Equivariant transversality for closed pseudo-holomorphic curves

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Symplectic manifolds are useful

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Can we find **global invariants**?
Yes: e.g., symplectic homology or Gromov–Witten invariants

Holomorphic curves are great

They played a role in breakthroughs such as

- proof of the Arnold conjecture (Floer et al)
- Conley conjecture (Salamon–Zehnder; Hingston, Ginzburg, . . .)
- Gromov's non-squeezing theorem; symplectic capacities
- symplectic filling problems

Behind these results: suitable symplectic invariants
defined using holomorphic curves

Defining an invariant involves dealing with a **transversality** question

What is a holomorphic curve?

(M,ω) closed $2n$-dimensional symplectic manifold.

Definition

A smooth **almost complex structure** on (M,ω) is a smooth section $J \in \Gamma(\mathrm{End}(TM))$ such that $J^2 = -\mathrm{id}$. J is **tame** if $g_J := \omega(\cdot, J\cdot) > 0$ and **compatible** if additionally g_J is symmetric.

Definition

Given an acs J on M, a closed genus g **pseudo-holomorphic curve** is a smooth map $u\colon (\Sigma_g, j) \to M$ such that $J \circ du = du \circ j$.

Definition: moduli space

Given acs J and data $g \in \mathbb{Z}_{\geq 0}$, $C \in H_2(M)$

$$\mathcal{M}(J) := \mathcal{M}_g(C,J) := \Big\{(\Sigma, j, u) \;\Big|\; u\colon (\Sigma, j) \to M \text{ closed genus } g \text{ holo. curve}, u_*[\Sigma] = C\Big\}/_{\text{reparametrisation}},$$

How to define an invariant: transversality

Dream transversality result

For generic compatible/tame a.c.s. J, the moduli space $\mathcal{M}(J)$ is a **compact smooth manifold** of **dimension** $(n-3)(2-2g)+2\langle c_1(TM), C\rangle$.

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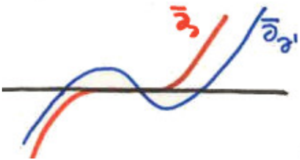
For generic compatible/tame a.c.s. J, the moduli space $\mathcal{M}(J)$ is a ~~compact~~ **compatifiable smooth** ~~manifold~~ **orbifold** of **dimension** $(n-3)(2-2g)+2\langle c_1(TM), C\rangle$ **if transversality holds**.

Deeper reason: inherent symmetry, through the automorphism group of multiple covers

Dealing with lack of transversality: common strategies



Is $\overline{\partial}_J \pitchfork 0$?

Common strategy 1: avoid it, by assuming suitable geometric hypotheses

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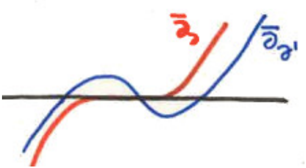

Is $\overline{\partial}_J \pitchfork 0$?

Common strategy 1: avoid it, by assuming suitable geometric hypotheses

Common strategy 2: use virtual techniques, e.g.
virtual fundamental classes, Kuranishi structures, global Kuranishi charts, domain-dependent perturbations or polyfolds

Bad news 1: no consensus which is best, or if equivalent

Bad news 2: perturbations destroy inherent symmetry

Equivariant transversality: a new approach

Accept and embrace the symmetry! Given a group acting on $\mathcal{M}(J)$:

1. Decompose $\mathcal{M}(J)$ into **iso-symmetric strata** according to the stabilisers of the action. Prove each iso-symmetric stratum is (generically) a smooth manifold.
2. To each curve u in a stratum S, associate an equivariant Fredholm operator F_u, varying smoothly with u.
   Decompose S further into **walls**

$$\{u \in S \mid \dim \ker F_u = k, \dim \operatorname{coker} F_u = c\}$$

3. Prove: each wall is (generically) a smooth submanifold of its iso-symmetric stratum.
4. Compute the dimension of each stratum and wall.

