Formalising bordism theory: a test of mathlib's differential geometry library

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Formalised mathematics seminar
April 28, 2025

Slides will be at `https://www.math.uni-bonn.de/people/rothgang/`

Outline of today's talk

1. What is bordism theory?
2. Motivation
   - Existence of exotic spheres
   - Homology theories
3. Formalisation overview
   - Existing work and new contribution
   - Formalisation design decisions
4. Outlook
5. Formalising smooth embeddings and submanifolds

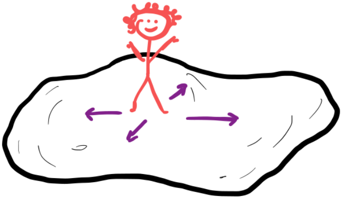
Before we begin: some timeline

- July 2024: PhD thesis submitted, learned about bordism theory
- August 2024: first formalisation attempt, failed badly
- January 2025: better definition, made great progress
- March 2025: mostly done; Berlin talk
- March/April: need smooth embeddings to be mathlib-ready
- April 2025: define smooth embeddings and submanifolds (in progress)

What is bordism theory? A non-answer

The study of smooth manifolds up to bordism

What is a manifold?



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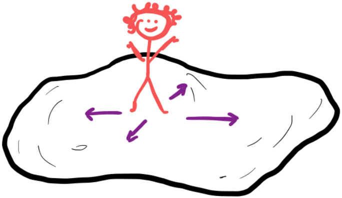
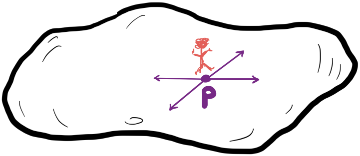
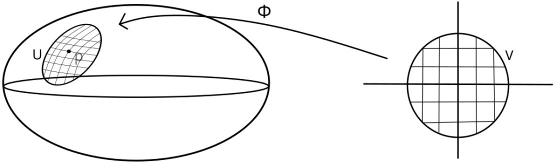




surface of a potato is a manifold: locally looks like a disk

Smooth manifolds

- topological **manifold**: second countable Hausdorff topological space M locally homeomorphic to open ball in $\mathbb{R}^n$
- every $p \in M$ has a coordinate chart: $p \in U \subset M$ open, homeomorphism $\phi \colon V \to U$ for $V \subset \mathbb{R}^n$ open ball
- **smooth manifold**: all coordinate transformations from overlapping charts are smooth



Picture courtesy of Dominik Gutwein.

Examples of smooth manifolds

- empty set (of any dimension)
- 0-dimensional: isolated points
- 1-dimensional: $\mathbb{R}$, $\mathbb{S}^1$
- n-dimensional: open disc $\mathbb{D} \subset \mathbb{R}^n$
- $n = 2$: $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{T}^2$, Σ_g for $g \geq 1$

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- $n = 2$: $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{T}^2$, Σ_g for $g \geq 1$

- $n \geq 3$: complicated; classification for $n \geq 4$ impossible
- **not** a manifold: letter "X"

Smooth manifolds with boundary

- interior points locally look like (open ball in) $\mathbb{R}^n$,
  boundary points look like (open ball in) upper half of $\mathbb{R}^n$
- closed manifold: compact and without boundary
- manifold with boundary and corners: details omitted
- examples: $\mathbb{S}^2$ is closed; $\overline{\mathbb{D}} \subset \mathbb{R}^2$ has boundary; $[0,1]^2 \subset \mathbb{R}^2$ has corners

Fact

The boundary ∂M of a smooth $n+1$-dimensional manifold M is a smooth n-manifold.

Question

Is every closed smooth n-dimensional manifold the boundary of a smooth $n+1$-dimensional manifold?

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Answer. Yes, for stupid reasons: $M = \partial([0, \infty) \times M)$.

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Answer. Yes, for stupid reasons: $M = \partial([0, \infty) \times M)$.

Better question

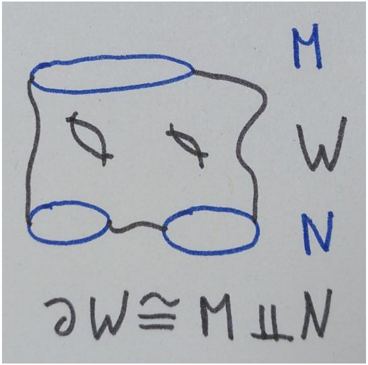
Is every closed smooth n-dimensional manifold M
the boundary of a compact smooth $n+1$-dimensional manifold?

