# Equivariant transversality for closed holomorphic curves

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# Abstract

Holomorphic curves are an important technical tool in symplectic geometry. They play an instrumental role in breakthroughs such as the proof of the Arnold and Conley conjectures, Gromov's non-squeezing theorem and subsequent work on symplectic capacities, as well as symplectic filling problems. In each case, holomorphic curves are used to define symplectic invariants which show the existence of periodic orbits or obstruct the symplectic embedding resp. the hypothetical filling. Underlying the definition of an invariant is always a question of transversality: proving that a suitable moduli space of holomorphic curves is a smooth manifold of the desired dimension.

A major headache in symplectic topology is the fact that transversality often does *not* hold. To circumvent this, common practice is to either avoid this issue by imposing suitable technical assumptions or to use virtual techniques — of which there are several kinds, with no clear consensus yet which approach is best. This is caused by an inherent symmetry, namely the action of the automorphism groups of multiply covered curves.

In this thesis, we investigate a setting with even more symmetry. We study equivariant closed holomorphic curves into symplectic $G$-manifolds. For generic $G$-equivariant $J$, we cannot expect the moduli space of $J$-holomorphic curves to be a smooth orbifold, let alone a smooth manifold. Instead, we accept this additional symmetry as a feature and pursue a different paradigm: we decompose the moduli space into countably many disjoint *iso-symmetric strata* and *walls*, using the stabiliser with respect to the group action. We prove that the number of strata and walls is always countable, and that for generic equivariant $J$, every stratum and wall is a smooth finite-dimensional manifold. The dimension of each strata and wall are explicitly computable using representation-theoretic data. This is inspired by Wendl's solution of the super-rigidity conjecture and bears resemblance to the orbit type and local action type decompositions of a smooth orbifold. We expect the strata and walls to possess a similar local structure, though our methods can only partially prove this.

Our proofs proceed by carefully analysing standard transversality proofs, and adapting them to the equivariant context as necessary. Often, finding the correct definition is the most difficult part, and the remaining proof is a relatively straightforward adaptation of the classical argument. In several places, our results apply to the general setting of proper smooth Lie group actions. In its analysis, we use structural results about Lie groups, such as the existence of the Haar measure and the countability of conjugacy classes of compact subgroups.

# Zusammenfassung

Pseudo-holomorphe Kurven sind ein wichtiges technisches Werkzeug in symplektischer Geometrie. Sie spielten bei verschiedenen Durchbrüchen eine wesentliche Rolle, etwa dem Beweis der Arnold- und der Conley-Vermutung, Gromovs nicht-Quetschbarkeitssatz (*non-squeezing theorem*) zu symplektischen Einbettungen und folgenden Arbeiten zu symplektischen Kapazitäten sowie Fragen symplektischer Füllbarkeit.

Bei allen diesen Fragen dienen holomorphe Kurven dazu, symplektische Invarianten zu definieren: Diese zeigen die Existenz periodischer Orbiten oder bilden eine Obstruktion für eine sympletische Einbettung bzw. die hypothetische Füllung. Einer Definition einer symplektischen Invariante liegt immer ein Transversalitätsproblem zugrunde: Es gilt zu zeigen, dass ein geeigneter Modulraum holomorpher Kurven eine glatte Mannigfaltigkeit der gewünschten Dimension ist.

Eine große Komplikation ist, dass solche Transversalitätsprobleme oft *nicht* lösbar sind. Viele Arbeiten vermeiden dieses Problem, etwa durch das fordern geeigneter technischer Voraussetzungen wie zum Beispiel Semipositivität. Eine andere Strategie ist das Verwenden verschiedener virtueller Techniken — derzeit herrscht kein klarer Konsens, welche der vorgeschlagenen Ansätze der beste ist. Grund dieser Schwierigkeiten ist eine inhärente Symmetrie, der Automorphismengruppe multipel überlagerter Kurven.

Diese Arbeit behandelt den Fall zusätzlicher Symmetrie: Wir untersuchen äquivariante abgeschlossene holomorphe Kurven in symplektische $G$-Mannigfaltigkeiten. Für generische $G$-äquivariante $J$ ist nicht zu erwarten, dass der Modulraum $J$-holomorpher Kurven eine glatte Orbifaltigkeit (oder gar eine Mannigfaltigkeit) ist. Daher akzeptiert diese Arbeit die Symmetrie als aussagekräftige Information und folgt einem anderen Paradigma: Der Modulraum wird in abzählbar viele disjunkte *isosymmetrische Straten* und *Wälle* zerlegt, abhängig von u.a. dem Stabilisator bezüglich der Gruppenwirkung. Wir beweisen, dass die Mengen der Straten bzw. Wälle stets abzählbar sind und zeigen dass, für generische äquivariante $J$, jedes Stratum und jeder Wall eine glatte endlich-dimensionale Mannigfaltigkeit ist. Die Kodimension jedes Stratums und Walles lässt sich explizit angeben. Dies ist inspiriert von Wendls Beweis der Superstarrheits-Vermutung und ähnelt den Zerlegungen einer glatten Orbifaltigkeit in Orbittypen und lokale Wirkungstypen. Wir erwarten, dass unser Straten und Wälle eine ähnliche lokale Struktur haben; dies können wir nur teilweise beweisen.

Zum Beweis dieser Ergebnisse analysieren wir klassische Transversalitätsbeweise sorgfältig und adaptieren sie für den äquivarianten Kontext. An mehreren Stellen ist das aufstellen der korrekten Definition der schwierigste Teil und der restliche Beweis ist eine relativ direkte Übertragung des klassischen Arguments. Teilweise

betreffen unsere Ergebnisse den allgemeinen Fall einer eigentlichen glatten Liegruppenwirkung. Dort verwenden wir strukturelle Eigenschaften von Liegruppen, beispielsweise die Existenz eines Haarmaßes und die Abzählbarkeit der Konjugationsklassen kompakter Untergruppen.

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# Contents

# 1. Introduction

**Background and motivation** Holomorphic curves are an important technical tool in symplectic geometry. They play an instrumental role in several recent and past breakthroughs, pertaining to various guiding questions. This includes the existence of periodic Hamiltonian orbits, such as the Arnold conjecture — proven in a series of papers from Floer's seminal work [Flo86; Flo89] until Abouzaid–Blumberg's recent solution of the version with finite field coefficients [AB21] — or the Conley conjecture (see [GG15] for a survey of the solution and recent progress in this area). Other areas include Gromov's non-squeezing theorem [Gro85] and subsequent work on symplectic embeddings and capacities (see e.g. Schlenk's survey [Sch17]) as well as symplectic filling problems (recent progress includes [Zho19], [BGM22] and [BGMZ24]).

In these results, holomorphic curves are witnesses of rigidity phenomena (following Gromov's terminology [Gro87] of flexibility and rigidity).[1] For instance, holomorphic curves underlie the definition of symplectic invariants such as various kinds of Floer homologies, symplectic homology and Gromov–Witten invariants. These invariants show the existence of periodic orbits or obstruct the symplectic embedding resp. the hypothetical filling.

Underlying the definition of a holomorphic curves invariant is always a question of transversality: whether a suitable moduli space of holomorphic curves is a smooth manifold of the expected virtual dimension. A positive answer enables defining the invariant. A major headache in symplectic topology is the fact that transversality often does *not* hold. The underlying reason for the failure of transversality is a well-known conflict of transversality and symmetry. To give an elementary example, a smooth section of a vector bundle $E \to M$ is generically transverse to the zero section — but if $E$ admits a non-trivial group action, generic equivariant sections need not be. Imposing symmetry constraints shrinks the space of possible perturbations, often to the effect of transversality being impossible. In the case of holomorphic curves, one inherent symmetry is the action of the automorphism groups of multiply covered curves.

Commonly, the failure of transversality is either avoided by imposing suitable technical assumptions so transversality is still satisfied, or worked around by the use of virtual techniques, such as virtual fundamental classes, Kuranishi structures, global Kuranishi charts, domain-dependent perturbations or polyfolds. At the time

[1] For completeness, let us mention that holomorphic curves also appear in some constructions of symplectic embeddings, providing upper bounds on symplectic capacities [Sch17, Section 7]. Most interesting results concern the boundary of flexibility and rigidity, and holomorphic curves can often tell us where the flexibility ends and rigidity begins.

of writing, there is no clear consensus yet if all of these approaches are equivalent or which one is best — an unsatisfactory situation.

In this thesis, we pursue a different paradigm, put forward by Wendl [Wen23d]: what if we accept the additional symmetry as a feature and study its repercussions instead? More specifically, the (generalised) automorphism group $G$ of a multiply covered curve $u$ induces a decomposition of the moduli space of holomorphic curves. The kernel and co-kernel of $u$'s *normal Cauchy–Riemann operator* define finite-dimensional $G$-representations. Wendl proved that prescribing the representations of kernel and co-kernel decomposes the moduli space into countably many *walls*, each of which is (for generic $J$) a smooth manifold of computable (co-)dimension.

This stratification result requires no virtual methods, yet is useful when classical transversality fails. Sometimes, it can be used to recover transversality (say, if each obstruction lives in a walls of negative dimension). For instance, for generic $J$, every unbranched closed holomorphic curve is Fredholm regular [Wen23d, Theorem B]. Beyond transversality, Wendl's result, via a theorem of Zinger [Zin11] (see also [LP12]) proves that each embedded index zero curve $u$ with $c_1(u) = 0$ in a six-dimensional closed symplectic manifold $(M, \omega)$ has a well-defined local obstruction bundle — this does not require virtual techniques. This obstruction bundle can be used to define local Gromov–Witten invariants.

Avoiding virtual techniques is also useful to preserve symmetry information. The symmetry inherent in the given setting may provide valuable information, which we do not wish to perturb away. In celestial mechanics, the equations of motion are often symmetric — breaking this symmetry by perturbing the equations of motion is not desirable. For Gromov–Witten invariants, there is a relation between invariants corresponding to a simple curve and its multiple covers, as exemplified by the Gopakumar–Vafa formula [GV; BP01; PT14; IP18; DIW21]. An inhomogeneous perturbation will destroy this symmetry.

**Main results** In this thesis, we extend Wendl's paradigm to incorporate additional kinds of symmetry. Wendl's solution of the super-rigidity conjecture dealt with multiply covered curves: we consider the case of an additional external group action on a symplectic manifold $(M, \omega)$. Consider a symplectic $G$-manifold $(M, \omega)$, i.e. the group $G$ acts by symplectomorphisms. For a generic $G$-equivariant almost complex structure $J$ on $M$, we cannot expect the moduli space of closed $J$-holomorphic curves into $(M, J)$ to be a smooth orbifold, let alone a smooth manifold. Instead, we decompose the moduli space into countably many disjoint *iso-symmetric strata* and *walls* and prove that these strata and walls are generically smooth manifolds.

Let us make this more precise. All results in this thesis hold similarly for tame and compatible almost complex structures; let us state just the compatible case for simplicity. Suppose $(M, \omega)$ is a symplectic manifold, let $\mathcal{U} \subset M$ be an open subset with compact closure. Fix a $G$-equivariant compatible almost complex structure $J_{\text{fix}}$ on $(M, \omega)$ (see Definition 2.26); denote by $\mathcal{J}^G(M, \omega; \mathcal{U}, J_{\text{fix}})$ the space of $G$-equivariant compatible almost complex structures $J$ which are equal to $J_{\text{fix}}$ on $M \setminus \mathcal{U}$ (see Definition 4.2). If $M$ is closed, we may simply take $\mathcal{U} = M$, so the last condition becomes vacuous.

Fix parameters $C \in H_2(M)$, integers $g, m, k, c \geq 0$ and an $l$-tuple $\mathbf{l}$ of positive integers, $1 \leq l \leq m$. For each $J \in \mathcal{J}^G(M, \omega)$, a marked closed genus $g$ surface $(\Sigma, \theta)$ with $|\theta| = m$ marked points and closed subgroups $A \subset \text{Diff}_+(\Sigma, \theta)$ and $H \leqslant A \times G$, we will define the iso-symmetric strata $\mathcal{M}^{A,H}_{\mathcal{U},\mathbf{l}}(J)$ (see Definition 3.35) and walls $\mathcal{M}(J; k, c) \subset \mathcal{M}^{A,H}_{\mathcal{U},\mathbf{l}}(J)$ (see Definition 5.1). Roughly speaking, the stratum $\mathcal{M}^{A,H}_{\mathcal{U},\mathbf{l}}(J)$ consists of simple $J$-holomorphic curves $(\Sigma, j, \theta, v)$ with an injective point mapped into $\mathcal{U}$, such that $A$ is the automorphism group of the domain $(\Sigma, j, \theta)$; the group $H$ is conjugate to the stabiliser of $v$ under the induced $A \times G$-action on the space of $J$-holomorphic curves, and $\mathbf{l}$ describes the orders of $v$'s critical points (if any). We consider also the (point-wise) stabiliser $G_u$ of $u$; each curve $v$ has an associated $G_u$-equivariant[2] Fredholm operator $D^{N,\text{restr}}_v$ (the restricted normal Cauchy–Riemann operator, see Definition 5.12). If $G_u$ is trivial, the operator $D^{N,\text{restr}}_v$ coincides with the normal Cauchy–Riemann operator of $v$ (see Definition 5.4) and is $H$-equivariant. A curve $v$ belongs to $\mathcal{M}(J; k, c)$ if and only if its associated operator $D^{N,\text{restr}}_v$ has $k$-dimensional kernel and $c$-dimensional co-kernel.