Goes back to Taubes ('96, "Counting . . . submanifolds"), extended and generalised by Wendl ('23, "super-rigidity").

Setting and standing assumptions

- (M, ω) closed $2n$-dimensional symplectic manifold
- G finite group acting symplectically on M, via $g \mapsto \psi_g$
- Then G acts

  ...on smooth maps $u \colon \Sigma \to M$ by $g \cdot u := \psi_g \circ u$,

  ...on $\mathcal{M}(J)$ by $g \cdot [u] := [\psi_g \circ u]$.

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  …on smooth maps $u\colon \Sigma \to M$ by $g \cdot u := \psi_g \circ u$,
  …on $\mathcal{M}(J)$ by $g \cdot [u] := [\psi_g \circ u]$.
- Consider the space of G-invariant a.c.s. on (M, ω):

  $$\mathcal{J}^G(M, \omega) := \{J \in \mathcal{J}(M) \text{ compatible} \mid \psi_g^* J = J \text{ for all } g \in G\},$$

  Always assume $J \in \mathcal{J}^G(M, \omega)$.
- Easy to prove: $\mathcal{J}^G(M, \omega)$ is non-empty and contractible.

Definition of iso-symmetric strata

Fix a closed oriented genus g surface Σ
Consider the moduli space of parametrised curves

$$\widetilde{\mathcal{M}}(J) := \{(j, u) \in \mathcal{J}(\Sigma) \times C^\infty(\Sigma, M) \mid [(\Sigma, j, u)] \in \mathcal{M}(J)\}$$

Consider **orbit types** w.r.t. suitable group actions:

$$\widetilde{\mathcal{M}}^A_{g,m} := \{j \text{ complex structure on } \Sigma \mid \mathrm{Aut}(\Sigma, j) \cong_{\text{conj.}} A\}$$

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"(parametrised) pre-stratum", w.r.t. the $\mathrm{Diff}_+(\Sigma)$-action on $\widetilde{\mathcal{M}}(J)$ by $\phi \cdot (j,u) := (\phi_* j, u \circ \phi^{-1})$

$$\widetilde{\mathcal{M}}^A(J) := \{(j,u) \in \widetilde{\mathcal{M}}(J) \mid j \in \widetilde{\mathcal{M}}^A_{g,m}\}$$

Definition of iso-symmetric strata (cont.)

$A \times G$ acts on $\widetilde{\mathcal{M}}^A(J)$ by $(\phi, g) \cdot (j, u) := (\phi_* j, \psi_g \circ u \circ \phi^{-1})$.
Orbit type of $H \leqslant A \times G$ is

$$\begin{aligned}\widetilde{\mathcal{M}}^{A,H}(J) :=& \{(j,u) \in \widetilde{\mathcal{M}}^A(J) \mid (A \times G)_u \cong_{\text{conj.}} H\} \\ =& \{(j,u) \in \widetilde{\mathcal{M}}(J) \mid \operatorname{Aut}(\Sigma, j) \cong_{\text{conj.}} A \text{ and } (A \times G)_u \cong_{\text{conj.}} H\}\end{aligned}$$

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Full definition of parametrised iso-symmetric strata: positive integers $\mathbf{l} = (l_1, \ldots, l_k)$,

$$\widetilde{\mathcal{M}}^{A,H}_{*,\mathbf{l}}(J) := \Big\{ v \in \widetilde{\mathcal{M}}^{A,H}(J) \mid v \text{ somewhere injective, critical points of } v \text{ have orders } \mathbf{l} \Big\}$$

Unparametrised version:

$$\mathcal{M}^{A,H}_{*,\mathbf{l}}(J) := \{[u] \in \mathcal{M}(J) \mid \exists \text{ reparametrisation in } \widetilde{\mathcal{M}}^{A,H}_{*,\mathbf{l}}(J)\}$$

How to define iso-symmetric strata (cont.)

$$\widetilde{\mathcal{M}}^{A,H}(J) := \{[(\Sigma, j, u)] \in \mathcal{M}(J) \mid (j,u) \in \mathcal{J}(\Sigma) \times C^\infty(\Sigma, M), \\ \mathrm{Aut}(\Sigma, j) \cong_{\text{conj.}} A \text{ and } (A \times G)_u \cong_{\text{conj.}} H\}$$

1. Stabiliser of u differs as parametrised and unparametrised curve!
   - parametrised curve/point-wise: have $G_u \leqslant G_{u(z)}$ for all $z \in \Sigma$. Equality need not hold, but is true for almost every point.
   - unparametrised curve/set-wise: what if $g \cdot u$ is a reparametrisation of u?

