Formalising bordism theory

Michael B. Rothgang (he/him)

Formalised mathematics group
Universität Bonn

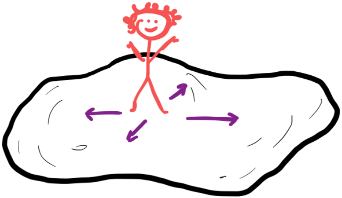
Leaning in! 2025
March 13, 2025

Outline of today's talk

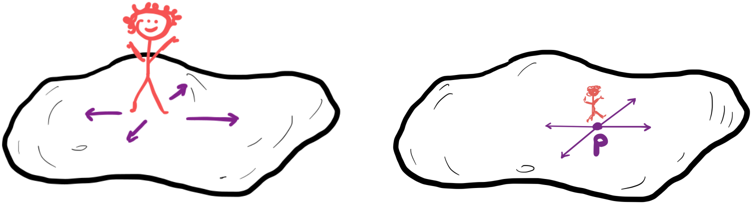
What is bordism theory?

The study of smooth manifolds up to bordism

Manifolds



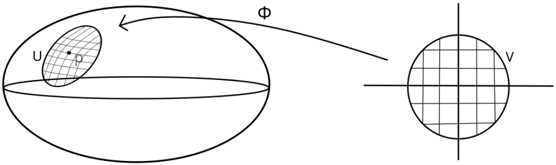
Manifolds



surface of a potato is a manifold: locally looks like a disk

Smooth manifolds

- manifold: second countable Hausdorff topological space M locally homeomorphic to open ball in $\mathbb{R}^n$
- every $p \in M$ has a coordinate chart: $p \in U \subset M$ open, homeomorphism $\phi\colon V \to U$ for $V \subset \mathbb{R}^n$ open ball
- smooth manifold: all coordinate transformations from overlapping charts are smooth



Picture courtesy of Dominik Gutwein.

Examples of smooth manifolds

- empty set (of any dimension)
- 0-dimensional: isolated points
- 1-dimensional: $\mathbb{R}$, $\mathbb{S}^1$
- n-dimensional: open disc $\mathbb{D} \subset \mathbb{R}^n$
- $n = 2$: $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{T}^2$, Σ_g for $g \geq 1$

Examples of smooth manifolds

- empty set (of any dimension)
- 0-dimensional: isolated points
- 1-dimensional: $\mathbb{R}$, $\mathbb{S}^1$
- n-dimensional: open disc $\mathbb{D} \subset \mathbb{R}^n$
- $n = 2$: $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{T}^2$, Σ_g for $g \geq 1$

- $n \geq 3$: complicated; classification for $n \geq 4$ impossible
- **not** a manifold: letter "X"

Manifolds with boundary

- interior points locally look like (open ball in) $\mathbb{R}^n$,
  boundary points look like (open ball in) upper half of $\mathbb{R}^n$
- closed manifold $=$ compact and without boundary
- manifolds with boundary and corners: details omitted
  examples: $\mathbb{S}^2$ is closed; $\overline{\mathbb{D}} \subset \mathbb{R}^2$ has boundary; $[0,1]^2 \subset \mathbb{R}^2$ has corners

Fact

The boundary ∂M of a smooth $n+1$-dimensional manifold M is a smooth n-manifold.

Question

Is every closed smooth n-dimensional manifold the boundary of a smooth $n+1$-dimensional manifold?

What is bordism theory?

Question

Is every closed smooth n-dimensional manifold the boundary of a smooth $n+1$-dimensional manifold?

Answer: Yes, for stupid reasons: $M = \partial([0,\infty) \times M)$.

What is bordism theory?

Question

Is every closed smooth n-dimensional manifold the boundary of a smooth $n+1$-dimensional manifold?

Answer: Yes, for stupid reasons: $M = \partial([0,\infty) \times M)$.

Better question

Is every closed smooth n-dimensional manifold M the boundary of a compact smooth $n+1$-dimensional manifold W?

