 2m)$.
Goal

Study admissible cover cycles like $[\overline{\text{Hyp}_{g,n,2m}}]$ and $[\overline{\mathcal{B}_{g,n,2m}}] \in H^*(\overline{\mathcal{M}_{g,n+2m}})$.
Theorem (Faber-Pandharipande 2005)

The fundamental class $[\text{Hyp}_{g,n,2m}] \in H^{2g+2n+2m-4}(\overline{M}_{g,n+2m})$ lies in the tautological ring $RH^{2g+2n+2m-4}(\overline{M}_{g,n+2m})$. 

Note: For small $(g,n,m)$ the cycle $[\text{Hyp}_{g,n,2m}]$ is tautological, since $H^*(\overline{M}_{g,n+2m}) = RH^*(\overline{M}_{g,n+2m})$. 

---

Johannes Schmitt  
Cycles on moduli spaces of curves  
May 2019 26 / 33
Admissible cover cycles

**Theorem (Faber-Pandharipande 2005)**

The fundamental class \([\text{Hyp}_{g,n,2m}] \in H^{2g+2n+2m-4}(\overline{M}_{g,n+2m})\) lies in the tautological ring \(RH^{2g+2n+2m-4}(\overline{M}_{g,n+2m})\).

**Theorem**

The fundamental class \([\text{B}_{g,n,2m}] \in H^{2g+2n+2m-2}(\overline{M}_{g,n+2m})\) does not lie in the tautological ring \(RH^{2g+2n+2m-2}(\overline{M}_{g,n})\) for

- \((g, n, m) = (2, 0, 10)\) (Graber-Pandharipande 2003)
- \(g \geq 2\) and \(g + m \geq 12\) (van Zelm 2016)
Admissible cover cycles

**Theorem (Faber-Pandharipande 2005)**

The fundamental class $[\text{Hyp}_{g,n,2m}] \in H^{2g+2n+2m-4}(\overline{M}_{g,n+2m})$ lies in the tautological ring $RH^{2g+2n+2m-4}(\overline{M}_{g,n+2m})$.

**Theorem**

The fundamental class $[\overline{B}_{g,n,2m}] \in H^{2g+2n+2m-2}(\overline{M}_{g,n+2m})$ does not lie in the tautological ring $RH^{2g+2n+2m-2}(\overline{M}_{g,n})$ for

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For small $(g, n, m)$ the cycle $[\overline{B}_{g,n,2m}]$ is tautological, since $H^*(\overline{M}_{g,n+2m}) = RH^*(\overline{M}_{g,n+2m})$. 

Strategy for computation

Lemma (Arbarello-Cornalba 1998)

For the inclusion \( i: \partial M_g^n \to M_g^n \), the pullback \( i^*: H^k(M_g^n) \to H^k(\partial M_g^n) \) is injective for \( k \leq d(g,n) \) with

\[
d(g,n) = \begin{cases} 
  n - 4 & \text{if } g = 0, \\
  2g - 2 & \text{if } n = 0, \\
  2g - 3 + n & \text{if } g > 0, n > 0.
\end{cases}
\]
Lemma (Arbarello-Cornalba 1998)

For the inclusion $i: \partial M_g, n \to M_g, n$ the pullback $i^*: H^k(M_g, n) \to H^k(\partial M_g, n)$ is injective for $k \leq d(g, n)$ with

$$d(g, n) = \begin{cases} n - 4 & \text{if } g = 0, \\ 2g - 2 & \text{if } n = 0, \\ 2g - 3 + n & \text{if } g > 0, n > 0. \end{cases}$$
Lemma (Arbarello–Cornalba 1998)

For the inclusion $i : \partial \bar{\mathcal{M}}_{g,n} \to \bar{\mathcal{M}}_{g,n}$ the pullback

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$$d(g,n) = \begin{cases} 
  n - 4 & \text{if } g = 0, \\
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  n > 0.
\end{cases}$$
Computer package admcycles

Written in Sage (Python) with Jason van Zelm, Vincent Delecroix; based on earlier implementation by Pixton

Features
- Computations with tautological classes (products and intersection numbers)
- Verification of tautological relations
- Pullbacks and pushforwards of tautological classes under gluing morphism
- Identification of admissible cover cycles in terms of tautological cycles
Computer package admcycles