Solution: consider the $\mathrm{Aut}(\Sigma, j) \times G$-action instead; is inherently parametrised.

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   Solution: consider the $\mathrm{Aut}(\Sigma, j) \times G$-action instead; is inherently parametrised.
2. $\mathrm{Aut}(\Sigma, j)$ is semi-continuous in j, so stratify by $\mathrm{Aut}(\Sigma, j)$ first
3. Using isomorphic instead of conjugate groups also works
4. Strata depend on A and H only up to conjugation

Defining walls

- $u\colon \Sigma \overset{C^\infty}{\to} M$ is J-holomorphic iff $\overline{\partial}_J(u) := du + J \circ du \circ j = 0$
- **linearised Cauchy–Riemann operator** of $u \in \widetilde{\mathcal{M}}(J)$ is $D_u := D\overline{\partial}_J(u)\colon W^{1,p}(u^*TM) \to L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma, u^*TM))$

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- Have splitting $u^*TM = T_u \oplus N_u$, where N_u is the **generalised normal bundle**
- induces the **normal Cauchy–Riemann operator** $D_u^N := \pi_N \circ D_u|_{\Gamma(N_u)}\colon W^{1,p}(N_u) \to L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma, N_u))$
- D_u and D_u^N are Fredholm operators, depend smoothly on u

Defining walls (cont.)

Normal Cauchy–Riemann operator
$D_u^N\colon W^{1,p}(N_u) \to L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma, N_u))$ of $u \in \widetilde{\mathcal{M}}(J)$

- If $u \in \widetilde{\mathcal{M}}^{A,H}_{*,\mathbf{l}}(J)$, then D_u and D_u^N are H-equivariant
- $\Rightarrow$ ker D_u^N and coker D_u^N are H-representations
- walls in $\mathcal{M}^{A,H}_{*,\mathbf{l}}(J)$ are defined by

$$\mathcal{M}(J;k,c) := \{u \in \mathcal{M}^{A,H}_{*,\mathbf{l}}(J) \mid \dim\ker D_u^N = k, \dim\operatorname{coker} D_u^N = c\}$$

Main results

Theorem A (R. '23)

Suppose A and G are finite. There exists a co-meagre subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}^G(M,\omega)$ such that for all $J \in \mathcal{J}_{\text{reg}}$, every iso-symmetric stratum $\mathcal{M}^{A,H}_{*,\mathbf{I}}(J)$ is a smooth finite-dimensional manifold.

Theorem B (R. '24)

There exists a co-meagre subset $\mathcal{J}'_{\text{reg}} \subset \mathcal{J}_{\text{reg}}$ such that each wall $\mathcal{M}(J;k,c) \subset \mathcal{M}^{A,H}_{*,\mathbf{I}}(J)$ is a smooth submanifold. Its codimension near $u \in \mathcal{M}(J;k,c)$ is $\dim_{\mathbb{R}} Hom_H(\ker D^N_u, \operatorname{coker} D^N_u)$, where D^N_u is the *normal Cauchy–Riemann operator* of u.

Proposition C (R. '23)

The number of distinct non-empty iso-symmetric strata is countable; same for the walls´.