Answer. No, e.g. $M = \mathbb{CP}^2$ is not (by Poincaré duality).

What is bordism theory?

Definition

A **smooth bordism** between smooth n-manifolds M and N is a compact $n+1$-dimensional manifold W such that $\partial W = M \sqcup N$.



We call M and N **bordant** if there exists a smooth bordism between them.

Fact

Being bordant is an equivalence relation.

The bordism groups

Definition

The n-th unoriented **bordism group** is

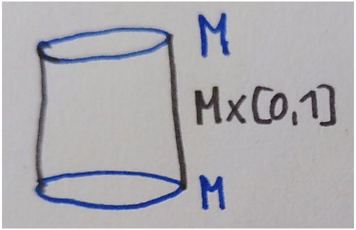
$$\Omega_n^O := \{\text{closed smooth } n\text{-manifolds}\}/\text{bordism}.$$

$\Omega_*^O := \oplus_{n \geq 0} \Omega_n^O$ is called the unoriented **bordism ring**. Binary operations pass to bordism classes: disjoint union resp. product of manifolds.

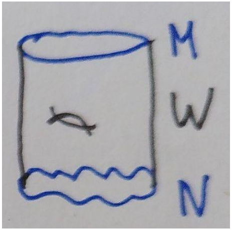
Theorem

Each Ω_n^O is an abelian group; Ω^O is a (graded commutative) ring.

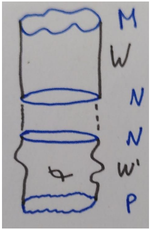
Some proofs by picture



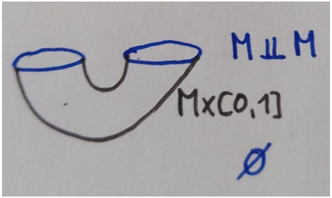
Reflexivity: the trivial bordism



Symmetry: turn upside down



Transitivity: glue bordisms along their common boundary



Every unoriented bordism class has order two.

Why study bordism theory?

- it's beautiful
- great test for differential geometry in mathlib´
- exotic spheres and the Hirzebruch signature theorem
- defines an (extraordinary) homology theory

Motivation: existence of exotic spheres

Question

Are there topological manifolds without a smooth structure?

- Low dimensions: no, e.g. by explicit classification
- Dimension $4k$: yes!

Theorem (Milnor '56)

There exists a smooth manifold S which is homeomorphic, but not diffeomorphic to $\mathbb{S}^7$.

A smooth manifold M has an **intersection form** with **signature** $\sigma(M) \in \mathbb{Z}$.

Theorem (Hirzebruch signature theorem for 8-manifolds)

Each closed oriented smooth 8-manifold M satisfies

$$\sigma(M) = \frac{1}{45}\langle 7p_2(M) - p_1(M) \cup p_1(M), [M]\rangle.$$

Existence of exotic spheres: outline of proof

Theorem (Milnor '56)

There exists a smooth manifold S which is homeomorphic, but not diffeomorphic to $\mathbb{S}^7$.

1. Clever construction ("plumbing of spheres") of a smooth 8-manifold X with simply connected boundary $Y = \partial X$ such that $\sigma(X) = 8$, $p_1(X) = p_2(X) = 0$ and $H_2(Y) = H_3(Y) = 0$
2. Compute: Y homotopy equivalent to $\mathbb{S}^7 \overset{\text{Smale}}{\Rightarrow} Y$ homeomorphic to $\mathbb{S}^7$
3. If Y were diffeomorphic to $\mathbb{S}^7$, consider $M := X \cup_{\mathbb{S}^7} \mathbb{D}^8$. Compute $\sigma(M) = 8$ and $p_1(M) = 0$, then Hirzebruch implies

$$45\,\sigma(M) = 45 \cdot 8 = 7\langle p_1(M), [M]\rangle \in 7\mathbb{Z},$$

contradiction!

Ingredients for the Hirzebruch signature theorem

- The signature defines a ring homomorphism $\Omega_*^{SO} \to \mathbb{Z}, [M] \mapsto \sigma(M)$.
- $\Omega_*^{SO} \otimes \mathbb{Q}$ is (graded ring) isomorphic to $\mathbb{Q}[x_4, x_8, \dots]$, where each generator x_{4k} is represented by $\mathbb{CP}^{2k}$
- Computation: $\sigma(\mathbb{CP}^{2n}) = 1$ for all n
- Corollary: any ring homomorphism $\Psi\colon \Omega_*^{SO} \to \mathbb{Q}$ satisfying $\Psi([\mathbb{CP}^{2n}]) = 1$ for all n satisfies $\Psi([M]) = \sigma(M)$ for every closed oriented smooth manifold M
- Algebraic trick ("L-genus") to deduce the theorem

Upshot: the existence of exotic spheres requires bordism theory

Motivation: homology theories

Question

When are two topological spaces "the same" (homeomorphic)?
How can we prove two spaces are different?