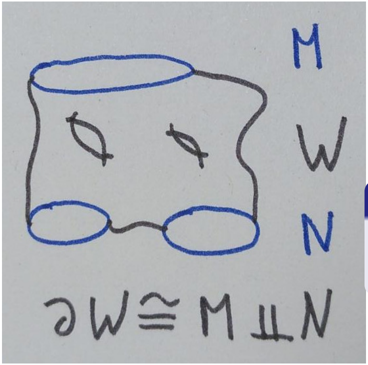
Answer

No, e.g. $M = \mathbb{CP}^2$ is not (by Poincaré duality).

What is bordism theory?

Definition

A **smooth bordism** between smooth n-manifolds M and N is a compact $n+1$-dimensional manifold W such that $\partial W = M \sqcup N$.



We call M and N **bordant** if there exists a smooth bordism between them.

Fact

Being bordant is an equivalence relation.

What is bordism theory?

Definition

The n-th unoriented **bordism group** is

$$\Omega_n^O := \{\text{closed smooth } n\text{-manifolds}\}/\text{bordism}.$$

$\Omega_*^O := \oplus_{n \geq 0} \Omega_n^O$ is called the unoriented **bordism ring**. Binary operations pass to bordism classes: disjoint union resp. product of manifolds.

Theorem

Each Ω_n^O is an abelian group; Ω^O is a (graded commutative) ring.

Why study bordism theory?

- it's beautiful
- great test case of the library
- exotic spheres and the Hirzebruch signature theorem
- defines an (extraordinary) homology theory

Motivation: existence of exotic spheres

Question

Are there topological manifolds without a smooth structure?

- Low dimensions: no, e.g. by explicit classification
- Dimension $4k$: yes!

Theorem (Milnor '56)

There exists a smooth manifold S which is homeomorphic, but not diffeomorphic to $\mathbb{S}^7$.

Every smooth manifold M has **intersection form** with **signature** $\sigma(M) \in \mathbb{Z}$

Theorem (Hirzebruch signature theorem)

Each closed oriented smooth 8-manifold M satisfies

$$\sigma(M) = \frac{1}{45}\langle 7p_2(M) - p_1(M) \cup p_1(M), [M]\rangle.$$

Existence of exotic spheres: outline of proof

Theorem (Milnor '56)

There exists a smooth manifold S which is homeomorphic, but not diffeomorphic to $\mathbb{S}^7$.

1. Clever construction ("plumbing of spheres") of a smooth 8-manifold X with simply connected boundary $Y = \partial X$ such that $\sigma(X) = 8$, $p_1(X) = p_2(X) = 0$ and $H_2(Y) = H_3(Y) = 0$
2. Compute: Y is homotopy equivalent to $\mathbb{S}^7 \overset{\text{Smale}}{\Rightarrow} Y$ homeomorphic to $\mathbb{S}^7$
3. If Y were diffeomorphic to $\mathbb{S}^7$, consider $M := X \cup_{\mathbb{S}^7} \mathbb{D}^8$. Compute $\sigma(M) = 8$ and $p_1(M) = 0$, so

$$45\sigma(M) = 45 \cdot 8 = 7\langle p_1(M), [M]\rangle \in 7\mathbb{Z},$$

contradiction!

Ingredients for the Hirzebruch signature theorem

- The signature defines a ring homomorphism $\Omega_*^{SO} \to \mathbb{Z}, [M] \mapsto \sigma(M)$.
- $\Omega_*^{SO} \otimes \mathbb{Q}$ is (graded ring) isomorphic to $\mathbb{Q}[x_4, x_8, \dots]$, where each generator x_{4k} is represented by $\mathbb{CP}^{2k}$
- Computation: $\sigma(\mathbb{CP}^{2n}) = 1$ for all n
- Corollary: any ring homomorphism $\Psi \colon \Omega_*^{SO} \to \mathbb{Q}$ satisfying $\Psi([\mathbb{CP}^{2n}]) = 1$ for all n satisfies $\Psi([M]) = \sigma(M)$ for every closed oriented smooth manifold M
- Algebraic trick ("L-genus") to deduce the theorem

Motivation: homology theories

Question

When are two topological spaces "the same" (homeomorphic)?
How can we prove two spaces are different?