Stating the dimension of each iso-symmetric stratum involves two more Fredholm operators. Section 4.2 will define the $A$-equivariant operator $D_{(j,\theta)}$, related to the variation of the complex structure $j$ on the domain. In particular, its co-kernel is a finite-dimensional $A$-representation; we denote by $m^A_1$ the multiplicity of the trivial $A$-representation in coker $D_{(j,\theta)}$. The second operator is the linearised Cauchy–Riemann operator $D_u$ (which is used to define $D^{N,\text{restr}}_u$). For $u \in \mathcal{M}^{A,H}_{\mathcal{U},\mathbf{l}}(J)$, it is an $H$-equivariant Fredholm operator; let $m^H_1$ denote the multiplicity of the trivial $H$-representation in its kernel. With all these definitions, the main results of this thesis are the following.

**Theorem A** (Smoothness of iso-symmetric strata)**.** *Suppose $2g + m \geq 3$ and $G$ is finite. For every open subset $\mathcal{U} \subset M$ with compact closure, there exists a co-meagre subset $\mathcal{J}_{reg} \subset \mathcal{J}^G(M, \omega; \mathcal{U}, J_{fix})$ such that for all $J \in \mathcal{J}_{reg}$, every iso-symmetric stratum $\mathcal{M}^{A,H}_{\mathcal{U},\mathbf{l}}(J)$ is a smooth finite-dimensional manifold, whose dimension near $u \in \mathcal{M}^{A,H}_{\mathcal{U},\mathbf{l}}(J)$ is given by*

$$\dim \mathcal{M}^{A,H}_{\mathcal{U},\mathbf{l}}(J) = m^A_1(\text{coker } D_{(j,\theta)}) + m^H_1(\ker D_u) - 2\sum_{i=1}^{l}(nl_i - 1).$$

*The analogous result holds for $G$-equivariant tame almost complex structures.*

**Theorem B** (Smoothness of walls)**.** *For $2g + m \geq 3$ and $G$ is finite, $\mathcal{J}_{reg}$ has a co-meagre subset $\mathcal{J}'_{reg}$ such that for all $J \in \mathcal{J}'_{reg}$, all walls $\mathcal{M}(J; k, c)$ are smooth submanifolds of $\mathcal{M}^{A,H}_{\mathcal{U},\mathbf{l}}(J)$. For each given curve $u$, the co-dimension of each wall $\mathcal{M}(J; k, c)$*

[2] The stabiliser $G_u$ is (isomorphic to) a subgroup of $H$: an element $g \in G$ lies in $G_u$ if and only if $(\phi, g) \in H$. In particular, there is no need to explicitly include $G_u$ in the definition of iso-symmetric strata, as it is determined by $H$.

*containing $u$ in $\mathcal{M}^{A,H}_{\mathcal{U},I}(J)$ is* $\mathrm{Hom}_H(\ker D^N_u, \operatorname{coker} D^N_u)$ *if $G_u$ is trivial, and is given by* $\mathrm{Hom}_{G_u}(\ker D^{N,restr}_u, \operatorname{coker} D^{N,restr}_u)$ *otherwise.*

The following two properties of the strata and walls are used to prove Theorems A and B, but are interesting on their own.

**Proposition C.** *The number of non-empty distinct iso-symmetric strata and walls is countable.*

**Proposition D.** *Suppose a smooth Lie group $G$ acts smoothly and properly on $M$. For each stable curve $u \in \mathcal{M}^{A,H}_{\mathcal{U},I}(J)$, the stabiliser $H$ is a compact Lie group.*

The author expects the conditions on $g$ and $m$ to be removable with a more involved proof. If $2g + m \geq 3$, each curve $u$ is *stable*, hence has finite automorphism group. When $G$ is also finite, this implies $A \times G$ is a finite group, which simplifies the argument significantly. The assumption on $G$ can probably be weakened: for instance, the author expects a proper Hamiltonian $G$-action and $G$ being abelian to also be sufficient. However, this requires significant technical effort. We will elaborate on this in Section 4.7.

**Related work** As mentioned, this thesis' overall framework goes back to Wendl's proof [Wen23d] of the super-rigidity conjecture. Doan and Walpuski [DW23] have rephrased Wendl's proof in more algebraic language (replacing, for instance, Wendl's use of the generalised automorphism group and "minimal regular presentations" by local systems). They also place the proof in a more abstract framework, introducing the terminology of "Petri's condition" and "flexibility". We mostly follow Wendl's approach, but will use this terminology. See also Bargalló i Gómez's exposition [Bar24] for a different phrasing of these proofs.

The core of Wendl's argument is a stratification theorem of the moduli space of holomorphic curves. This setting has a natural symmetry, due to the action of each curve's automorphism group. Their argument has three parts: firstly, the moduli space is decomposed into *iso-symmetric strata*: writing each curve as $u = v \circ \phi$ for $v$ simple and $\phi$ a holomorphic branched cover, the strata are characterised by the combinatorial type (number and order of branch points, orders of critical points) and (generalised) automorphism group $H$ of $\phi$. For generic $J$, each iso-symmetric stratum is a smooth manifold. In Wendl's setting, this result follows almost immediately from standard facts (this is very different from our setting).

Secondly, each iso-symmetric stratum is further decomposed into *walls*. The normal Cauchy–Riemann operator of $u$ is turned into a "twisted" operator, which is $H$-equivariant. Its kernel and co-kernel define finite-dimensional $H$-representations, which thus decompose as the sum of irreducible $H$-representations $\rho_i$. This splitting induces a corresponding splitting of the twisted Cauchy–Riemann operator into operators $D^{\rho_i}$. The wall corresponding to tuples $(k_i)$ and $(c_i)$ of non-negative integers consists of all curves such that $D^{\rho_i}$ has $k_i$-dimensional kernel and $c_i$-dimensional co-kernel.[3] Wendl proved that for generic $J$, each wall is a submanifold of its corresponding stratum (with an explicit expression for the co-dimension). This theorem is much harder, and requires proving that Petri's condition (see Chapter 5) is satisfied generically. Finally, results such as generic transversality of unbranched covers or the super-rigidity conjecture follow from this stratification theorem by dimension-counting arguments.

In hindsight, this stratification is very similar to the *orbit types* and *local action types* of a smooth and proper group action on a smooth manifold. If a smooth Lie group $G$ acts smoothly and properly on a smooth manifold $M$, the orbit type of a closed subgroup $H \leqslant G$ consists of all points $p \in M$ whose stabiliser (w.r.t. the $G$-action) is conjugate to $H$. Each orbit type is a smooth submanifold of $G$. Each orbit type splits further into local action types: if $p$ has stabiliser $G_p = H$, the tangent space $T_pM$ is a representation of the group $H$ — we decompose each orbit type according to the isomorphism class of the $H$-representation on $T_pM$.[4]

Equivariant transversality problems have also been studied in equivariant Morse theory, previous to and after Wendl's work. To some extent, this is a finitedimensional analogue of the holomorphic curves situation; some features (such as the finite-dimensionality of the space acted on) make this situation easier to handle. For instance, in a smooth manifold with a smooth $G$-action, generic $G$-equivariant smooth functions are still Morse [Was69; Hep09], but generic $G$-equivariant gradient-like vector fields need not be Morse–Smale [HHM19]. Kirilova's master's thesis [Kir21] applies Wendl's ideas to equivariant Morse theory, proving (for $G$ finite) a stratification result similar to this case. There is also in-progress work of Fauck, about equivariant Morse theory (and symplectic homology) *without* recourse to such a stratification [Fau].

Bai and Zhang [BZ] discuss equivariant transversality problems in the context of bifurcation theory. The main result of the paper is a construction of perturbative SU$(n)$ Casson invariants on integer homology spheres for $n \geq 3$, but they also prove an equivariant version of Cerf's theorem in Morse theory. The key step of their argument involves proving that a generic 1-parameter family (in the Morse case, of equivariant functions), crosses a number of walls (on which bifurcations occur) transversely. Finally, Hirschi [Hir23] has defined an equivariant version of the Gromov–Witten invariants, using an equivariant global Kuranishi chart.

**Outline of this thesis** After this introduction, we begin by reviewing the relevant background about holomorphic curves in Chapter 2. We define the spaces of tame and compatible equivariant almost complex structures and prove some of their basic properties. In Chapter 3, we discuss how to define iso-symmetric strata of (simple

[3] For the pedantic reader, let us emphasize that the $\rho_i$ are *real* representations, hence their endomorphism algebra $W_i = \mathrm{End}(\rho_i)$ is one of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. The dimensions $k_i$ and $c_i$ are taken as $W_i$-algebras, not real dimensions.

[4] For the pedantic reader: this statement is strictly correct if $G$ is discrete; if has positive dimension, we quotient out the tangent space $\alpha_p \subset T_pM$ of the $G$-orbit $G \cdot p$ at $p$ and consider the representation on the quotient $T_pM/\alpha_p$ instead.

and multiply covered) curves in our setting. This definition is more subtle than one might think. We also prove some basic properties of the stratification, including Propositions C and D from the introduction. This uses some results from Lie theory, including Montgomery–Zippin's neighbouring subgroups theorem. In Chapter 4, we prove that for generatic equivariant $J$, the iso-symmetric strata are smooth, and determine their dimensions in terms of representation-theoretic data. Next, we turn to decomposing each iso-symmetric stratum into walls and prove that walls are generically smooth (Chapter 5). In the final chapter, we mention possible next steps.

# 2. Background and setting

Let us begin by reviewing some background and pre-requisites. This thesis is concerned with pseudo-holomorphic curves in the presence of a symplectic group action. Hence, we review the definition of pseudo-holomorphic curves and their moduli space first (Section 2.1). Next, we proceed to the equivariance aspect and review basic aspects of symplectic group actions (Section 2.2). Finally, in Section 2.3 we study equivariant almost complex structures, providing the basis for studying equivariant pseudo-holomorphic curves.

## 2.1. Holomorphic curves

Let us review the definitions of holomorphic curves. All material in this section is standard and can be found in e.g. [MS12] or [Wen15]. This is also why we omit proofs.

Holomorphic curves are an instrumental tool for studying symplectic manifolds; they are defined more generally for almost complex manifolds.

**Definition 2.1.** *A smooth* almost complex structure *$J$ on a smooth manifold $M$ is a smooth section $J \in \Gamma(\mathrm{End}(TM))$ such that $J^2 = -\mathrm{id}$, i.e. $J$ defines a smooth family of linear maps $\{J_p\colon T_pM \to T_pM\}_{p\in M}$ with $J_p^2 = \mathrm{id}$ for each $p \in M$.*

**Definition 2.2.** *An* almost complex manifold *is a smooth manifold $M$ together with a smooth* almost complex structure *$J$ on $M$. Note that the tangent bundle $TM$ of an almost complex manifold $(M, J)$ is naturally a complex vector bundle.*

Complex manifolds are the easiest examples of almost complex manifolds, as a complex structure induces an almost complex structure: each tangent space $T_pM$ is naturally a complex vector space, and $J_p$ is given by multiplication by $i$ on that tangent space.[1]

An almost complex structure induced from a complex structure is called *integrable*. In (real) dimension two, every almost complex structure is integrable [Wen15, Theorem 2.1.6]. In higher dimensions, this is generically false, i.e. an almost complex

[1]More precisely: a complex chart yields an identification of $T_pM$ with $\mathbb{C}^{\dim_{\mathbb{C}}(M)}$, and this identification endows $T_pM$ with a complex vector space structure. This a priori depends on the choice of chart, but any two charts yield the same map on $T_pM$ as transition maps between complex charts are holomorphic.

manifold is generally not complex.[2] Every symplectic manifold is almost complex (see Lemma 2.7 below); the converse is not true.

Pseudo-holomorphic curves generalise holomorphic maps to almost complex manifolds (hence the modifier "pseudo"). We will often drop the prefix "pseudo"; this will not cause confusion since we almost never speak about actually holomorphic maps. The terms "curve" signifies their domain is complex one-dimensional.

**Recall.** A *Riemann surface* is a complex manifold of complex dimension one.

For our purposes, we will consider Riemann surfaces as pairs $(\Sigma, j)$ of a smooth surface with an almost complex structure on $\Sigma$; since $j$ is automatically integrable, this is an equivalent definition. In general, we will always speak of dimension as *real* dimensions, unless specified otherwise.

Recall that a function $f\colon \mathbb{C} \to \mathbb{C}$ is *holomorphic* if and only if it is complex differentiable. Equivalently, $f$ is smooth (between real manifolds) and its differential $df$ is complex linear. This definition generalises to almost complex manifolds, yielding pseudo-holomorphic curves.

**Definition 2.3.** *Let $(\Sigma, j)$ be a Riemann surface and $(M, J)$ be an almost complex manifold. A* pseudo-holomorphic curve *is a smooth map $u\colon \Sigma \to M$ such that $J \circ du = du \circ j$. When we want to emphasize $J$, we will also speak of a $J$-*holomorphic curve.

To study symplectic manifolds, it is usually more helpful to study all its holomorphic curves, as this provides geometric information. (For instance, in some cases $M$ is foliated by holomorphic curves.) Thus, one studies *moduli spaces* of all curves (with certain prescribed topological data). Making this precise requires refining the above definition.

Firstly, allowing any Riemann surfaces as the domain of holomorphic curves is far too general. Most commonly, one considers *closed* holomorphic curves, whose domain is a closed connected Riemann surface: this will also be the scope of this thesis. (Another common setting are *punctured* curves, defined on a finite type Riemann surface, i.e. a closed Riemann surface with finitely many punctures.) Recall that a closed connected Riemann surface $\Sigma$ has a well-defined genus and a fundamental class.