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Computer package `admcycles`

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Results

$$[\text{Hyp}_2] = 1 \quad (\approx 19\text{th century})$$
\[ [\text{Hyp}_2] = 1 \]
\[ [\text{Hyp}_3] = \frac{3}{4} \kappa_1 \]
\[ -\frac{9}{4} \begin{array}{c}
    \circ \\
    2
\end{array} - \frac{1}{8} \begin{array}{c}
    \circ \\
    2
\end{array} \]
\[ \approx 19th \text{ century} \]
\[ \left( \begin{array}{c}
    \text{Harris-Mumford} \\
    1982
\end{array} \right) \]
\[
[Hyp_2] = 1 \\
[Hyp_3] = \frac{3}{4} \kappa_1 \\
[Hyp_4] = \frac{17}{2} \kappa_2 \\
\]

\[
+ \frac{11}{12} \begin{bmatrix} \kappa_1 \\ 2 \\ 2 \end{bmatrix} - \frac{49}{8} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \frac{31}{24} \begin{bmatrix} \kappa_1 \\ 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} - \frac{163}{24} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \\
+ \frac{1}{12} \begin{bmatrix} \kappa_1 \\ 3 \\ 2 \\ 1 \end{bmatrix} + \frac{5}{8} \begin{bmatrix} 3 \\ 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{12} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \\
- \frac{3}{8} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}
\]

\(\approx 19th \text{ century} \) \\
\(\text{Harris-Mumford } 1982\) \\
\(\text{Faber-Pandharipande } 2005\)
Results

\[
[Hyp_5] = \frac{13307}{380} \kappa_3 - \frac{1583}{288} \kappa_2 \kappa_1 + \frac{37}{144} \kappa_3^3 - \frac{1943}{288} \kappa_2 \kappa_1 + \frac{5}{72} \kappa_1^2 \kappa_1 + \frac{407}{96} \kappa_2
\]
$[\text{Hyp}_6] =$
\[ [\text{Hyp}_{6}] = \left( \text{sum of } 376 \text{ terms} \right) \]
Results

\[
[Hyp_6] = \left( \text{sum of} \right. \\
\left. 376 \text{ terms} \right) \left( \text{van Zelm-S.} \right. \\
\left. 2018 \right)
\]
Other hyperelliptic and bielliptic cycles

Using `admcycles` one can compute the following cycles

### Hyperelliptic cycles $[\text{Hyp}_{g,n,2m}]$

<table>
<thead>
<tr>
<th>g</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
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<td>3</td>
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<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>m</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- (Faber-Pagani '12)
- (Harris-Mumford '82)
- (Faber-Pandharipande '05)
- (Vermeire '02)
- (Cavalieri-Tarasca '17: $n=1, \ldots, 5$)
- (Chen-Tarasca '15)

### Bielliptic cycles $[\overline{B}_{g,n,2m}]$

<table>
<thead>
<tr>
<th>g</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>m</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- (Faber '96)
- (Faber-Pagani '12)
\( \overline{M}_{g,n} \) smooth, compact moduli space

\( RH^*(\overline{M}_{g,n}) \subset H^*(\overline{M}_{g,n}) \) tautological ring, explicit generators \([\Gamma, \alpha]\)

\( \tilde{H}^k_g(\mu) \) moduli space of twisted \( k \)-differentials

- generalizes condition \( \omega_C^\otimes k \cong \mathcal{O}_C(\sum_i m_i p_i) \)
- Theorem about dimension of the components of \( \tilde{H}^k_g(\mu) \)
- Conjecture about formula for weighted fundamental class of \( \tilde{H}^k_g(\mu) \) as tautological classes

\( Hyp_{g,n,2m} \), example of admissible cover cycle

- generalizes condition \( C \) hyperelliptic with ramification points \( p_i \), conjugate pairs \( q_j, q'_j \)
- Algorithm for restriction of \( [Hyp_{g,n,2m}] \) to boundary of \( \overline{M}_{g,n} \)
- Computation of new examples of formulas for \( [Hyp_{g,n,2m}] \)

Crucial ingredient: recursive boundary structure of moduli spaces

Thank you for your attention!