Algebraic invariants: different values means spaces are non-homeomorphic
Common algebraic invariants

- homotopy groups: really hard to compute
- (singular, simplicial, cellular, Morse) homology groups:
  $(X, A) \mapsto \{H_n(X, A)\}_{n\in\mathbb{N}}$, abelian

Motivation: homology theories

Question

When are two topological spaces "the same" (homeomorphic)?
How can we prove two spaces are different?

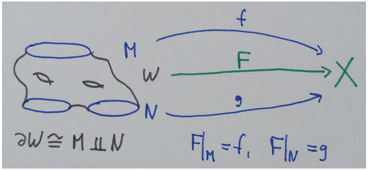
Algebraic invariants: different values means spaces are non-homeomorphic
Common algebraic invariants

- homotopy groups: really hard to compute
- (singular, simplicial, cellular, Morse) homology groups: $(X, A) \mapsto \{H_n(X, A)\}_{n \in \mathbb{N}}$, abelian
- Eilenberg-Steenrod axioms characterise homology theories
- Singular homology: widely used, but proving the axioms is painful
- Bordism theory: proving the axioms is really easy

Dream goal

Bordism theory as first proven homology theory in mathlib

Bordism theory as a homology theory



Fix a topological space X.
A **singular** n**-manifold** on X is a pair (M, f) of a smooth closed n-manifold M and a continuous map $f\colon M \to X$.

A **bordism** between singular n-manifolds (M, f) and (N, g) is a compact $n+1$-manifold W with a continuous map $F\colon W \to X$ such that $\partial W \cong M \sqcup N$, $F|_M = f$ and $F|_N = g$.

Definition

The n-th unoriented bordism group of X is $\Omega_n^O(X) := \{\text{singular } n\text{-manifolds on } X\}/\text{bordism}$.

Example. For $X = \{*\}$, we recover the bordism groups Ω_n^O.

Existing work and new contribution

- Building on mathlib's differential geometry library
- Everything in a branch of mathlib/aiming for mathlib
- Lots of ground-work already existed
    - general theory of smooth manifolds
    - interval $[a, b]$ (for $a < b$) is a manifold; products of manifolds
    - disjoint unions of top. spaces

New contributions to mathlib

- discrete spaces are 0-dimensional manifolds (and conversely)
- disjoint union of manifolds
- interior and boundary of a manifold
- boundary of a disjoint union, product; $\partial[a, b] = a, b$
- disjoint union of two embeddings is an embedding (with Aaron Liu)
- new notion "this manifold has smooth boundary", basic instances

New contributions to mathlib (cont.)

- singular n-manifolds and basic constructions
- unoriented cobordisms and bordism classes
- bordism relation is an equivalence relation: done except transitivity
- (absolute) bordism groups; proof of abelian group: virtually done

Missing/next steps

- differential of the inclusion, at a product (easy)
- proof of the collar neighbourhood theorem (hard; omit)
- transitivity of the bordism relation
- finish proving the group laws (easy)

Mathlib's manifold design

- mathlib has a very general definition of manifolds
  - infinite-dimensional case included (e.g. Banach manifolds)
  - over any field: e.g. $\mathbb{R}$, $\mathbb{C}$ or p-adic numbers
  - allows boundary, corners (and even more)

Mathlib's manifold design

- mathlib has a very general definition of manifolds
  - infinite-dimensional case included (e.g. Banach manifolds)
  - over any field: e.g. $\mathbb{R}$, $\mathbb{C}$ or p-adic numbers
  - allows boundary, corners (and even more)
- the data of a manifold (example: $\overline{\mathbb{D}}$)
  - M: the manifold (e.g. $\overline{\mathbb{D}}$)
  - H: the local model, a topological space (e.g. $\mathbb{H}$)
  - E: normed space (e.g. $\mathbb{R}^2$)
  - I: model with corners, continuous map $H \to E$ (e.g. canonical inclusion)
  - charts on M (one preferred chart at each point)
  - compatibility condition: transition maps lie in structure groupoid
- why? abstract to clarify, re-usability