It is often easier to assume the target $M$ to be a closed manifold. This is, however not required for our thesis: all arguments only need small modifications from the closed setting to apply in general. Hence, we do *not* assume $M$ to be closed.

Finally, in all practical applications some constraint on the almost complex structure $J$ is required: depending on the application, one demands that $J$ be *tame* or *compatible*. These notions require $M$ to be symplectic, as it depends on the symplectic form. Compatibility implies tameness; in most applications, either condition can be

[2] More precisely: each almost complex structure $J$ on $M$ has an associated *Nijenhuis tensor* $N_J$ defined by $N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$ for smooth vector fields $X$ and $Y$ on $M$. The Nijenhuis tensor $N_J$ vanishes identically if and only if $J$ is integrable [DK90, Chapter 2]. It is not hard to show that $\dim(M) = 2$ implies $N_J = 0$. In higher dimension, generically $N_J \neq 0$.

used and the difference does not matter much. Sometimes, compatibility makes certain formulas easier to read; on the other hand, tameness is an open condition, and this flexibility is occasionally useful. There is perhaps one situation where compatibility is actually more convenient: in symplectic field theory, the analysis is quite a bit simpler for compatible almost complex structures — which is why, to date, the setup has only been fully developed in this context. It seems likely that tame almost complex structure can also be permitted, but the technical effort to verify this has not been performed. Further details can be found in e.g. Wendl's upcoming book [Wen20, Section 6.7.2]. This thesis will apply to both settings.

Tameness is required for Gromov's compactness theorem, showing that the moduli space of holomorphic curves has a natural compactification. On a symplectic vector space $E$, if a complex structure on $E$ is tame, every complex line (i.e., one-dimensional complex subspace) is a symplectic subspace. Hence, if $J$ is tamed by $\omega$, the image of any $J$-holomorphic curve is a symplectic submanifold (possibly with singularities). This implies non-constant curves have positive *energy*, which is important for the proof.

**Definition 2.4.** *A smooth almost complex structure $J$ on a symplectic manifold $(M, \omega)$ is called $\omega$-*tame *or* tamed by $\omega$ *if and only if for all $p \in M$ and non-zero $X \in T_pM$, we have $\omega_p(X, JX) > 0$. The almost complex structure $J$ is called $\omega$-*compatible *or* compatible with $\omega$ *if and only if $\omega(\cdot, J\cdot)$ defines a Riemannian metric on $M$. Equivalently, $J$ is compatible if and only if $J$ is tame and $\omega(X, JY) = \omega(Y, JX)$ for all $X, Y \in T_pM$.*
*When there is no risk of confusion, we will often omit $\omega$ from the notation and simply speak of* tame *or* compatible *almost complex structures.*

Almost complex structures are an auxiliary object: the exact choice of $J$ is usually not important; it is merely important that a choice can be made. To make this precise, let us consider the space of all tame resp. compatible almost complex structures.

**Notation.** Let $(M, \omega)$ be a symplectic manifold. We denote the spaces of tame resp. compatible almost complex structures on $(M, \omega)$ by $\mathcal{J}_\tau(M, \omega)$ resp. $\mathcal{J}(M, \omega)$.

Both $\mathcal{J}_\tau(M, \omega)$ and $\mathcal{J}(M, \omega)$ can be equipped with a topology, as subspaces of $\Gamma(\mathrm{End}(TM))$. For our purposes, the weak $C^\infty$-topology is the correct choice. We refer the reader to e.g. Hirsch's differential topology textbook [Hir76, Chapter 2] for the details, and just mention the definition and its key properties. (We will not explicitly use these properties in this thesis.)

**Recall.** Let $X$ and $Y$ be topological spaces. The *compact-open topology* on the space $C^0(X, Y)$ of continuous functions from $X$ to $Y$ is generated by the subsets $O(K, V) := \{f \in C^0(M, N) \mid f(K) \subset V\}$ for $K \subset M$ compact and $V \subset N$ open: neighbourhoods are all sets containing the intersection of finitely many such sets. A sequence $(f_n)$ converges to $f$ in the compact-open topology if and only if $(f_n)$ converges to $f$ uniformly on each compact subset of $X$.

Intuitively speaking, convergence in the $C^\infty_{\mathrm{loc}}$-topology demands uniform convergence on compact sets, of the functions and each of its derivatives.

**Definition 2.5** (Weak $C^k$-topology)**.** *Let $M$ and $N$ be $C^k$-manifolds for $1 \leq k \leq \infty$; consider the space $C^k(M,N)$ of all $C^k$ maps $f\colon M \to N$. The* compact-open $C^k$ topology, *also known as* weak $C^k$ topology *or $C^k_{loc}$-topology, is the metrizable topology on $C^k(M,N)$ such that a sequence $(f_n)$ converges to $f$ if and only if $f_n \to f$ in $C^0(M,N)$ and for all $K \subset M$ compact, charts $(\phi, U \subset M)$ of $M$ and $(\psi, V \subset N)$ of $N$ with $K \subset U$ and $f_n(K) \subset V$,[3] the local coordinate representatives $\psi \circ f_n \circ \phi^{-1}|_{\phi(K)}\colon \phi(K) \to \mathbb{R}^{\dim(N)}$ and their derivatives of order $\leq k$ converge uniformly to $\psi \circ f \circ \phi|_{\phi(K)}$.*

**Proposition 2.6** ([Hir76, p. 35, Theorem 2.4.4])**.** *The weak $C^k$-topology has a complete metric and a countable base. If $M$ is compact, it is locally contractible.* □

The above defines a topology on the space of $C^k$ maps; it remains to discuss spaces of sections of a $C^k$ vector bundle. There are two ways to endow the space $C^k(E)$ of $C^k$ sections of a $C^k$ vector bundle $E \to M$ with a topology. Firstly, every section of $E$ is also a $C^k$ map $M \to E$, hence we could endow $C^k(E)$ with the subspace topology. Secondly, we could choose a system of local trivialisations of $E$ with compact closures and define a sequence $(s_n)$ in $C^k(E)$ to converge to $s \in \Gamma(E)$ if and only if in each trivialisation, the local representatives of $(s_n)$ converge to the local representative of $s$. Hence, for smooth sections, $s_n \to s$ if and only if $s_n \to s$ in the $C^k$-norm on $\Gamma(E)$, for all $k$. Over a compact base, all auxiliary choices yield equivalent norms, hence the same topology. The second option matches matches how the space of $W^{k,p}$-sections of a vector bundle of Sobolev class $W^{k,p}$ obtains its topology; see e.g. [Wen15, Section 3.1] for the details in this case.

If $M$ is compact, these definitions yield the same topology, and the topology on the spaces $\mathcal{J}_\tau(M,\omega)$ and $\mathcal{J}(M,\omega)$ is the subspace topology in $\Gamma(\mathrm{End}(TM))$.[4] We conclude that $\mathcal{J}_\tau(M,\omega)$ and $\mathcal{J}(M,\omega)$ are complete metric spaces. If $M$ is non-compact, the two approaches differ: for this and other reasons, we will work with a smaller space instead (which is again complete); see Chapter 4 for the details.

A priori, $\mathcal{J}_\tau(M,\omega)$ and $\mathcal{J}(M,\omega)$ could be empty or have very non-trivial topology. (For instance, they are generally infinite-dimensional spaces.) The condition of tameness or compatibility, however, ensures that they are non-empty and contractible.[5]

**Lemma 2.7** ([Wen15, Theorem 2.2.8])**.** *$\mathcal{J}_\tau(M,\omega)$ and $\mathcal{J}(M,\omega)$ are non-empty contractible topological spaces.* □

There are (at least) two qualitatively different proofs of this fact. The first proof is due to Gromov [Gro85]. The case of compatible almost complex structures is

[3] By definition of the compact-open topology, $f_n \to f$ in $C^0(M,N)$ implies that for any sub-basic set $O(K,V)$ containing $f$, we have $f_n \in O(K,V)$ for all $n$ sufficiently large. Thus, the second condition is not a real restriction.

[4] In particular, the topology on $\mathcal{J}(M,\omega)$ coincides with the subspace topology induced from $\mathcal{J}_\tau(M,\omega)$.

[5] They are even infinite-dimensional manifolds in a precise sense; we will discuss this slightly subtle question further in Section 4.1.

treated by an elegant argument, presenting an equivalence to the space of Riemannian metrics (which is non-empty and convex, hence contractible). Contractibility of tame almost complex structures is reduced to this case using tools of homotopy theory, including the homotopy exact sequence and Serre fibrations. An alternative and more elementary proof due to Sévennec [AL94, Corollary 2.1.1.7] allows treating both spaces on the same footing: it exhibits $\mathcal{J}_\tau(M,\omega)$ resp. $\mathcal{J}(M,\omega)$ as the Cayley transform (at some fixed $J_0$) of a convex subset of a suitable normed space, which implies contractibility. This perspective is also useful for studying the manifold structure on $\mathcal{J}_\tau(M,\omega)$ and $\mathcal{J}(M,\omega)$, as this provides a description of the tangent space at $J_0$. We will revisit this in Section 4.1.

Lemma 2.7 implies that many constructions do not essentially depend on the particular choice of $J$.

**Example 2.8.** Let $J$ and $J'$ be two compatible almost complex structures on $(M,\omega)$. Then $(TM,J)$ and $(TM,J')$ are isomorphic complex vector bundles.

Without further ado, this is the definition of the moduli spaces of unparametrised curves. One can also consider spaces of parametrised holomorphic curves. These are useful auxiliary objects (we will consider one in Section 3.3), but less relevant for eventual geometric applications: in practice, one usually cares about the *image* of a holomorphic curves, but not about the choice of its parametrisation.

**Definition 2.9** (Moduli space of unparametrised holomorphic curves)**.** *Consider integers $g, m \geq 0$, a homology class $C \in H_2(M)$ and an almost complex structure $J$ on $M$. The* moduli space *of genus $g$ closed connected*[6] *unparametrised holomorphic curves with $m$ marked points of class $C$ is given by*

$$\mathcal{M}_{g,m}(C,J) := \Big\{(\Sigma,j,\theta,u) \mid (\Sigma,j) \text{ closed connected genus } g \text{ Riemann surface}, \\ \theta \subset \Sigma \text{ ordered subset with } |\theta| = m, u\colon \Sigma \to M \text{ smooth} \\ \text{such that } J \circ du = du \circ j, u_*[\Sigma] = C\Big\}/\sim,$$

*where $\sim$ denotes the equivalence relation defined by $(\Sigma,j,\theta,u) \sim (\Sigma',j',\theta',u')$ if and only if there exists a biholomorphic map $\phi\colon (\Sigma,j) \to (\Sigma',j')$ such that $\phi(\theta) = \theta'$ as ordered sets and $u = u' \circ \phi$, and $u_*[\Sigma]$ denotes the push-forward of the fundamental class $[\Sigma]$ of $\Sigma$ by the smooth map $u$. When $g$, $m$ and $C$ are understood, we will abbreviate $\mathcal{M}(J) := \mathcal{M}_{g,m}(C,J)$.*

If $M$ is a single point, this reduces to the moduli space $\mathcal{M}_{g,m}$ of *pointed Riemann surfaces*: its elements are equivalence classes $[(\Sigma,j,\theta)]$ of a closed connected genus $g$ Riemann surface $(\Sigma,j)$ with an ordered set of $m$ marked points, up to biholomorphic maps preserving the points in $\theta$ in order. Both moduli spaces $\mathcal{M}_{g,m}(C,J)$ and $\mathcal{M}_{g,m}$ have natural topologies, which we will explain in Subsection 3.3.2: for now, we just consider them as sets.

[6]In this thesis, we only consider connected curves. This is mainly for bookkeeping reasons; including multiple components will not make proofs meaningfully harder.

We need to introduce one last feature about holomorphic curves: the distinction between simple and multiply covered curves.

**Definition 2.10** ([Wen15, Definition 2.15.3, Corollary 2.15.4]). *A non-constant connected closed holomorphic curve* $[(\Sigma, j, \theta, u)]$ *is called* simple *if and only if there exists a finite set* $\Gamma \subset \Sigma$ *such that* $u$ *is an embedding on* $\Sigma \setminus \Gamma$*. Otherwise,* $u$ *is called* multiply covered*.*

**Lemma 2.11** ([Wen15, Theorem 2.15.2]). *If* $(\Sigma, j, \theta, u)$ *is multiply covered, there exists a factorisation* $u = v \circ \phi$*, where* $(\Sigma', j', \theta', v)$ *is a simple curve on some closed connected Riemann surface* $(\Sigma', j')$ *and* $\phi\colon (\Sigma, j) \to (\Sigma', j')$ *is a holomorphic branched covering map of degree* $\deg(\phi) > 1$*, such that* $\phi(\theta') = \theta$ *as ordered sets.* □

This distinction is meaningful because of the *automorphism group* of holomorphic curves.

**Definition 2.12** (Automorphism group of a holomorphic curve). *The* automorphism group *of a pointed Riemann surface* $(\Sigma, j, \theta) \in \mathcal{M}_{g,m}$ *is*

$$\mathrm{Aut}(\Sigma, j, \theta) := \{\phi \in \mathrm{Diff}(\Sigma, \theta) \mid \phi \text{ is biholomorphic}\}.$$

*A holomorphic curve* $(\Sigma, j, \theta, u) \in \mathcal{M}_{g,m}(C, J)$ *has* automorphism group

$$\mathrm{Aut}(u) := \{\phi \in \mathrm{Aut}(\Sigma, j, \theta) \mid u = u \circ \phi\}.$$

If a curve has non-trivial automorphism group, it must be multiply covered. The converse is *very false*; it becomes a true statement when replacing $\mathrm{Aut}(u)$ by the "generalised automorphism group" (see [Wen23d, Definition 2.6]) of $u$.