Algebraic invariants: different values means spaces are non-homeomorphic
Common algebraic invariants

- homotopy groups: really hard to compute
- (singular, simplicial, cellular, Morse) homology groups: $(X, A) \mapsto$ abelian groups $\{H_n(X, A)\}_{n \in \mathbb{N}}$

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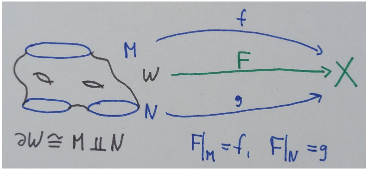
- (singular, simplicial, cellular, Morse) homology groups: $(X, A) \mapsto$ abelian groups $\{H_n(X, A)\}_{n \in \mathbb{N}}$
- Eilenberg-Steenrod axioms characterise homology theories
- Singular homology: widely used, but proving the axioms is painful
- Bordism theory: proving the axioms is really easy

Dream goal

Bordism theory as first proven homology theory in mathlib

Bordism theory as a homology theory



Fix a topological space X.
A **singular** ***n*-manifold** on X is a pair (M, f) of a smooth closed n-manifold M and a continuous map $f\colon M \to X$.

A **bordism** between singular n-manifolds (M, f) and (N, g) is a compact $n+1$-manifold W with a continuous map $F\colon W \to X$ such that $\partial W \cong M \sqcup N$, $F|_M = f$ and $F|_N = g$.

Definition

The n-th unoriented **bordism group** of X is $\Omega_n^O(X) := \{\text{singular } n\text{-manifolds on } X\}/\text{bordism}$.

Example. For $X = \{*\}$, we recover the bordism groups Ω_n^O.

Existing work and new contribution

- Building on mathlib's differential geometry library
- Everything in a branch of mathlib/aiming for mathlib
- Lots of ground-work already existed
  - general theory of smooth manifolds
  - interval $[a, b]$ (for $a < b$) is a manifold; products of manifolds
  - disjoint unions of top. spaces

New contributions to mathlib: pre-requisites

- discrete spaces are 0-dimensional manifolds (and conversely)
- disjoint union of manifolds
- interior and boundary of a manifold
- boundary of a disjoint union, product; $\partial[a, b] = \{a, b\}$
- disjoint union of two embeddings is an embedding (with Aaron Liu)
- new notion "this manifold has smooth boundary", basic instances

New contributions to mathlib (cont.)

- singular n-manifolds and basic constructions
- unoriented bordisms and bordism classes
- bordism relation is an equivalence relation: done except transitivity
- (absolute) bordism groups; proof of abelian group: virtually done

Missing/next steps

- differential of the inclusion, differential at a product (easy)
- proof of the collar neighbourhood theorem: hard/large
- transitivity of the bordism relation
- remaining group properties

Mathlib's manifold design

- mathlib has a very general definition of manifolds
  - infinite-dimensional case included (e.g. Banach manifolds)
  - over any field: e.g. $\mathbb{R}$, $\mathbb{C}$ or p-adic numbers
  - allows boundary and corners (and even more)

Mathlib's manifold design

- mathlib has a very general definition of manifolds
  - infinite-dimensional case included (e.g. Banach manifolds)
  - over any field: e.g. $\mathbb{R}$, $\mathbb{C}$ or p-adic numbers
  - allows boundary and corners (and even more)
- the data of a manifold (example: $\overline{\mathbb{D}}$)
  - M: the manifold (e.g. $\overline{\mathbb{D}}$)
  - H: the local model, a topological space (e.g. $\mathbb{H}$)
  - E: normed space (e.g. $\mathbb{R}^2$)
  - I: model with corners, continuous map $H \to E$ (e.g. canonical inclusion)
  - charts on M (one preferred chart at each point)
  - compatibility condition: transition maps lie in structure groupoid
- why? abstract to clarify, re-usability

Design decisions: singular manifolds

Abridged code:

```
structure SingularManifold.{u} (X : Type*) [TopologicalSpace X] (k : WithTop ℕ∞)
  {E H : Type*} [NormedAddCommGroup E] [NormedSpace ℝ E] [FiniteDimensional ℝ E]
  [TopologicalSpace H] (I : ModelWithCorners ℝ E H) where
  M : Type u
  [CompactSpace M] [BoundarylessManifold I M]
  f : M → X
  hf : Continuous f
```