Design decisions: singular n-manifolds

Abridged code:

```
structure SingularNManifold.{u} (X : Type*) [TopologicalSpace X] (k : WithTop ℕ∞)
  {E H : Type*} [NormedAddCommGroup E] [NormedSpace ℝ E] [FiniteDimensional ℝ E]
  [TopologicalSpace H] (I : ModelWithCorners ℝ E H) where
  M : Type u
  f : M → X
  hf : Continuous f
```

- bundled design, to allow using in the definition of bordism groups
- include smoothness exponent explicitly:
  allow smooth manifolds, but also C^k or analytic
- model with corners as a type explicit parameter
  disjoint union and bordism needs matching model on components
- non-ideal: type parameter in the definition with new universe variable
  but: X need not be related to M, want to enable functoriality

```
def map (s : SingularNManifold X k I)
    {φ : X → Y} (hφ : Continuous φ) : SingularNManifold Y k I where
  f := φ ∘ s.f
  hf := hφ.comp s.hf
```

Design decisions: manifolds with smooth boundary

- initial design: consider the set of boundary points, endow with smooth structure
- painful to work with, because of propositional equality of types
  - e.g. if M is closed, $\partial(M \times N) = M \times \partial N$ is not def-eq, so cannot re-use a general product construction
  - closed manifolds have empty boundary: only propositionally
- better design: consider boundary as **embedded smooth submanifold**, i.e. choose a smooth manifold M_0 with a smooth embedding $f : M_0 \to M$ s.t. $\mathrm{range} f = \partial M$

Design decisions: manifolds with smooth boundary (cont.)

Abridged definition:

```
structure BoundaryManifoldData.{u} (M : Type u) [TopologicalSpace M] [ChartedSpace H M]
    (I : ModelWithCorners ℝ E H) (k : WithTop ℕ∞) [IsManifold I k M]
    {E₀ H₀: Type*} [NormedAddCommGroup E₀] [NormedSpace ℝ E₀]
    [TopologicalSpace H₀] (I₀ : ModelWithCorners ℝ E₀ H₀) where
  /-- A `C^k` manifold `M₀` which describes the boundary of `M` -/
  M₀: Type u
  [isManifold : IsManifold I₀ k M₀]
  /-- A `C^k` map from the model manifold into `M`, which is required to be a smooth embedding,
  i.e. a `C^k` immersion which is also a topological embedding -/
  f: M₀ → M
  isEmbedding: Topology.IsEmbedding f
  contMDiff: ContMDiff I₀ I k f
  /-- If `f` is `C¹`, it is an immersion: this condition is vacuous for `C⁰` maps. -/
  isImmersion: (1 : WithTop ℕ∞) ≤ k → ∀ x, Function.Injective (mfderiv I₀ I f x)
  range_eq_boundary: Set.range f = I.boundary M
```

- no mathlib definition of immersions/smooth embeddings yet; infinite-dimensional definition is different
- type field is needed; choose to align universe to M

Definition of unoriented bordisms

```
structure UnorientedBordism.{u, v} {X E H E' H' : Type*}
    [TopologicalSpace X] [TopologicalSpace H] [TopologicalSpace H']
    [NormedAddCommGroup E] [NormedSpace ℝ E] [NormedAddCommGroup E'] [NormedSpace ℝ E']
    (k : WithTop ℕ∞) {I : ModelWithCorners ℝ E H} [FiniteDimensional ℝ E]
    (s : SingularNManifold.{u} X k I) (t : SingularNManifold.{v} X k I)
    (J : ModelWithCorners ℝ E' H') where
  /-- The underlying compact manifold of this unoriented bordism -/
  W : Type (max u v)
  [compactSpace : CompactSpace W]
  [isManifold: IsManifold J k W]
  /-- The presentation of the boundary `W` as a smooth manifold -/
  -- Future: we could allow bd.M₀ to be modelled on some other model, not necessarily I:
  -- we only care that this is fixed in the type.
  bd: BoundaryManifoldData W J k I
  /-- A continuous map `W → X` of the bordism into the topological space we work on -/
  F : W → X
  hF : Continuous F
  /-- The boundary of `W` is diffeomorphic to the disjoint union `M ⊔ M'`. -/
  φ : Diffeomorph I I (s.M ⊕ t.M) bd.M₀ k
  /-- `F` restricted to `M ↪ ∂W` equals `f`: this is formalised more nicely as
  `f = F ∘ ι ∘ φ⁻¹ : M → X`, where `ι : ∂W → W` is the inclusion. -/
  hFf : F ∘ bd.f ∘ φ ∘ Sum.inl = s.f
  /-- `F` restricted to `N ↪ ∂W` equals `g` -/
  hFg : F ∘ bd.f ∘ φ ∘ Sum.inr = t.f
```