Proof outline of Theorem A

1. Local model: $\widetilde{\mathcal{M}}_{*,\mathbf{l}}^{A,H}(J)$ described by $(\overline{\partial}_J^H)^{-1}(0)$, for

$$\overline{\partial}_J^H : \mathcal{T} \times \mathcal{B} \to \mathcal{E}, (j, u) \mapsto du + J \circ du \circ j,$$

where $\mathcal{T}$ is an A-adapted Teichmüller slice through j,
$\mathcal{B} = \mathrm{Fix}(H) \subset W^{1,p}(\Sigma, M)$ and
$\mathcal{E}_{(j,u)} = L^p_H(\overline{\mathrm{End}}_{\mathbb{C}}((T\Sigma, j), u^* TM))$

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2. Universal moduli space
$\mathcal{U}^*(\mathcal{J}_\epsilon) = \{(u,J) \mid J \in \mathcal{J}_\epsilon, u \in \mathcal{M}^{A,H}_{*,\mathbf{I}}(J)\}$ is a smooth separable metrisable Banach manifold, $(u,J) \to J$ is smooth

3. Thus: for $J \in \mathcal{J}_\epsilon$ a regular value, $(\overline{\partial}^H_J)^{-1}(0)$ is a smooth manifold, and $\mathcal{M}^{A,H}_{*,\mathbf{I}}(J) \cong (\overline{\partial}^H_J)^{-1}(0)/A$ is a smooth manifold (as simple curves)

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4. Sard–Smale theorem: regular values are co-meagre in $\mathcal{J}_\epsilon$
5. Taubes' trick: upgrade to a co-meagre subset of $\mathcal{J}^G(M, \omega)$

Proof outline of Theorem A (cont.)

- Key step: $\mathcal{U}^*(\mathcal{J}_\epsilon)$ is smooth
- Local model: $\bar\partial_J\colon \mathcal{T} \times \mathcal{B} \times \mathcal{J}_\epsilon \to \mathcal{E}, (j, u, J) \mapsto du + J \circ du \circ j$
- Fix $(j, u, J) \in \mathcal{U}^*(\mathcal{J}_\epsilon)$ and consider

$$L\colon W^{1,p}_H(u^*TM) \oplus C^G_\epsilon(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J)) \to L^p_H(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma, u^*TM)),$$
$$(\eta, Y) \mapsto D_u\eta + Y \circ du \circ j$$

Key Lemma

If u has an injective point, then L is surjective.

Proof outline of Theorem A (cont.)

- Key step: $\mathcal{U}^*(\mathcal{J}_\epsilon)$ is smooth
- Local model: $\bar\partial_J\colon \mathcal{T}\times\mathcal{B}\times\mathcal{J}_\epsilon \to \mathcal{E}, (j,u,J)\mapsto du + J\circ du\circ j$
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Proof sketch (part 1).

1. Use Hahn-Banach theorem: suppose $\alpha\in(L^p_H)^*$ with $\alpha\neq 0$ but $\alpha|_{\mathrm{im}\,L}=0$
2. averaging: extend α to $(L^p)^*\cong L^q$, s.t. α is H-invariant

□

Proof of Key Lemma, cont.

3. $\alpha|_{\mathrm{im}\, L} = 0$ implies

$$\langle D_u\eta, \alpha\rangle = 0 \text{ for all } \eta \in W^{1,p}_H(u^*TM) \quad (1)$$

$$\langle Y \circ du \circ j, \alpha\rangle = 0 \text{ for all } Y \in C^G_\epsilon(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J)) \quad (2)$$

4. (1) implies $\alpha^{-1}(0)$ is discrete
   (H-invariance of α and pairing, unique continuation)
5. choose Y so $\langle Y \circ du \circ j, \alpha\rangle > 0$, contradiction to (2)
   - choose a good injective point $z_0 \in \Sigma$
   - choose $Y(u(z_0)) = \alpha(z_0)$
   - multiply with bump function so $\langle Y \circ du \circ j, \alpha\rangle > 0$
6. choose auxiliary sequence ϵ so $Y \in C^G_\epsilon(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J))$

Proof of Key Lemma, cont.