- bundled design, to allow using in the definition of bordism groups
- include smoothness exponent explicitly:
  allow smooth manifolds, but also C^k or analytic
- model with corners as a type explicit parameter
  disjoint union and bordism needs matching model on components
- non-ideal: type parameter in the definition with new universe variable
  but: X need not be related to M, want to enable functoriality

```
def map.{u} (s : SingularManifold.{u} X k I) {φ : X → Y} (hφ : Continuous φ) :
    SingularManifold.{u} Y k I where
  f := φ ∘ s.f
  hf := hφ.comp s.hf
```

Design decisions: manifolds with smooth boundary

- initial design: consider the set of boundary points, endow with smooth structure
- painful to work with, because of propositional equality of types
  - e.g. if M is closed, $\partial(M \times N) = M \times \partial N$ is not definitionally equal thus cannot re-use a general product construction
  - closed manifolds have empty boundary: only propositionally
- better design: consider boundary as **embedded smooth submanifold**, i.e. choose a smooth manifold M_0 with a smooth embedding $f : M_0 \to M$ s.t. $\operatorname{range} f = \partial M$

Design decisions: manifolds with smooth boundary (cont.)

Abridged definition:

```
structure BoundaryManifoldData.{u} (M : Type u) [TopologicalSpace M] [ChartedSpace H M]
    (I : ModelWithCorners ℝ E H) (k : WithTop ℕ∞) [IsManifold I k M]
    {E₀ H₀: Type*} [NormedAddCommGroup E₀] [NormedSpace ℝ E₀]
    [TopologicalSpace H₀] (I₀ : ModelWithCorners ℝ E₀ H₀) where
  /-- A `C^k` manifold `M₀` which describes the boundary of `M` -/
  M₀: Type u
  [isManifold : IsManifold I₀ k M₀]
  /-- A `C^k` map from the model manifold into `M`, which is required to be a smooth embedding,
  i.e. a `C^k` immersion which is also a topological embedding -/
  f: M₀ → M
  isEmbedding: Topology.IsEmbedding f
  contMDiff: ContMDiff I₀ I k f
  /-- If `f` is `C¹`, it is an immersion: this condition is vacuous for `C⁰` maps. -/
  isImmersion: (1 : WithTop ℕ∞) ≤ k → ∀ x, Function.Injective (mfderiv I₀ I f x)
  range_eq_boundary: Set.range f = I.boundary M
```

- type field is needed; choose to align universe to M
- real definition asks for `IsSmoothEmbedding I₀ I k f` instead in finite dimension, is equivalent the snippet above

Definition of unoriented bordisms

```
structure UnorientedBordism.{u, v} {X E H E' H' : Type*}
    [TopologicalSpace X] [TopologicalSpace H] [TopologicalSpace H']
    [NormedAddCommGroup E] [NormedSpace ℝ E] [NormedAddCommGroup E'] [NormedSpace ℝ E']
    (k : WithTop ℕ∞) {I : ModelWithCorners ℝ E H} [FiniteDimensional ℝ E]
    (s : SingularNManifold.{u} X k I) (t : SingularNManifold.{v} X k I)
    (J : ModelWithCorners ℝ E' H') where
  /-- The underlying compact manifold of this unoriented bordism -/
  W : Type (max u v)
  [compactSpace : CompactSpace W]
  [isManifold: IsManifold J k W]
  /-- The presentation of the boundary `W` as a smooth manifold -/
  -- Future: we could allow bd.M₀ to be modelled on some other model, not necessarily I:
  -- we only care that this is fixed in the type.
  bd: BoundaryManifoldData W J k I
  /-- A continuous map `W → X` of the bordism into the topological space we work on -/
  F : W → X
  hF : Continuous F
  /-- The boundary of `W` is diffeomorphic to the disjoint union `M ⊔ M'`. -/
  φ : Diffeomorph I I (s.M ⊕ t.M) bd.M₀ k
  /-- `F` restricted to `M ↪ ∂W` equals `f`: this is formalised more nicely as
  `f = F ∘ ι ∘ φ⁻¹ : M → X`, where `ι : ∂W → W` is the inclusion. -/
  hFf : F ∘ bd.f ∘ φ ∘ Sum.inl = s.f
  /-- `F` restricted to `N ↪ ∂W` equals `g` -/
  hFg : F ∘ bd.f ∘ φ ∘ Sum.inr = t.f
```