- bundled design, like `SingularNManifold`
- note: no requirement $\dim W = \dim M + 1$ yet (just for transitivity)
- model parameters I (for the boundary) and J (for the bordism)
  later applications take J as the product of I and the model for $[0, 1]$
- universe choice: take W in universe `max u v`

Outlook: future possibilities

- define the bordism ring, multiplication: not hard
- prove it's a ring: distributivity requires the inverse function theorem (in progress)
- relative bordism groups
    - generalise both singular n-manifolds and bordisms
    - describe the boundary of manifolds with corners
    - define a homology functor (probably easy)
    - show the Eilenberg-Steenrod axioms: mostly easy
      interesting: boundary is a smooth manifold
      (false without co-dimension condition)
- *oriented* bordism groups: mostly straightforward, but requires oriented manifolds and induced boundary orientation (missing)
- for mathlib: need a general definition of smooth immersions and embeddings

Summary

1. Bordism theory is an *extra-ordinary* homology theory.
2. Applications: Hirzebruch signature theorem, existence of exotic spheres
3. Formalisation is a good test of mathlib's differential geometry section
4. Be patient and prepared to fill in missing API.
   Avoid propositional equality of types.
   Be careful with your universes.

Thanks for listening! Any questions?

Thanks for listening! Any questions?

Thanks for listening! Any questions?

Where did I cheat?

Thanks for listening! Any questions?

Where did I cheat?

Dream goal

Bordism theory as first proven homology theory in mathlib

Thanks for listening! Any questions?

Where did I cheat?

Dream goal

Bordism theory as first proven homology theory in mathlib

Answer: boundary map for homology requires proving "∂M is a $\dim M - 1$-dimensional manifold".

- Uses: interior and boundary are independent of the chosen chart.

Thanks for listening! Any questions?

Where did I cheat?

Dream goal

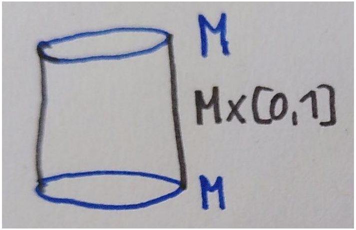
Bordism theory as first proven homology theory in mathlib

Answer: boundary map for homology requires proving "∂M is a $\dim M - 1$-dimensional manifold".

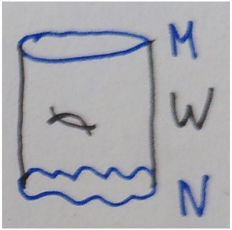
- Uses: interior and boundary are independent of the chosen chart.
- Uses: invariance of domain, e.g. via singular homology of spheres

Upshot: this requires singular homology (or similar) first

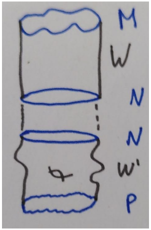
Some proof sketches about bordism classes



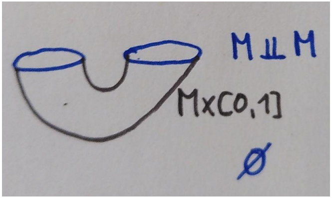
Reflexivity: the trivial bordism



Symmetry: turn upside down



Transitivity: glue bordisms along their common boundary



Every element has order two in $\Omega_n^O(X)$