Simple curves can equivalently be characterised by having an injective point. We will use this in Chapter 4.

**Definition 2.13** (Injective points and somewhere injective maps). *Let* $f\colon M \to N$ *be a differentiable map of* $C^1$ *manifolds. Then* $p \in M$ *is called an* injective point *of* $f$ *if and only if the differential* $df_p\colon T_pM \to T_{f(p)}N$ *is injective and* $f^{-1}(f(p)) = \{p\}$*. If* $f$ *has an injective point, it is called* somewhere injective*.*

**Remark 2.14.** For a pseudo-holomorphic curve $u\colon \Sigma \to M$, each differential $du_z$ is complex linear, hence either $du_z = 0$ or $du_z$ is injective. Since $\dim M > 2$ in most cases, $du_z$ is usually never surjective.

The crucial fact is that simple and somewhere injective holomorphic curves are, in our setting, equivalent.

**Proposition 2.15** ([Wen15, Proposition 4.1.3]). *A closed connected holomorphic curve* $u$ *is simple if and only if it has an injective point.* □

**Remark 2.16.** If $u$ is not a closed curve, this equivalence is *false* in general. A somewhere injective curve is always simple; the converse need not be true, e.g. for holomorphic curves with totally real boundary [Laz00; KO00].

## 2.2. Symplectic group actions

Let us now review symplectic group actions. Similarly to Section 2.1, virtually all of this material is classical and can be found in, for example, McDuff–Salamon's book [MS12] or Pelayo's survey [Pel17].[7]

**Recall.** A (*left*) *action* of a group $G$ on a set $X$ is a group homomorphism $A\colon G \to \mathrm{Sym}(X)$ into the space of bijections on $X$; in particular, $A(g) \circ A(h) = A(gh)$ for all $g, h \in G$. A *right action* is a group anti-homomorphism $A\colon G \to \mathrm{Sym}(X)$, i.e. $A(g) \circ A(h) = A(hg)$ for all $g, h \in G$. Every left action $A$ can be converted to a right action by $g \mapsto A(g^{-1})$, and vice versa. We will often denote this action by $A(g, x) =: g \cdot x$.

**Remark 2.17.** Whether to work with left or right actions is mostly a question of notation. In this document, we choose to use symplectic left actions, as this matches the order of function composition. Unfortunately, the automorphism group $\mathrm{Aut}(u)$ of a holomorphic curve acts on the right, meaning we have to convert it to a left action. There is no free lunch.

We actually want our group actions to respect the smooth and symplectic structure on $M$, hence we consider group actions by diffeomorphisms and symplectomorphisms.

**Definition 2.18.** *A* smooth left action *of a smooth Lie group G on a smooth manifold G is a group homomorphism $G \to \mathrm{Diff}(M)$; smooth right actions are defined similarly. Equivalently, a G-action by diffeomorphisms is a smooth action if and only if the corresponding map $G \times M \to M$ is smooth.*[8]

**Observation 2.19.** A smooth $G$-action $g \mapsto \psi_g$ on $M$ by diffeomorphisms induces a $G$-action on its tangent bundle by $G \ni g \mapsto d\psi_g \in \mathrm{Sym}(TM)$; in fact, each map $d\psi_g$ is a smooth bundle isomorphism over the map $\psi_g$.

A smooth action on $M$ induces an action on the space of almost complex structures. If $J$ is *equivariant* (see Definition 2.26), it also induces an action on the moduli space of $J$-holomorphic curves: we will see this in Chapter 3.

**Recall** (Pull-back and push-forward of almost complex structures)**.** Let $f\colon M \to N$ be a diffeomorphism of smooth manifolds. Let $J$ be an almost complex structure on $M$. Then, $f_*J := df \circ J \circ df^{-1}$ (i.e., $(f_*J)_{f(p)} = df_p \circ J_p \circ df_p^{-1}$ for all $p \in M$) defines an almost complex structure on $N$, called the *push-forward* of $J$ under $f$. Similarly, if $J'$ is an almost complex structure on $N$, $f^*J' := df^{-1} \circ J' \circ df$ defines an almost complex structure on $M$, called the *pull-back* of the almost complex structure $J'$.

[7] This survey is certainly biased towards Pelayo's interests (and references a *lot* of their own work), but to the best of this author's knowledge, this is the most recent survey on symplectic group actions.

[8] The "only if" direction uses the finite-dimensionality of $M$ and holds essentially because a continuously partially differentiable function is totally differentiable. If $M$ were e.g. an infinite-dimensional Banach manifold, smoothness of each map $A(g)$ a weaker condition smoothness of the map $G \times M \to M$; the latter condition is the correct one.

**Lemma 2.20.** *Suppose $A\colon G \to \mathrm{Diff}(M), g \mapsto A_g$ is a left action on a smooth manifold $M$. Then $g \cdot J := (A_g)_* J$ defines a left action on the space of almost complex structures on $M$. Similarly, $g \cdot J := (A_g)^* J$ defines a right action.*

*Proof.* Since $\psi^* J = (\psi^{-1})_* J$ for any diffeomorphism $\psi$ on $M$, it suffices to consider the pull-back case. Slightly more generally, we show that $\phi^*(\psi^* J) = (\psi \circ \phi)^* J$ for any two diffeomorphisms $\phi$ and $\psi$ on $M$. To this end, we simply compute

$$\begin{aligned}\phi^*(\psi^* J) &= d\phi^{-1} \circ (d\psi^{-1} \circ J \circ d\psi) \circ d\phi = d(\phi^{-1} \circ \psi^{-1}) \circ J \circ d(\psi \circ \phi) \\ &= d((\psi \circ \psi)^{-1}) \circ J \circ d(\psi \circ \phi) = (\psi \circ \phi)^* J.\end{aligned}$$

□

Compatible almost complex structures correspond to Riemannian metrics; we will use this in Section 2.3. To make use of this correspondence, we note that a $G$-action on $M$ by diffeomorphisms induces an action on the space of Riemannian metrics on $M$.

**Recall** (Pull-back and push-forward of Riemannian metrics)**.** Let $f\colon M \to N$ be a smooth map of smooth manifolds. Let $h$ be Riemannian metric on $N$. Then $f^* h := h(df\cdot, df\cdot)$ defines a Riemannian metric on $M$, called the *pull-back* of $g$ w.r.t. $f$. If $f$ is a diffeomorphism and $g$ a Riemannian metric on $M$, $f_* g := (f^{-1})^* g = g(df^{-1}\cdot, df^{-1}\cdot)$ defines a Riemannian metric on $N$, called the *push-forward* of $g$ w.r.t. $f$.

**Notation.** Denote the space of smooth Riemannian metrics on $M$ by $\mathfrak{M}(M)$.

**Lemma 2.21.** *Suppose $A\colon G \to \mathrm{Diff}(M), g \mapsto A_g$ is a (left) action of a group $G$ on a smooth manifold $M$. Then $g \cdot h := (A_g)^* h$ defines a left action on the space $\mathfrak{M}(M)$ of Riemannian metrics on $M$, and $g \cdot h := (A_g)_* h$ defines a right action on $\mathfrak{M}(M)$.*

*Proof.* Analogously to Lemma 2.20, we only show $\phi^*(\psi^* g) = (\phi \circ \psi)^* g$ for all Riemannian metrics $g \in \mathfrak{M}(M)$ and diffeomorphisms $\phi, \psi$ of $M$. To this end, we compute

$$\begin{aligned}\phi^*(\psi^* g) &= \phi^*(g(d\psi\cdot, d\phi\cdot)) = g(d\phi(d\psi\cdot), d\phi(d\psi\cdot)) = g(d(\phi \circ \psi)\cdot, d(\phi \circ \psi)\cdot) \\ &= (\phi \circ \psi)^* g.\end{aligned}$$

□

In general, a smooth group action need not preserve compatibility or tameness of an almost complex structure: in contrast, symplectic actions preserve tameness and compatibility.

**Definition 2.22** (Symplectic group action [MS12; Pel17, Definition 3.4])**.** *A left group action $\psi$ of a group $G$ on a symplectic manifold $(M, \omega)$ is called* symplectic *if and only if $G$ acts by symplectomorphisms, i.e. $\psi$ is defined by a group homomorphism $G \to \mathrm{Symp}(M, \omega)$.* Symplectic right actions *are defined analogously. A* symplectic $G$-manifold *is a tuple $(M, \omega, G, \psi)$ of a symplectic manifold $(M, \omega)$, a group $G$ and a symplectic $G$-action $\psi\colon G \to \mathrm{Symp}(M, \omega)$.*

**Lemma 2.23.** *If $G$ acts symplectically on $(M, \omega)$ by $G \ni g \mapsto \psi_g \in \mathrm{Symp}(M, \omega)$, the spaces $J_\tau(M, \omega)$ and $J(M, \omega)$ are invariant under the $G$-action: if an almost complex structure $J$ on $M$ is $\omega$-tame or $\omega$-compatible, respectively, so is $\psi_g^* J$.*

*Proof.* Let $J$ be an arbitrary almost complex structure on $M$. For any diffeomorphism $\phi$ of $M$, we show that $\phi^* J$ is $\phi^*\omega$-tame resp. $\phi^*\omega$-compatible if $J$ is $\omega$-tame resp. $\omega$-compatible. In particular, if $G$ acts symplectically, $\omega = \psi_g^*\omega$ and $\psi_g^* J$ is $\omega$-tame resp. $\omega$-compatible once $J$ is.

This is an easy computation: for any $X, Y \in T_pM$, we have

$$(\phi^*\omega)(X, (\phi^*J)Y) = \omega(d\phi_p X, d\phi_p((\phi^*J)Y)) = \omega(d\phi_p X, d\phi_p \circ d\phi_p^{-1} \circ J(d\phi_p Y)) \\ = \omega(d\phi_p X, J(d\phi_p Y)). \tag{2.1}$$

Suppose $J$ is $\omega$-tame; let $X \in T_pM$ be arbitrary. Then we deduce

$$\phi^*\omega(X, (\phi^*J)X) = \omega(d\phi_p X, J(d\phi_p X)) > 0,$$

hence $\phi^*J$ is indeed $\phi^*\omega$-tame. If $J$ is $\omega$-compatible, for all $X, Y \in T_pM$ we have

$$(\phi^*\omega)(X, (\phi^*J)Y) \overset{(2.1)}{=} \omega(d\phi_p X, J(d\phi_p Y)) = \omega(d\phi_p Y, J(d\phi_p X)) \overset{(2.1)}{=} (\phi^*\omega)(Y, (\phi^*J)X),$$

hence $\phi^*J$ is indeed $\phi^*\omega$-compatible. □

## 2.3. Equivariant almost complex structures

We aim to study moduli spaces of holomorphic curves in a symplectic $G$-manifold $M$. This only makes sense if the almost complex structure on $M$ is compatible with the $G$-action: this is captured by the following:

**Definition 2.24.** *Suppose a group $G$ acts symplectically on a smooth manifold $M$ by $g \mapsto \psi_g$. An almost complex structure $J$ on $M$ is called* $G$-equivariant *if and only if it commutes with the induced $G$-action on $M$, that is $\psi_g^* J = J$ for all $g \in G$. More explicitly, this means $d\psi_g \circ J = J \circ d\psi_g$ for all $g \in G$, i.e. for all $g \in G$, $p \in M$ and $X \in T_pM$ we have $d\psi_g(J_pX) = J_{\psi_g(p)} d\psi_g(X)$.*

**Remark 2.25.** An almost complex structure $J$ on $M$ is $G$-equivariant if and only if it is invariant under the $G$-action by push-forward (or pull-back) of almost complex structures.

Thus, in this thesis we will consider the spaces of $G$-equivariant tame resp. compatible almost complex structures.

**Definition 2.26.** *Let $(M, \omega)$ be a symplectic manifold. Consider*

$$\mathcal{J}_\tau^G(M, \omega) := \{J \in \mathcal{J}_\tau(M, \omega) \mid J \text{ is } G\text{-equivariant}\}$$
$$\mathcal{J}^G(M, \omega) := \{J \in \mathcal{J}(M, \omega) \mid J \text{ is } G\text{-equivariant}\},$$

*endowed with the subspace topology on $\mathcal{J}_\tau(M, \omega)$.*

Like their non-equivariant counterparts, both spaces are contractible: our proof follows the argument in the non-equivariant case (outlined in Section 2.1). Recall that there were two proofs for these facts, due to Gromov and Sévennec. The natural question is whether Gromov's or Sévennec's proofs can be carried out equivariantly: on its own, each proof is tricky to adapt, but a combination works.

Gromov's proof for compatible almost complex structures (including existence) transfers well to the equivariant setting. There is an issue with transforming his reduction of compatible to tame almost complex structures, as it is based on the following lemma.

**Lemma 2.27** ([Wen15, Lemma 2.2.14])**.** *Suppose $\pi\colon E \to M$ is a smooth locally trivial fibre bundle over a manifold $M$, and the fibres are contractible. Then the space $\Gamma(E)$ of smooth sections is non-empty and contractible (in the $C^\infty_{loc}$-topology).* □

In our setting, $E$ is endowed with an action by smooth bundle isomorphisms (over corresponding diffeomorphisms on the base) and we are considering the space $\Gamma^G(E)$ of $G$-equivariant smooth sections. This makes the picture non-local and breaks the argument as written. Sévennec's argument for existence also uses this lemma, hence suffers from the same issue. Fortunately, Sévennec's contractibility proof works for both $\mathcal{J}^G_\tau(M,\omega)$ and $\mathcal{J}^G(M,\omega)$.