3. $\alpha|_{\text{im } L} = 0$ implies

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   - multiply with bump function so $\langle Y \circ du \circ j, \alpha \rangle > 0$
6. choose auxiliary sequence ϵ so $Y \in C^G_\epsilon(\overline{\text{End}}_{\mathbb{C}}(TM, J))$

Proof outline of Theorem B

- If $u \in \mathcal{M}^{A,H}_{*,\mathbf{I}}(J)$, the formal adjoint $(D^N_u)^*$ is also H-equivariant
- Implicit function theorem: present $\mathcal{M}(J;k,c)$ near $u \in \mathcal{M}(J;k,c)$ as $F^{-1}(0)$ for a suitable map

$$F\colon \text{nbhd of } u \to \mathrm{Hom}_H(\ker D^N_u, \operatorname{coker} D^N_u)$$

- **Flexibility:** if $u \in \mathcal{M}^{A,H}_{*,\mathbf{I}}(J)$ is simple, any H-equivariant section $A \in \Gamma^H(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma, N_u))$ with support within a set of good injective points satisfies $A\eta = \partial_\tau D^N_{v,\tau}\eta|_{\tau=0}$, where $D^N_{v,\tau}$ are defined w.r.t. a smooth family $(J_\tau) \subset \mathcal{J}^G(M,\omega)$ with $J_\tau = J$ along v.
- **Petri's condition:** for a co-meagre subset of $\mathcal{J}^G(M,\omega)$, the operators D^N_u for $u \in \mathcal{M}(J;k,c)$ satisfy Petri's condition Proof by reduction to the non-equivariant case

Proof outline of Proposition C

To show: number of non-empty distinct iso-symmetric strata/walls is countable

- Iso-symmetric strata: depends on the genus g
  - $g = 0$, i.e. spheres: uniformisation theorem implies $(\Sigma, j) \cong (\mathbb{S}^2, i)$, exactly one stratum
  - $g = 1$, i.e. tori: analyse model surface carefully
    - after reparametrisation, $(\Sigma, j) = (\mathbb{C}/(\mathbb{Z} + \lambda\mathbb{Z}), j_\lambda)$ for $\lambda \in \mathbb{H}$
    - conjugation and translation: reduce to $G_\lambda := \mathrm{Aut}(\mathbb{T}^2, j_\lambda, \{0\})$ details on next slide
  - $g = 2$: stable surface, so $\mathrm{Aut}(\Sigma, j)$ is finite $\mathcal{J}(\Sigma)$ is Lindelöf; finiteness, $\mathrm{Aut}(\Sigma, j)$ is semi-continuous
- For walls, is clear (as $\mathbb{N}$ is countable)

Details for the $g = 1$ case

To show: only countably many groups G_λ for $\lambda \in \mathbb{H}$

- Lefschetz fixed point theorem: G_λ injects into the mapping class group $M(\mathbb{T}^2)$
- the map $M(\mathbb{T}^2) \to \mathrm{End}(H_1(\mathbb{T}^2) \cong SL(2,\mathbb{Z}), \phi \mapsto \phi_*$
  is a group isomorphism
  thus, G_λ is discrete
- G_λ is compact: $A \in G_\lambda$ preserves basis B_λ,
  so A lies in some $\mathrm{U}(1)$
- only countably many finite subsets of a given countable set

Outlook and next steps

- strata and walls of multiply covered curves
- compute the dimensions of iso-symmetric strata and walls
- allow infinite groups A, i.e. unstable domains
- punctured holomorphic curves
  challenge for applicability: compute Conley–Zehnder indices of multiply covered Reeb orbits
- generalise to infinite groups G
- applications, e.g. equivariant super-rigidity, equivariant Gromov invariant, equivariant Gromov–Witten invariants
- beyond symplectic actions

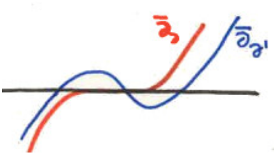
Beyond symplectic actions

- anti-symplectic involutions: $\phi \in \mathrm{Diff}(M)$ with $\phi^*\omega = -\omega$ then, want $\phi^* J = -J$ instead; $\mathcal{J}^G(M,\omega)$ is still contractible
- anti-symplectic actions: $G = \langle S \rangle$, each $s \in S$ acts by an anti-symplectic involution