- bundled design, like `SingularManifold`
- note: no requirement $\dim W = \dim M + 1$ yet (just for transitivity)
- model parameters I (for the boundary) and J (for the bordism)
  later applications take J as the product of I and the model for $[0,1]$
- universe choice: take W in universe `max u v`

Outlook: future possibilities

- define the bordism ring with ring operation
  need to rewrite models with corners, using $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$
- prove ring axioms: distributivity requires the inverse function theorem
- relative bordism groups
  - generalise both singular manifolds and bordisms
  - describe the boundary of manifolds with corners
  - define a homology functor (probably easy)
  - show the Eilenberg-Steenrod axioms: mostly easy
    interesting: boundary is a smooth manifold
    (false without co-dimension condition)
- *oriented* bordism groups: mostly straightforward, but requires oriented manifolds and induced boundary orientation (missing)
- for mathlib: need a general definition of smooth immersions and embeddings

Immersions and smooth embeddings

Let M and N be finite-dimensional smooth manifolds.

Definition

A map $f: M \to N$ is an **immersion** iff each differential df_p, $p \in M$ is injective. f is a **smooth embedding** iff it is an immersion and a topological embedding.

Caution about smooth embeddings

- injective immersion does not imply embedding
- smooth map and topological embedding does not imply embedding

Immersions in infinite dimensions

Let $f\colon M \to N$ be a smooth map between smooth (Banach) manifolds.

Definition

f is an immersion iff each differential df_p for $p \in M$ is injective.

Caution: too weak in infinite dimensions.

Better definition 1

f is an immersion iff each differential df_p for $p \in M$ **splits**, i.e. is an injective continuous linear map whose range is closed with a closed complement.

Better definition 2

f is an immersion for each $p \in M$, there are charts ϕ and ψ around p and $f(p)$ in which f looks like $u \mapsto (u, 0)$.

Fact

If M and N are finite-dimensional, these definitions are all equivalent.

Immersions in Banach manifolds

Caution

Banach manifolds require additional conditions are boundary points.
Currently, smoothness of immersions follows only at interior points.

Comparing these definitions

- **Fact.** Are equivalent over Banach manifolds.
- Definition 2 is nicer to work with: implies smoothness, similar to constant rank theorem.
- Definition 1 is easier to check (just compute differentials).
  Proving that composition of immersions is an immersion is *much* easier!

Formalisation status: immersions and smooth embeddings

1000–1500 lines of code already: work in progress

- find the right definition (in progress)
- reduce to the standard finite-dimensional definition
- prove: composition of immersions is an immersion
- prove: composition of split linear maps is split
- f immersion implies differential splits
- split differential implies immersion: requires inverse function theorem
- immersion is C^n (need better definition)

Formalisation status: smooth embeddings

- inverse function theorem first version done, “proper version” in progress
- define smooth embeddings
- prove composition of smooth embeddings is a smooth embedding

Formalising embedded submanifolds

Green items mean "sorry-free and mathlib-ready"; depend on smooth embeddings.

- define a suitable class of models with corners
- candidate definition of embedded submanifolds
- construction and properties of slice charts
- $f : M \to N$ smooth embedding implies $M \subset N$ embedded submanifold
- open subset is an embedded submanifold
- $\mathbb{D}^2 \subset \mathbb{R}^2$ is an embedded submanifold
- sanity check: M as submanifold of $M \times N$ (easy)
- future: construct submanifolds via constant rank theorem

Outlook and next steps

- correct the definition of immersions, prove smoothness
- prove: split linear maps compose
- polish inverse function theorem; prove “split differential $\to$ immersion”
- open a definition of embedded submanifolds for discussion

Summary

1. Bordism theory is an *extra-ordinary* homology theory.
2. Applications: Hirzebruch signature theorem, existence of exotic spheres
3. Formalisation is a good test of mathlib's differential geometry section
4. Immersions, smooth embeddings and submanifolds are missing, but within reach.
5. Be patient and prepared to fill in missing API.
   Avoid propositional equality of types.
   Be careful with your universes.

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Answer: boundary map for homology requires proving "∂M is a $\dim M - 1$-dimensional manifold".

- Uses: interior and boundary are independent of the chosen chart.

Thanks for listening! Any questions?

Where did I cheat?

Dream goal

Bordism theory as first proven homology theory in mathlib

Answer: boundary map for homology requires proving "∂M is a $\dim M - 1$-dimensional manifold".

- Uses: interior and boundary are independent of the chosen chart.
- Uses: invariance of domain, e.g. via singular homology of spheres

Upshot: this requires singular homology (or similar) first