Let us embark on the equivariant proof combining these arguments. We begin the proof by presenting the equivariant version of Gromov's proof for $\mathcal{J}(M,\omega)$: this will be used to show non-emptiness.

The core observation for Gromov's proof that $\mathcal{J}(M,\omega)$ is contractible is the following. Let $\mathfrak{M}(M)$ denote the space of Riemannian metrics on $M$, endowed with the $C^\infty_{\text{loc}}$-topology. This is a non-empty convex subset of a vector space, hence contractible.

**Lemma 2.28** ([Wen15, p. 34])**.** *There exists a continuous map*

$$\mathcal{J}(M,\omega) \to \mathfrak{M}(M),\, J \mapsto g_J := \omega(\cdot, J\cdot), \tag{2.2}$$

*which has a continuous left inverse given by*

$$\mathfrak{M}(M) \to \mathcal{J}(M,\omega),\, g \mapsto J_g := A_g \sqrt{A_g^* A_g}^{\,-1}, \tag{2.3}$$

*where the bundle map $A_g\colon TM \to TM$ is defined by $g(A_g\cdot,\cdot) = \omega$.*[9] *Hence, contractibility of $\mathfrak{M}(M)$ implies contractibility of $\mathcal{J}(M,\omega)$.* □

In the remainder of this section, we assume the following.

**Convention.** Let $\psi$ be a symplectic group action of $G$ on $(M,\omega)$.

[9] The operator $A_g^*A_g$ is positive definite on each fibre w.r.t. $g$, and $\sqrt{A_g^*A_g}$ is its unique positive definite square root on each fibre.

We claim that (2.2) and (2.3) descend to a correspondence between $\mathcal{J}^G(M,\omega)$ and $G$-invariant Riemannian metrics (with respect to the induced $G$-action of Lemma 2.21). The proof is a sequence of small computations, none of them particularly difficult.

**Lemma 2.29.** *If a compatible almost complex structure $J \in \mathcal{J}(M,\omega)$ is $G$-equivariant, the corresponding Riemannian metric $g_J = \omega(\cdot, J\cdot)$ is $G$-invariant.*

*Proof.* Let $J \in \mathcal{J}^G(M,\omega)$, $h \in G$ and $X, Y \in T_pM$ be arbitrary. We directly compute

$$\begin{aligned}(\psi_h^* g_J)(X,Y) &= g_J(d\psi_h X, d\psi_h Y) = \omega(d\psi_h X, J(d\psi_h Y)) \\ &\overset{(*)}{=} \omega(d\psi_h X, d\psi_h(JY)) = (\psi_h^*\omega)(X, JY) \\ &\overset{(**)}{=} \omega(X, JY) = g_J(X,Y),\end{aligned}$$

using the $G$-equivariance of $J$ in step $(*)$ and the fact that $G$ acts symplectically in step $(**)$. □

**Lemma 2.30.** *If a Riemannian metric $g \in \mathfrak{M}(M)$ is $G$-invariant, the corresponding compatible almost complex structure $J_g$ is $G$-equivariant.*

*Proof.* Let $h \in G$ be arbitrary; we want to show $\psi_h^* J_g = J_g$. Since $g$ is $G$-invariant, we have $A_{\psi_h^* g} = A_g$ and $J_{\psi_h^* g} = J_g$. It remains to show $\psi_h^* J_g = J_{\psi_h^* g}$.

**Claim 1.** $A_{\psi_h^* g} = d\psi_h \circ A_g \circ d\psi_h^{-1}$

*Proof of Claim 1.* Let $p \in M$ and $X, Y \in T_pM$ be arbitrary. Note $X = d\psi_h X'$ and $Y = d\psi_h Y'$ for $X' = d\psi_h^{-1} X$ and $Y' = d\psi_h^{-1} Y$. We compute

$$\begin{aligned}g(d\psi_h \circ A_g \circ d\psi_h^{-1} X, Y) &= g(d\psi_h A_g(X'), d\psi_h Y') = (\psi_h^* g)(A_g X', Y') \\ &\overset{(\dagger)}{=} g(A_g X', Y') = \omega(d\psi_h^{-1} X, d\psi_h^{-1} Y) \\ &= ((\psi_h^{-1})^*\omega)(X,Y) \overset{(\ddagger)}{=} \omega(X,Y) \\ &= \omega(X,Y) = g(A_g X, Y),\end{aligned}$$

using the $G$-invariance of $g$ in step $(\dagger)$ and $G$ acting symplectically in step $(\ddagger)$. This implies $d\psi_h \circ A_g \circ d\psi_h^{-1} = A_g$, and we conclude

$$A_{\psi_h^* g} = A_g = d\psi_h \circ A_g \circ d\psi_h^{-1}. \tag{2.4}$$

△

Let us abbreviate $B := d\psi_h \circ A_g \circ d\psi_h^{-1}$. Since $g$ is $G$-invariant, we have $\psi_h^* g = g$, thus $d\psi_h^* = d\psi_h^{-1}$. In particular, $(d\psi_h^{-1})^* = (d\psi_h^*)^{-1} = d\psi_h$. Thus, we compute

$$\begin{aligned}B^*B &= (d\psi_h \circ A_g \circ d\psi_h^{-1})^*(d\psi_h \circ A_g \circ d\psi_h^{-1}) \\ &= (d\psi_h^{-1})^* \circ A_g^* \circ (d\psi_h^* \circ d\psi_h) \circ A_g \circ d\psi_h^{-1} \\ &= d\psi_h \circ A_g^* \circ (d\psi_h^{-1} \circ d\psi_h) \circ A_g \circ d\psi_h^{-1} \\ &= d\psi_h \circ A_g^* \circ A_g \circ d\psi_h^{-1}.\end{aligned} \tag{2.5}$$

To compute $J_{\psi_h^* g}$, by Claim 1 we need to determine the positive definite square root of $B^*B$ w.r.t. $\psi_h^* g$.

**Claim 2.** $\sqrt{B^*B} = d\psi_h \sqrt{A_g^* A_g} d\psi_h^{-1}$ *is the positive definite square root w.r.t.* $\psi_h^* g$.

*Proof of Claim 2.* Since $A_g^* A_g$ is positive definite w.r.t. $g$, by (2.5) $B^*B$ is positive definite w.r.t. $\psi_{h^{-1}}^* g = g = \psi_h^* g$. Analogously, $d\psi_h \sqrt{A_g^* A_g} d\psi_h^{-1}$ is positive definite w.r.t. $\psi_{h^{-1}}^* g = \psi_h^* g$. Finally, a short computation yields

$$(d\psi_h \sqrt{A_g^* A_g} d\psi_h^{-1})^2 = d\psi_h \sqrt{A_g^* A_g}^2 d\psi_h^{-1} \overset{(2.5)}{=} B^*B. \qquad \triangle$$

**Claim 3.** $\psi_h^* J_g = J_{\psi_h^* g}$

*Proof of Claim 3.* Claim 2 implies

$$\sqrt{B^*B}^{-1} = d\psi_h \sqrt{A_g^* A_g}^{-1} d\psi_h^{-1}. \tag{2.6}$$

Altogether, we obtain

$$\begin{aligned} J_{\psi_h^* g} &= A_{\psi_h^* g} \sqrt{A_{\psi_h^* g}^* A_{\psi_h^* g}}^{-1} \overset{(2.4)}{=} B\sqrt{B^*B}^{-1} \\ &\overset{(2.6)}{=} (d\psi_h \circ A_g \circ d\psi_h^{-1}) d\psi_h \sqrt{A_g^* A_g}^{-1} d\psi_h^{-1} \\ &= d\psi_h \circ A_g \sqrt{A_g^* A_g}^{-1} \circ d\psi_h^{-1} = d\psi_h \circ J_g \circ d\psi_h^{-1} = \psi_h^* J_g. \qquad \triangle \end{aligned}$$

Claim 3 completes the proof. □

Let $\mathfrak{M}(M)^G$ denote the space of $G$-invariant Riemannian metrics on $M$. Then, combining Lemmas 2.29 and 2.30 yields the following.

**Proposition 2.31.** *Suppose* $\psi$ *is a symplectic group action on* $(M, \omega)$*. Then* (2.2) *restricts to a continuous map*

$$\mathcal{J}^G(M, \omega) \to \mathfrak{M}(M)^G,$$

*and* (2.3) *restricts to a continuous map*

$$\mathfrak{M}(M)^G \to \mathcal{J}^G(M, \omega). \qquad \square$$

Thus, the proof that $\mathcal{J}^G(M, \omega)$ is contractible is concluded by the following.

**Lemma 2.32.** *The space* $\mathfrak{M}(M)^G$ *of* $G$*-invariant Riemannian metrics on* $M$ *is convex.*

*Proof.* Since the space $\mathfrak{M}(M)$ of *all* Riemannian metrics is convex, it suffices to check that $G$-invariance is preserved under convex combinations. Indeed, for each $h$, the map $g \mapsto \psi_h^* g$ is linear, hence linear combinations (in particular, convex combinations) of $G$-invariant metrics are $G$-invariant. □

**Corollary 2.33.** *If $G$ acts symplectically, the space $\mathcal{J}^G(M,\omega)$ is contractible when non-empty.* □

Non-emptiness of $\mathcal{J}^G(M,\omega)$ and $\mathcal{J}^G_\tau(M,\omega)$ requires an additional hypothesis on the $G$-action.

**Recall** (Proper actions)**.** A continuous action $A$ of a topological group $G$ on a Hausdorff topological space $X$ is called *proper* if and only if the map $G \times X \to X \times X, (g,x) \mapsto (g \cdot x, x)$ is a proper map (i.e., preimages of compact sets are compact). We say $G$ acts *properly at* $x \in X$ if and only if for every sequence $(x_n)$ in $X$ and $(g_n)$ in $G$ such that $\lim_n x_n = x_0$ and $\lim_n g_n \cdot x_n = x_0$, there exists a subsequence $n = n(k)$ such that $g_{n(k)}$ converges in $G$ as $k \to \infty$. If $G$ acts properly, it acts properly at each $x \in x$.
If $G$ is compact, its action on $X$ is always proper.

**Proposition 2.34.** *If $G$ acts symplectically and properly on $(M,\omega)$, the spaces $\mathcal{J}(M,\omega)$ and $\mathcal{J}_\tau(M,\omega)$ are non-empty.*

*Proof.* Since $G$ acts smoothly and properly, there exists a $G$-equivariant Riemannian metric on $M$ (see e.g. [DK00, Proposition 2.5.2; AB15, Theorem 3.65]). In light of Proposition 2.31, the corresponding compatible almost complex structure $J := J_g$ is $G$-equivariant. Since every compatible almost complex structure is tame, this completes the proof. □

This completes our discussion of compatible equivariant almost complex structures. Let us now turn to tameness. As indicated, we adapt Sévennec's argument (as presented by Wendl [Wen15]). Its starting point is the following.

**Observation 2.35.** Let $J_0$ be an almost complex structure on $M$. For all invertible $\phi \in \mathrm{End}_{\mathbb{R}}(TM)$, setting $J_\phi := \phi \circ J_0 \circ \phi^{-1}$ defines an almost complex structure on $M$.

*Proof.* Clearly, $J_\phi \in \mathrm{End}_{\mathbb{R}}(TM)$ is smooth, so it only remains to show $J_\phi^2 = -\mathrm{id}$. This is an easy computation: we have

$$J_\phi^2 = (\phi \circ J_0 \circ \phi^{-1}) \circ (\phi \circ J_0 \circ \phi^{-1}) = \phi \circ J_0^2 \circ \phi^{-1} = -\phi \circ \phi^{-1} = -\mathrm{id}\,. \qquad \square$$

We are interested in the following special case.

**Corollary 2.36.** *Let $J_0$ be an almost complex structure on $M$. For all $Y \in \overline{\mathrm{End}}_{\mathbb{C}}(TM, J_0)$, whenever $\mathrm{id} + \frac{1}{2}J_0 Y \in \mathrm{End}_{\mathbb{R}}(TM)$ is invertible,*

$$J_Y := (\mathrm{id} + \frac{1}{2}J_0 Y) J_0 (\mathrm{id} + \frac{1}{2}J_0 Y)^{-1} \tag{2.7}$$

*defines an almost complex structure on $M$.* □

In the following statement, we call a subset $U \subset E$ of a vector bundle $E$ *fibre-wise convex* if its intersection with every fibre is convex, and we denote by $\Gamma(U)$ the space of (smooth) sections of $E$ that are everywhere contained in $U$.

**Proposition 2.37** ([Wen15, Proposition 2.2.17])**.** *Suppose $J \in \mathcal{J}_\tau(M,\omega)$ is tame. There exists an open and fibre-wise convex subset $U \subset \overline{\mathrm{End}}_{\mathbb{C}}(TM, J_0)$ such that (2.7) defines a bijective continuous map $\Gamma(U) \to \mathcal{J}_\tau(M,\omega)$. In particular, each tame almost complex structure on $(M,\omega)$ is of the form $J_Y$ for a unique $Y \in \Gamma(U)$.*
*Let $\mathrm{End}^S_{\mathbb{R}}(TM,\omega,J_0) \subset \mathrm{End}_{\mathbb{R}}(TM)$ denote the sub-bundle of linear maps that are symmetric with respect to the metric $\omega(\cdot, J_0\cdot)$, then $\mathcal{J}(M,\omega) = \{J_Y \mid Y \in \Gamma(U \cap \mathrm{End}^S_{\mathbb{R}}(TM,\omega,J_0))\}$ with $Y$ being unique.* □

**Corollary 2.38** ([Wen15, Proposition 2.2.8])**.** *The spaces $\mathcal{J}_\tau(M,\omega)$ and $\mathcal{J}(M,\omega)$ are contractible.* □

A very similar argument applies to $\mathcal{J}^G_\tau(M,\omega)$; we merely consider $G$-equivariant sections instead.