Beyond symplectic actions

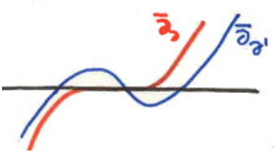
- anti-symplectic involutions: $\phi \in \mathrm{Diff}(M)$ with $\phi^*\omega = -\omega$ then, want $\phi^* J = -J$ instead; $\mathcal{J}^G(M, \omega)$ is still contractible
- anti-symplectic actions: $G = \langle S \rangle$, each $s \in S$ acts by an anti-symplectic involution
- motivation 1: celestial mechanics
- motivation 2: real Gromov–Witten theory studies *real* holomorphic curves $u \colon \Sigma \to M$, with $u \circ \sigma = \phi \circ u$, where σ is an anti-holo. involution on Σ

Summary/take-home message



1. Using holomorphic curves involves dealing with a transversality problem.
2. Traditionally, transversality and symmetry are incompatible; virtual techniques. New paradigm: equivariant transversality, through stratification of the moduli space.
3. Implemented for simple curves, w.r.t. a finite symplectic group action.

Summary/take-home message



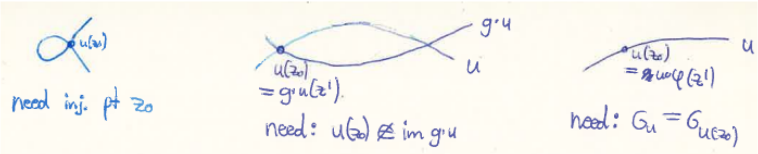
1. Using holomorphic curves involves dealing with a transversality problem.
2. Traditionally, transversality and symmetry are incompatible; virtual techniques. New paradigm: equivariant transversality, through stratification of the moduli space.
3. Implemented for simple curves, w.r.t. a finite symplectic group action.

Thanks for listening! Any questions?

Bonus slides

Detail: choosing a good injective point

Choose a good injective point $z_0 \in \Sigma$



Several conditions are required

- z_0 is an injective point
- $\alpha(z_0) \neq 0$
- if $g \cdot u$ is not a reparametrisation of u, then $u(z_0) \notin \mathrm{im}(g \cdot u)$
- $u(z_0)$ is not fixed by some reparametrisation

overall, obtain $G_u = G_{u(z_0)}$

Proof outline: Petri's condition (corrected)

- Choose $z \in \Sigma$ with $G_u = G_{u(z)}$ (open dense set)
- Prove: D_u^N satisfies Petri's condition to infinite order at all such z
- If G_u is trivial: reduce to Wendl's result (as submitted)
- G_u non-trivial: an open subset nbhd of z maps into fixed point set $M^{G_u} \subset M$
- G_u-action on M^{G_u} is trivial $\rightarrow$ no equivariance constraint then apply the argument above (careful to preserve countability)

Dimension of the iso-symmetric strata

See blackboard

Iso-symmetric strata of multiply covered curves

Suppose $u = v \circ \psi$ is multiply covered: v is simple, ψ a holomorphic branched cover

Candidate definition

The iso-symmetric stratum $\mathcal{M}^{A,H;K}_{\mathbf{b},d,\mathbf{l}}(J) \subset \mathcal{M}(J)$ consists of all curves $u = v \circ \psi$ such that

- $v \in \mathcal{M}^{A,H}_{*,\mathbf{l}}(J)$
- ψ is a degree d holomorphic branched cover, with **branching data b** $= (b_1, \dots, b_r)$
- ψ has generalised automorphism group K

This is a $2r + \dim \mathcal{M}^{A,H}_{*,\mathbf{l}}(J)$-dimensional smooth manifold.
What about $(A \times G)_u$? Can $g \in G$ act by an automorphism of ψ?
Is $(A \times G)_u$ smaller or larger than $(A \times G)_v$?