**Proposition 2.39.** *Suppose $G$ acts smoothly on $M$; let $J_0 \in \mathcal{J}^G_\tau(M,\omega)$ be arbitrary. Choose a fibre-wise convex set $U \subset \overline{\mathrm{End}}_{\mathbb{C}}(TM, J_0)$ as in Proposition 2.37. Then the map $Y \to J_Y$ from (2.7) restricts to continuous bijections*

$$\Gamma^G(U) \to \mathcal{J}^G_\tau(M,\omega) \tag{2.8}$$

$$\Gamma^G(U \cap \mathrm{End}^S_{\mathbb{R}}(TM,\omega,J_0)) \to \mathcal{J}^G(M,\omega). \tag{2.9}$$

*Proof.* As the restriction of a continuous bijection, $Y \to J_Y$ is still continuous and injective. It remains to show this is well-defined and surjective in both cases. Throughout this proof, we abbreviate $\phi_Y := \mathrm{id} + \frac{1}{2}J_0 Y$.

Consider the tame case first. For well-definedness of (2.8), let $Y \in \Gamma^G(U)$ be given. We need to show $J_Y \in \mathcal{J}^G_\tau(M,\omega)$. We know $J_Y \in \mathcal{J}_\tau(M,\omega)$ by Proposition 2.37, hence only need to show $G$-equivariance of $J_Y$. Since $J_0$ and $Y$ are $G$-equivariant, so is $\phi_Y$. This implies $\phi_Y^{-1}$ is $G$-equivariant: for each $g \in G$, we have

$$(d\psi_g \circ \phi_Y^{-1}) \circ \phi_Y = d\psi_g = (\phi_Y^{-1} \circ \phi_Y) \circ d\psi_g = (\phi_Y^{-1} \circ d\psi_g) \circ \phi_Y,$$

hence $d\psi_g \circ \phi_Y^{-1} = \phi_Y^{-1} \circ d\psi_g$ follows. Combining these, we deduce that each $J_Y = \phi_Y \circ J_0 \circ \phi_Y^{-1}$ is $G$-equivariant.

For surjectivity of (2.9), suppose $J \in \mathcal{J}^G_\tau(M,\omega)$. By Proposition 2.37, we have $J = J_Y$ for a unique $Y \in \Gamma(U)$; we want to show $Y$ is $G$-equivariant. To this end, observe the following.

**Claim 1.** *For all $g \in G$ and $Y \in \Gamma(U)$, we have $\psi_g^* J_Y = J_{\psi_g^* Y}$, where $\psi_g^* Y := d\psi_g^{-1} \circ Y \circ d\psi_g$.*

*Proof.* First, we compute (since $J_0$ is $G$-equivariant)

$$\begin{aligned} d\psi_g^{-1} \circ \phi_Y \circ d\psi_g &= d\psi_g^{-1} \circ (\mathrm{id} + \frac{1}{2}J_0 Y) \circ d\psi_g = \mathrm{id} + \frac{1}{2}J_0(d\psi_g^{-1} \circ Y \circ d\psi_g) \\ &= \mathrm{id} + \frac{1}{2}J_0(d\psi_g^* Y) = \phi_{\psi_g^* Y}. \end{aligned}$$

This implies $d\psi_g^{-1} \circ \phi_Y \circ d\psi_g = \phi^{-1}_{\psi_g^* Y}$ since

$$\phi_{\psi_g^* Y} \circ (d\psi_g^{-1} \phi_Y^{-1} d\psi_g) = d\psi_g^{-1} \circ \phi_Y \circ d\psi_g \circ (d\psi_g^{-1} \circ \phi_Y^{-1} \circ d\psi_g) = \mathrm{id}\,.$$

Putting this all together, we compute

$$\begin{aligned} d\psi_g^{-1} J_Y d\psi_g &= d\psi_g^{-1} \phi_Y d\psi_g \circ d\psi_g^{-1} J_0 \phi_Y^{-1} d\psi_g = \phi_{\psi_g^* Y} \circ J_0 \circ d\psi_g^{-1} \phi_Y^{-1} d\psi_g \\ &= \phi_{\psi_g^* Y} J_0 \phi^{-1}_{\psi_g^* Y} = J_{\psi_g^* Y}\,. \end{aligned}$$ △

Since $J = J_Y$ is $G$-equivariant, we have $J_Y = J = \psi_g^* J = J_{\psi_g^* Y}$ for all $g \in G$. By uniqueness, this implies $\psi_g^* Y = Y$ for all $g \in G$. Therefore, $Y$ is $G$-equivariant. This completes the proof in the case case.

This basically proves the compatible case as well. For well-definedness, for $Y \in \Gamma^G(U \cap \mathrm{End}^S_{\mathbb{R}}(TM, \omega, J_0))$, then $J_Y \in \mathcal{J}^G_\tau(M, \omega)$ by the tame case. By Proposition 2.37, $U \in \Gamma(U \cap \mathrm{End}^S_{\mathbb{R}}(TM, \omega, J_0))$ implies $J_Y$ is also compatible. For surjectivity, if $J \in \mathcal{J}^G(M, \omega)$, we have $J = J_Y$ for a unique $Y \in \Gamma(U \cap \mathrm{End}^S_{\mathbb{R}}(TM, \omega, J_0))$: the same argument as in the tame case shows $Y$ is $G$-equivariant. □

**Corollary 2.40.** *If $G$ acts symplectically on $M$, the space $\mathcal{J}^G_\tau(M, \omega)$ is contractible.*

*Proof.* Proposition 2.39 yields a correspondence with $\Gamma^G(U)$. Since $U$ is fibre-wise convex, $\Gamma^G(U)$ is convex, hence contractible. The remaining argument is the exactly the same as for Corollary 2.38. □

**Remark 2.41.** This proof also shows that $\mathcal{J}^G(M, \omega)$ is contractible.

To summarize: altogether, in this section we have shown the following.

**Proposition 2.42.** *If a Lie group $G$ acts symplectically and properly on $(M, \omega)$, then $\mathcal{J}^G_\tau(M, \omega)$ and $\mathcal{J}^G(M, \omega)$ are non-empty and contractible.* □

# 3. Defining the iso-symmetric strata

In this chapter, we explain the definition of iso-symmetric strata and prove some basic properties. Recall from the introduction what the purpose of iso-symmetric strata is: we want to split the moduli space of holomorphic curves into countably many disjoint finite-dimensional smooth manifolds, taking into account the additional symmetry through the symplectic $G$-action on $M$. More precisely, the symplectic $G$-action on $(M, \omega)$ induces an action on the moduli space of holomorphic curves; we decompose the moduli space according to the stabilisers of this action. In the next step, each stratum is further split into countably many *walls* (which are smooth manifolds as well) according to the representation theory of the stabilisers. This representation theory allows computing the co-dimension of the walls, in principle.

Implementing this idea in practice is somewhat more involved. The correct definition of the strata is subtle; let us mention three features which hint at some of the underlying complexity. Firstly, our setting involves two group actions: the $G$-action on $M$, but in addition the automorphism group $\mathrm{Aut}(\Sigma, j, \theta)$ of a pointed Riemann surface $(\Sigma, j, \theta)$ acts on the space of $J$-holomorphic curves $(\Sigma, j) \to M$ by reparametrisation. The correct definition of strata needs to take this action into account as well. Secondly, to make the splitting into walls work, we also need to include some topological information, about the critical points and their orders. Finally, the stratification also depends on whether we have a simple or multiply covered curve: the latter involves extra data. All of this will be detailed in this chapter.

Let us focus on the "stabiliser" aspect of the definition first. In hindsight, iso-symmetric strata are inspired by the orbit type stratification of proper Lie group actions. For a proper smooth group action on a smooth finite-dimensional manifold, such a decomposition is a classical result: the manifold decomposes into so-called *orbit types*, determined by the group action's stabiliser. Each orbit type is a smooth manifold, and orbit types even form a Whitney stratification of the manifold. We aim to transfer this construction to our setting: the moduli space $\mathcal{M}_{g,m}(C, J)$ has finite virtual dimension, but is generally not a manifold. It is (locally) contained in an infinite-dimensional Banach manifold of maps: this means a more elaborate argument is needed.

Thus, we begin by reviewing the definition of orbit types in the finite-dimensional setting (Section 3.1). Turning to our setting, Section 3.2 explains some pitfalls transferring this to the moduli space $\mathcal{M}_{g,m}(C, J)$. Then, we are prepared to appreciate the full and correct definition of iso-symmetric strata (Section 3.3).

We prove that the number of distinct non-empty iso-symmetric strata is countable.[1] This (and the proof that iso-symmetric strata are smooth, for generic equivariant $J$) proceeds differently for stable and unstable holomorphic curves: to some extent, unstable curves are a simpler edge case, which we treat in Section 3.4. For stable curves (this is *false* for unstable curves), the group action is proper (Section 3.5) — this is a crucial ingredient for proving countability of the strata in Section 3.6. We close with some elementary properties of our strata (Section 3.7).

The properties in the last sections are *much* easier to prove for $A \times G$ finite (for instance, each stabiliser $(A \times G)_u$ is always compact, and countability of the number of strata simplifies as $A \times G$ has only finitely many subgroups overall). They are included as further motivation and justification for the given definitions, by means of proving they have reasonable properties. While the author believes that Theorems A and B extend to infinite groups with the current definitions (perhaps translating them to an equivalent phrasing in the process), this is not confirmed yet. We view these properties as evidence that these definitions are a useful starting point for further investigation. A reader only interested in the finite case may mostly skip their proofs (only reading Section 3.7), and continue with the next chapter directly.

## 3.1. Review: orbit type stratification

Before we dive into the moduli space of holomorphic curves, let us review the orbit type stratification for proper Lie group actions. This material is classical and well-known; all results in this section can be found in standard textbooks (e.g. [DK00; AB15]).

Throughout this section, assume that $G$ is a smooth Lie group acting smoothly and properly on a smooth manifold $M$. This induces a decomposition of $M$ called the *orbit type stratification*: loosely speaking, $M$ decomposes according to the stabiliser of the $G$-action into a locally finite collection of smooth submanifolds which fit together nicely.

**Definition 3.1** (Orbit type)**.** *Let $H \leqslant G$ be a closed subgroup. The* orbit type *of $H$ is*

$$M_{(H)} := \{p \in M \mid G_p \cong H\},$$

*where $G_p$ denotes the stabiliser of $p$ under the $G$-action, and $\cong$ denotes conjugate subgroups of $G$.*

Note that the stabiliser of every point is always a closed subgroup, as the $G$-action is continuous. In fact, it must be a Lie subgroup.

**Lemma 3.2.** *Suppose a Lie group $G$ acts continuously on a topological space $X$. Each stabiliser subgroup $G_x$ is a closed Lie subgroup of $G$.*

[1] Just to clarify terminology: in this document, a "countable" set is one which is bijective to a subset of the natural numbers. In other words, a countable set is either finite or countably infinite.

*Proof.* Since $G$ acts continuously on $X$, the stabiliser $G_x = \{g \in G \mid g \cdot x = x\} \subset G$ is a closed subset. By the closed subgroup theorem [Car30], closed subgroups of Lie groups are Lie subgroups. $\square$

The *principal orbit type* $M_{\text{reg}} \subset M$ is the orbit type of $H := \bigcap_{x \in M} G_x$. If $G$ acts effectively, $M_{\text{reg}}$ consists of all points with trivial stabiliser.

**Proposition 3.3** ([AB15, p. 80, Theorem 3.82])**.** *Let $G$ be a Lie group acting smoothly and properly on a smooth manifold $M$.*

(1) *Each orbit type is a $G$-invariant smooth submanifold of $M$; different connected components may have different dimensions.*

(2) *The principal orbit type is the unique open orbit type in $M$. The principal orbit type is open and dense.* $\square$

In addition, the collection of orbit types behaves nicely: if $G$ acts properly on $M$, the connected components of the orbit types of $G$ form a topological stratification and a Whitney stratification (the latter implies the former). We will postpone the definitions and mention the result and a few corollaries first.

**Proposition 3.4** ([AB15, Theorem 3.102; DK00, Theorem 2.7.4])**.** *Let $G$ be a Lie group acting smoothly and properly on a smooth manifold $M$. The connected components of the orbit types of $M$ form a Whitney stratification (see Definition 3.10 below).* $\square$

In particular, this implies

**Proposition 3.5.** *Let $G$ be a Lie group acting smoothly and properly on a smooth manifold $M$.*

(1) *The collection of orbit types $\{M_{(H)}\}_{H \leqslant G \text{ closed}}$ is locally finite.*

(2) *For each $H \leqslant G$ closed, $\overline{M}_{(H)}$ is the union of finitely many orbit types $\{M_{(H_i)}\}_{i \in I}$ of dimension at most $\dim M_{(H)}$.* $\square$

The proof of Proposition 3.4 also shows the following.

**Proposition 3.6.** *Let $G$ be a Lie group acting smoothly and properly on a smooth manifold $M$. Every point $p \in M$ has a neighbourhood $U \subset M$ such that each $q \in U$ has stabiliser conjugate to a subgroup of $G_p$.* $\square$

If G is finite, an even stronger version holds.

**Lemma 3.7.** *If $G$ is finite, every $p \in M$ has a neighbourhood $U \subset M$ such that every $q \in U$ has stabiliser* contained *in $G_p$ and $M_{(G_p)} \cap U = U \cap \operatorname{Fix}(G_p)$.* $\square$

Finally, here are the definitions of topological and Whitney stratifications.

**Definition 3.8** (Topological stratification [AB15, Definition 3.100]). *A* topological stratification *of a topological space $X$ is a partition of $X$ into topological manifolds $\{M_i\}_{i \in I}$, called* strata, *such that*

(1) *The partition is locally finite, i.e. each compact subset of $X$ intersects only finitely many strata.*

(2) *For each stratum $M_i$, there exists a finite set $I_i \subset I \setminus \{i\}$ such that $\overline{M_i} = M_i \cup \bigcup_{j \in I_i} M_j$.*

(3) *For all $j \in I_i$, we have $\dim M_j < \dim M_i$.*

**Definition 3.9** (Smooth stratification, [Sch03, Remark 4.0.1]). *If $X$ is a closed subset of a smooth manifold $M$, a* smooth stratification *of $X$ is a topological stratification $\{M_i\}_{i \in I}$ of $X$ such that each stratum is a smooth embedded submanifold of $M$.*

**Definition 3.10** (Whitney stratification [DK00, Definition 2.74; Sch03, Def. 4.1.2]). *A* Whitney stratification *of a smooth manifold $M$ is a smooth stratification such that for any two strata $M_i$ and $M_j$ with $j \in I_i$ (i.e., $M_j \subset \overline{M_i} \setminus M_i$), the following two conditions are met.*

(a) *For each sequence $(x_n)$ in $M_i$ such that $\lim_{n \to \infty} x_n = x \in M_j$ and $\lim_{n \to \infty} T_{x_n} M_i = L$ in the Grassmann bundle of $TM$, we have $T_x M_j \subset L$.*

(b) *For each sequence $(x_n)$ as in (a) and each sequence $(y_n)$ in $M_j$ converging to $x \in M_j$, the secant lines $l_n = \overline{x_i, y_i}$ (with respect to some local coordinates) converge to some limiting line contained in $L$.*

*It turns out that condition (b) implies condition (a), but not conversely.*

**Remark 3.11.** Whitney stratifications can also be defined for non-smooth spaces. (Then, conditions (a) and (b) need to be phrased in terms of a local embedding of $M$ into Euclidean space.) This generality is not necessary in our context.

Whitney stratifications were introduced by Whitney [Whi65] and since then, many further kinds of stratifications have been considered, differing in the local properties required of the strata. Whitney stratifications are among the strongest possible conditions. See Schürmann's monograph [Sch03, Section 4.2] for an overview of contemporary definitions. Our Definition 3.10 is a "Whitney b-stratification"' in Schürmann's terminology.

## 3.2. Finding the right definition of iso-symmetric strata

In this section, we discuss two less obvious aspects of the correct definition, thus motivating its eventual shape. Readers who are happy to accept the definition may skip this section. To keep the discussion manageable, in this section we ignore critical points, their orders and the distinction of simple and multiply covered curves:

we focus solely on the "stabiliser under the action" aspect. Throughout this section, let $G$ be a (smooth) Lie group and $(M, \omega, \psi)$ be a symplectic $G$-manifold such that the action $\psi$ of $G$ on $M$ is smooth and proper. Let $J$ always be a $G$-equivariant almost complex structure.

Let us begin with an easy observation, relating the stabilisers of a curve $u$ with the stabilisers of the points in its image.

**Observation 3.12.** Let $H \leqslant G$ be a closed[2] subgroup and suppose $[(\Sigma, j, \theta, u)] \in \mathcal{M}_{g,m}(C, J)$ such that $G_u = H$. Then $H \leqslant G_p$ for all $p \in \operatorname{im}(u)$.

*Proof.* Let $p \in \operatorname{im}(u)$ be arbitrary; write $p = u(z)$ for some $z \in \Sigma$. For each $h \in H$, we have $\psi_h \circ u = u$ as functions $\Sigma \to M$, hence $h \cdot p = \psi_h(u(z)) = (\psi_h \circ u)(z) = u(z) = p$. □

This looks like a very nice result, relating the $G$-action on $\mathcal{M}(J)$ with the orbit type stratification of $M$. However, let us caution the reader: this observation does *not* imply that $\operatorname{im}(u) \subset M_{(H)}$. There may be "exceptional points" $p \in \operatorname{im}(u)$ whose stabiliser is a superset of $G_u$. If $G$ is finite, this set of exceptional points is discrete, so an open dense set of $\operatorname{im}(u)$ belongs to $M_{(H)}$; this will be proven in Lemma 4.76. While this viewpoint does not lead directly to a definition of iso-symmetric strata, it will be useful in Chapter 5.

Observe (by Lemma 3.13 in the next section) that the $G$-action on $M$ induces a $G$-action on parametrised holomorphic curves $u\colon \Sigma \to M$ by $g \cdot u := \psi_g \circ u$, which descends to unparametrised curves as $g \cdot [u] := [\psi_g \circ u]$. The second key observation is that the stabilisers w.r.t. this action can differ between parametrised and unparametrised curves. As a parametrised curve, one would consider the following definition.

**False definition 3.1.** *Let $H \leqslant G$ be a closed subgroup. The corresponding (parametrised) iso-symmetric stratum of the moduli space of parametrised holomorphic curves consists of all curves $u$ such that the stabiliser $G_u$ of $u$ under the $G$-action is conjugate to $H$.*

In contrast, $g \in G$ stabilises the *unparametrised* curve $[u] \in \mathcal{M}(J)$ if $[g \cdot u] = [u]$. If $[g \cdot u] = [u]$ while $g \cdot u \neq u$, then $g$ acts by a reparametrisation of $u$. This is geometrically meaningful; considering this will be necessary for proving smoothness of the iso-symmetric strata. The proof of Lemma 4.74, a key lemma for proving smoothness, fails without including reparametrisations.

[2]In this thesis, we only consider closed subgroups of Lie groups. (Some people even argue that the only reasonably Lie subgroups to consider are closed, since non-closed subgroups can certainly be non-well behaved. For instance, taking a line with irrational slope in the torus $\mathbb{T}^2$ yields $\mathbb{R}$ as an immersed non-closed submanifold — which is dense, so there are no well-behaved slice charts.) While most definitions, including this, would generalise as written to non-closed groups, this additional generality is not useful: in practice, we are only interested in $H$ being the a stabiliser subgroup, which is always closed (by Lemma 3.2). Hence, if $H$ were not closed, the corresponding set is always empty, which is not interesting. Henceforth, we may mention the word "closed", but will implicitly assume it throughout this text.

Taking a step back, this mismatch occurs because our setting in fact features *two* group actions on the moduli space $\mathcal{M}(J)$, both of which matter: $G$ acts on $\mathcal{M}(J)$, but there is also a local action on moduli spaces of parametrised curves, by automorphism groups of the domain. Choosing representatives whose domain is a fixed oriented surface $\Sigma$ (see Lemma 3.22 later), automorphisms of the domain locally act by $\phi \circ [u] := [u \circ \phi]$ for each $\phi \in \mathrm{Aut}(\Sigma, j, \theta) \leqslant \mathrm{Diff}_+(\Sigma)$.[3] These actions commute; the crucial point is that they interact: the stabiliser of $u$ w.r.t. the $\mathrm{Aut}(\Sigma, j, \theta) \times G$-action can be larger than just $G_u$. This happens precisely whenever some curve $g \cdot u$ is just a reparametrisation of $u$, i.e. $\mathrm{im}(u)$ is $\psi_g$-invariant *as a set*, but *not point-wise*. The solution is to consider the stabiliser w.r.t. this joint action instead of just the $G$-action.

Because of the local action by automorphisms of the domain, the iso-symmetric strata intrinsically depend on the parametrisation of holomorphic curves: thus, we proceed by first defining a version of the iso-symmetric strata on parametrised curves, and take the quotient to stratify the space $\mathcal{M}(J)$ of unparametrised curves.

As a final twist, note that the automorphism group $\mathrm{Aut}(\Sigma, j, \theta)$ of a marked closed Riemann surface also depends on the complex structure $j$; this dependence is upper semi-continuous, but not continuous. In other words, for $j$ fixed, for each $j'$ close to $j$ the group $\mathrm{Aut}(\Sigma, j', \theta)$ is conjugate to a subgroup of $\mathrm{Aut}(\Sigma, j, \theta)$ — but it can "get smaller suddenly". Hence, we first stratify the moduli space of marked Riemann surfaces by the automorphism group $\mathrm{Aut}(\Sigma, j, \theta)$. This induces a decomposition of the moduli space of (parametrised) holomorphic curves, which we then refine to the iso-symmetric strata.

## 3.3. Definition of iso-symmetric strata

Let us finally give the correct definition of iso-symmetric strata. From this section onwards, we will have the following standing assumptions. (For clarity, we still mention them in important lemmas or theorems.)

**Convention.** Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold, $G$ be a smooth and $\psi$ be a symplectic and proper $G$-action on $(M, \omega)$. Fix positive integers $g$ and $m$ and a homology class $C \in H_2(M)$. Almost complex structures $J$ are always required to be $G$-equivariant and tame or compatible.

The starting observation for our analysis is that $G$-equivariance of almost complex structures has a useful consequence: if $J$ is $G$-equivariant, the $G$-action on $M$ induces a $G$-action on the space of all $J$-holomorphic curves.

**Lemma 3.13.** *Let $J$ be a $G$-equivariant almost complex structure on $M$. Then $G$ acts on $\bigcup_{C \in H_2(M)} \mathcal{M}_{g,m}(C, J)$ by $g \cdot [u] = [\psi_g \circ u]$. If $(\psi_g)_* C = C$ for all $g \in G$, this $G$-action restricts to $\mathcal{M}_{g,m}(C, J)$.*

[3] This local action is already present in the classical setting. As it is free for stable curves, this poses no issue in that context.

*Proof.* Let $g \in G$ and $[u] \in \mathcal{M}_{g,m}(C, J)$ be arbitrary; we show $[\psi_g \circ u] \in \mathcal{M}_{g,m}((\psi_g)_* C, J)$. Since $G$ acts by smooth maps, $\psi_g \circ u$ is smooth. For any almost complex structure $J$, a brief computation shows $\psi_g \circ u$ is $(\psi_g)_* J$-holomorphic:

$$\begin{aligned}(\psi_g)_* J \circ d(\psi_g \circ u) &= (\psi_g)_* J \circ d\psi_g \circ du = (d\psi_g \circ J \circ d\psi_g^{-1}) \circ d\psi_g \circ du \\ &= d\psi_g \circ J \circ du = d\psi_g \circ du \circ j = d(\psi_g \circ u) \circ j\end{aligned}$$

Since $J$ is $G$-equivariant, we conclude $\psi_g \circ u$ is $J$-holomorphic. We compute $(\psi_g \circ u)_* [\Sigma] = (\psi_g)_* (u_* [\Sigma]) = (\psi_g)_* C$.

It remains to show that this action is well-defined, i.e. independent of the parametrisation of $u$: indeed, for any biholomorphic map $\phi \colon \Sigma \to \Sigma'$ we have $[g \cdot (u \circ \phi)] = [(g \cdot u) \circ \phi] = [g \cdot u]$ since reparametrisation and the $G$-action commute. □

The strategy of this section was explained in the previous section. First, we stratify the moduli space of marked Riemann surfaces by the automorphism group $\mathrm{Aut}(\Sigma, j, \theta)$, in Subsection 3.3.1: this is necessary as $\mathrm{Aut}(\Sigma, j, \theta)$ is not continuous in $j$. As the second step, we introduce a suitable moduli space of parametrised holomorphic curves (Subsection 3.3.2). In Subsection 3.3.3, we define iso-symmetric strata of this parametrised space and of the moduli space of unparametrised curves.

### 3.3.1. Stratifying the moduli space of Riemann surfaces

We begin by splitting the moduli space of pointed Riemann surfaces according to their automorphism group. Recall the moduli space $\mathcal{M}_{g,m}$ of closed connected genus $g$ Riemann surfaces with $m$ marked points, up to biholomorphic equivalence which fixes the marked points in order. To be compatible with the parametrised version of iso-symmetric strata later, we make two bookkeeping choices.

**Convention.** Fix a closed connected genus $g$ surface $\Sigma$, an orientation of $\Sigma$ and an ordered set $\theta \subset \Sigma$ with $|\theta| = m$. All complex structures on $\Sigma$ will be taken to match the chosen orientation on $\Sigma$.

This convention will be in effect for the remainder of the thesis: $\Sigma$ will be the domain of parametrised curves, $\theta$ will be their set of marked points.

**Definition 3.14.** *Let $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$ be a closed subgroup. The corresponding* iso-symmetric stratum *of $\mathcal{M}_{g,m}$ is*

$$\mathcal{M}_{g,m}^A := \{[(\Sigma, j, \theta)] \in \mathcal{M}_{g,m} \mid \mathrm{Aut}(\Sigma, j, \theta) \cong A\},$$

*where $\cong$ denotes conjugate subgroups of* $\mathrm{Diff}_+(\Sigma, \theta)$.

This definition makes sense and is well-defined: by Lemma 3.22 below, any $[\Sigma', j, \theta'] \in \mathcal{M}_{g,m}$ has a reparametrisation $[(\Sigma, j', \theta)]$, and different parametrisations of $(\Sigma, j, \theta)$ have conjugate automorphism groups.

Each iso-symmetric stratum of $\mathcal{M}_{g,m}$ is a smooth manifold in a natural way. This follows from general nonsense, since the moduli space $\mathcal{M}_{g,m}$ is a smooth global quotient orbifold. A folklore fact is that "orbifolds are stratified spaces", for suitable interpretations of both words. The author is not aware of any written reference, perhaps due to the fact that both terms have several definitions. For our purposes, it suffices to know that every orbit type of a smooth global quotient orbifold is naturally a smooth manifold; this is not difficult to prove by hand.