Iso-symmetric strata of multiply covered curves (cont.)

How do $(A \times G)_u$ and $(A \times G)_v$ relate?

$G_v \subset G_u$ (easy); $g \in G_u$ implies $g \circ v$ is a reparametrisation of v as

$$v \circ \psi = u = g \cdot u = \underbrace{(g \circ v)}_{\textit{simple}} \circ \psi,$$

Standard Fact

If u is multiply covered, u decomposes as $u = v \circ \psi$ for v simple and ψ a holomorphic branched cover. v is unique up to reparametrisation.

Compare $(A \times G)_u$ and $(A \times G)_v$:

- $g \circ u = u \circ \phi$ implies $(g \cdot v) \circ (\psi \circ \phi^{-1}) = v \circ \psi$, so $g \cdot v$ and v are reparametrisations
- Conversely, $g \circ v = v \circ \phi$ implies $g \circ u = v \circ (\phi \circ \psi)$...

Upshot: candidate definition looks promising

Teichmüller slices and A-adapted Teichmüller slices

Recall: Teichmüller slices

A **Teichmüller slice** through j is parametrised by an injective smooth map $\mathcal{O} \to \mathcal{J}(\Sigma), \tau \mapsto j_\tau$ such that

$$\operatorname{im} D_j \oplus T_j\mathcal{T} = L^p(\overline{\operatorname{End}}_{\mathbb{C}}(T\Sigma)),$$

with $\dim \mathcal{O} = \dim T_j\mathcal{T}$, where
$D_j := D\bar{\partial}_J(\mathrm{id})\colon W^{1,p}(T\Sigma) \to L^p(\overline{\operatorname{End}}_{\mathbb{C}}(T\Sigma))$ and $T_j\mathcal{T}$ are ...

Definition: *A*-adapted Teichmüller slices

$A \leqslant \mathit{Diff}_+(\Sigma)$ closed subgroup, suppose $j \in \mathcal{J}(\Sigma)$ has $\operatorname{Aut}(\Sigma, j) = A$. An ***A*-adapted Teichmüller slice** through j is parametrised by an injective smooth map $\mathcal{O} \to \mathcal{J}(\Sigma), \tau \mapsto j_\tau$ such that

$$D_j(W^{1,p}_A) \oplus T_j\mathcal{T} = L^p_A(\overline{\operatorname{End}}_{\mathbb{C}}(T\Sigma)),$$

with $\dim \mathcal{O} = \dim T_j\mathcal{T}$, where D_j and $T_j\mathcal{T}$ are as above.

Existence of adapted Teichmüller slices

Claim. adapted Teichmüller slices always exist.

Intuition/"moral proof"

Choose a Teichmüller slice $\mathcal{T}$ through j
which is A-invariant (as a set).

Then $\mathcal{T}_A := \mathrm{Fix}(A) \subset \mathcal{T}$ is a candidate for an A-adapted T. slice.

Rigorous proof

Assume $\mathcal{T}$ is given by the exponential map

$$\mathcal{O} \to \mathcal{J}(\Sigma), y \mapsto j_y := (\mathrm{id} + \frac{1}{2}jy)j(\mathrm{id} + \frac{1}{2}jy)^{-1}$$

for $\mathcal{O} \subset T_j\mathcal{T}$ sufficiently small, contained in some smooth complement of $\mathrm{im}\, D_j$. Then the above holds.

Infinite groups: new challenges

1. Finding a local model: fixed point set no longer works; slice theorem does not hold (reparametrisation action is not smooth)
2. Finding a large set of good injective points
3. Counterexample 1: G acts transitively
   Then $\mathcal{J}^G(M, \omega)$ is finite-dimensional, too small
4. Counterexample 2: $\mathrm{SO}(2n+1)$ acts on $\mathbb{S}^{2n} \times \mathbb{S}^2$
   all equivariant acs are biholomorphic (uniformisation theorem)
5. Candidate condition: Hamiltonian action of abelian Lie group