The correct definition of smooth orbifolds is somewhat subtle; some older definitions have now been accepted as not quite correct. What is more, most texts in symplectic geometry gloss over the details of the definition. This thesis is not the place to rectify this either. However, we refer the interested reader to [McD06] for the definition of orbifolds that is commonly accepted in symplectic topology, to Moerdijk's survey [Moe02] or to Wendl's blog [Wen23b] for a less intimidating overview and introduction to this conundrum. Another common reference is Adem–Leida–Ruan's *Orbifolds and string topology* [ALR07].

The classical, more hands-on definition of orbifolds is via an *orbifold atlas*, a collection of compatible *orbifold charts*. Each point $p \in X$ of an $n$-dimensional topological (resp. smooth) orbifold $X$ has a neighbourhood which is homeomorphic (resp. diffeomorphic) to an open subset of a quotient $\mathbb{R}^n/H$ by a smooth action of a finite group $H$. The orbifold $X$ is called *effective* if it admits an orbifold atlas for which all these group actions are effective. Many orbifolds are effective, yet non-effective orbifolds also occur naturally.

A smooth orbifold is called a *global quotient* if has a presentation as the quotient space $M/G$ of a smooth manifold $M$ by the smooth action of a compact Lie group $G$, with the smooth structure on $M/G$ being uniquely determined by the smooth structure on $M$. The moduli space $\mathcal{M}_{g,m}$ is a global quotient orbifold, as the quotient of the Teichmüller space by its mapping class group. It is a well-known fact that all effective orbifolds are global quotient orbifolds (e.g. [ALR07, Corollary 1.24]). Recently, Pardon showed that every orbifold satisfying a mild geometric assumption admits a description as a global quotient orbifold [Par22, Corollary 1.3]. Using this abstract description would likely be bad mathematical style; we mention this merely to emphasize that global quotient orbifolds form a fairly large class.

Every point $p \in X$ in an orbifold has an associated *isotropy group*, the equivalence class of a finite group. If $X = M/G$ is a global quotient orbifold, the isotropy group is unique up to conjugation. Suppose from now on that $X$ is a global quotient orbifold. Suppose $\phi\colon U \subset X \to V \subset \mathbb{R}^n/H$ is an orbifold chart near $p \in X$, we define the isotropy group of $p$ as the conjugacy class of the stabiliser $H_{\phi(p)} \leqslant H$: choosing two different orbifold charts yields conjugate subgroups of $G$, hence this is well-defined. The *orbit type* of $H \leqslant G$ in $X$ is the set $X_{(H)} = \{x \in X : \mathrm{Stab}(x) \cong H\}$.

**Lemma 3.15.** *Every orbit type of a smooth $n$-dimensional global quotient orbifold $X$ admits an induced smooth manifold structure.*

While this lemma it not used in this thesis, it is an immediate consequence of Lemma 3.16 below, so we indicate a proof anyway. Lemma 3.16 is used to construct adapted Teichmüller slices in Section 4.2, and to apply Taubes' trick in Section 4.5.

**Lemma 3.16.** *Let $X$ be a smooth global quotient orbifold. Each $p \in X$ has a neighbourhood $U$ such that for all $q \in U$, the isotropy group* $\mathrm{Stab}(q)$ *is conjugate to a subgroup of* $\mathrm{Stab}(p)$*.*

*Proof.* Let $H \leqslant G$ and $p \in X_{(H)}$ be arbitrary. Let $n := \dim X$. Choose an orbifold chart $\phi\colon U \subset X \to V \subset \mathbb{R}^n/H_0$ around $p$. Since $p$ is a fixed point of $H$ by construction, we know $H$ is isomorphic to some subgroup of $H$. Since $H$ is finite, it acts properly discontinuously: thus, shrinking $V$, we may assume $H_0$ is isomorphic to $H$. Thus, every $\tilde{q} \in V$ has stabiliser contained in $H = V_{\phi(p)}$ and every $q \in U$ has stabiliser conjugate to a subgroup of $H$. □

*Proof of Lemma 3.15.* Let $H \leqslant G$ and $p \in X_{(H)}$ be arbitrary. Choose a neighbourhood $U$ of $p$ as in the previous Lemma 3.16. Then every $q \in U$ has stabiliser conjugate to a subgroup of $H$, and the orbit type $X_{(H)}$ is locally given as the fixed point set of $H$ on $\mathbb{R}^n$. This fixed point set is a smooth submanifold of $\mathbb{R}^n$ (e.g. [AB15, Proposition 3.93]). □

**Proposition 3.17.** *For every group $A$, the stratum $\mathcal{M}^A_{g,m}$ admits a smooth manifold structure, induced from the quotient orbifold structure on $\mathcal{M}_{g,m}$.*

*Proof of Proposition 3.17.* The moduli space $\mathcal{M}_{g,m}$ is a smooth orbifold; the isotropy group at a point $[(\Sigma, j, \theta)]$ is $\mathrm{Aut}(\Sigma, j, \theta)$ (see e.g. [Wen15, Theorem 4.2.10]). Hence, each stratum $\mathcal{M}^A_{g,m}$ is precisely the orbit type for the group $A$. Hence, Lemma 3.15 implies that each stratum is a smooth manifold. □

**Remark 3.18.** A lot more can be said about the strata $\mathcal{M}^A_{g,m}$: for each $(g, m)$, only few strata $\mathcal{M}^A_{g,m}$ are non-empty and there is a clear description of these groups $A$ and the dimension of the corresponding stratum.

**Definition 3.19.** *For any group $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$, the corresponding* pre-stratum *of the moduli space $\mathcal{M}_{g,m}(C, J)$ of unparametrised holomorphic curves is*

$$\begin{aligned}\mathcal{M}^A(J) :=& \{[(\Sigma, j, \theta, u)] \in \mathcal{M}_{g,m}(C, J) \mid [(\Sigma, j, \theta)] \in \mathcal{M}^A_{g,m}\} \\ =& \{[(\Sigma, j, \theta, u)] \in \mathcal{M}_{g,m}(C, J) \mid \mathrm{Aut}(\Sigma, j, \theta) \cong A\}.\end{aligned}$$

### 3.3.2. Parametrised moduli spaces

In this subsection, we define suitable parametrised versions of the moduli space $\mathcal{M}_{g,m}$, its stratum $\mathcal{M}^A_{g,m}$ and the pre-stratum $\mathcal{M}^A(J)$. We begin with the parametrised moduli space of pointed Riemann surfaces. Recall that we fixed a closed oriented connected genus $g$ surface $\Sigma$ and an ordered set $\theta \subset \Sigma$ of $m$ points.

**Definition 3.20.** *The* parametrised moduli space *of pointed Riemann surfaces is given by $\widetilde{\mathcal{M}}_{g,m} := \{(\Sigma, j, \theta) \mid j \in \mathcal{J}(\Sigma)\}$.*

We endow the space $\widetilde{\mathcal{M}}_{g,m}$ with the Gromov–Hofer topology. The details of the construction are a pain (see interested reader may consult [Hum97] or [Abb14; BEHWZ] if they are interested in punctured curves also); for our purposes, it suffices to know the following properties.

**Lemma 3.21** ([Abb14; BEHWZ; Hum97]). *The space $\widetilde{\mathcal{M}}_{g,m}$ admits a separable metrizable topology such that a sequence $(\Sigma, j_k, \theta)$ in $\widetilde{\mathcal{M}}_{g,m}$ converges to $(\Sigma, j, \theta) \in \widetilde{\mathcal{M}}_{g,m}$ if and only if $j_k \to j$ in the $C^\infty_{loc}$-topology.*

*The space $\mathcal{M}_{g,m}$ admits a separable metrizable topology such that a sequence $[(\Sigma_k, j_k, \theta_k)] \in \mathcal{M}_{g,m}$ converges to $[(\Sigma, j, \theta)] \in \mathcal{M}_{g,m}$ if and only if there are reparametrisations $[(\Sigma, j'_k, \theta)]$ of $[(\Sigma_k, j_k, \theta_k)]$ such that $j'_k \to j$ in the $C^\infty_{loc}$-topology.* □

The following result is easy to show; we omit the details.

**Lemma 3.22.** *The canonical projection $\phi\colon \widetilde{\mathcal{M}}_{g,m} \to \mathcal{M}_{g,m}$ is continuous and surjective.* □

For each closed subgroup $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$, we define the corresponding stratum of $\widetilde{\mathcal{M}}_{g,m}$ by

$$\widetilde{\mathcal{M}}^A_{g,m} := \{(\Sigma, j, \theta) \in \widetilde{\mathcal{M}}_{g,m} \mid \mathrm{Aut}(\Sigma, j, \theta) = A\}.$$

The projection $\phi$ restricts to a map $\widetilde{\mathcal{M}}^A_{g,m} \to \mathcal{M}^A_{g,m}$. The stratum $\mathcal{M}^A_{g,m}$ is captured by the parametrised version $\widetilde{\mathcal{M}}^A_{g,m}$:

**Lemma 3.23.** *The restriction $\phi|_{\widetilde{\mathcal{M}}^A_{g,m}} : \widetilde{\mathcal{M}}^A_{g,m} \to \mathcal{M}^A_{g,m}$ is surjective.*

*Proof.* Let $[(\Sigma, j, \theta)] \in \mathcal{M}^A_{g,m}$ be arbitrary. By definition, $A' := \mathrm{Aut}(\Sigma, j, \theta) \leqslant \mathrm{Diff}(\Sigma, \theta)$ is conjugate to $A$, i.e. $A' = \phi \circ A \circ \phi^{-1}$ for some $\phi \in \mathrm{Diff}(\Sigma, \theta)$. Consider the reparametrisation $(\Sigma, \phi_* j, \theta)$ of $(\Sigma, j, \theta)$. Since $\phi\colon (\Sigma, j) \to (\Sigma, \phi_* j)$ biholomorphic, Lemma 3.24 below shows that $\mathrm{Aut}(\Sigma, \phi_* j, \theta) = A$, hence $(\Sigma, \phi_* j, \theta) \in \widetilde{\mathcal{M}}^A_{g,m}$. □

This proof used the following simple computation, whose details we omit.

**Lemma 3.24.** *Suppose $(\Sigma, j, \theta)$ is a pointed Riemann surface and $\Psi\colon (\Sigma, j, \theta) \to (\Sigma', j', \theta')$ is biholomorphic with $\Psi|_\theta = \theta'$. Then $F(\Psi)\colon \mathrm{Aut}(\Sigma, j, \theta) \to \mathrm{Aut}(\Sigma', j', \theta'), \phi \mapsto \Psi \circ \phi \circ \Psi^{-1}$ is a group isomorphism.* □

Next, we come to the moduli space of holomorphic curves.

**Definition 3.25.** *The moduli space of* parametrised closed $J$-holomorphic curves *of genus $g$ with $m$ marked points is*

$$\widetilde{\mathcal{M}}(J) := \{(\Sigma, j, \theta, u) \mid j \in \mathcal{J}(\Sigma), u\colon (\Sigma, j) \to (M, J) \text{ is } J\text{-holomorphic}, [u] = C\}.$$

**Remark 3.26.** This space $\widetilde{\mathcal{M}}(J)$ is infinite-dimensional (probably a Fréchet manifold); in subsequent analysis, we will describe the moduli space locally, using a Teichmüller slice for $j$ — the analytical setup will use a finite-dimensional local model.

Note that we prescribe both the domain $\Sigma$ and the set $\theta$ of marked points. This is not conceptually relevant: both choices are merely bookkeeping measures to keep the parametrised space simpler. Fixing $\Sigma$ is technically necessary (otherwise, $\widetilde{\mathcal{M}}(J)$ were a proper class, which is an unnecessary complication); fixing $\theta$ avoids having to differentiate w.r.t. changes in the marked points, and incurring a loss of derivatives which would make certain maps non-smooth. Again, we emphasize that the parametrised space is an auxiliary object; its precise definition is not important.

Again, the topologies on the modul spaces of parametrised and unparametrised holomorphic curves are standard; we omit the details.

**Fact** ([Abb14; BEHWZ; Hum97])**.** The space $\widetilde{\mathcal{M}}(J)$ admits a separable metrizable topology such that a sequence $(\Sigma, j_k, \theta, u_k)$ in $\widetilde{\mathcal{M}}(J)$ converges to $(\Sigma, j, \theta, u) \in \widetilde{\mathcal{M}}(J)$ if and only if $j_k \to j$ and $u_k \to u$ in the $C^\infty_{\text{loc}}$-topology.

The space $\mathcal{M}(J)$ admits a separable metrizable topology such that a sequence of curves $[(\Sigma_k, j_k, \theta_k)] \in \mathcal{M}(J)$ converges to $[(\Sigma, j, \theta, u)] \in \mathcal{M}(J)$ if they can be reparametrised as a sequence $[(\Sigma, j_k, \theta, u_k)]$ such that $j_k \to j$ and $u_k \to u$ in $C^\infty_{\text{loc}}$. □

If $M$ is a single point, this reduces to the topologies on $\widetilde{\mathcal{M}}_{g,m}$ and $\mathcal{M}_{g,m}$, respectively.

**Lemma 3.27.** *The canonical projection $\widetilde{\mathcal{M}}(J) \to \mathcal{M}(J)$ is continuous.* □

**Lemma 3.28.** *The following diagram of canonical projections commutes.* □

$$\begin{array}{ccc} \widetilde{\mathcal{M}}(J) & \longrightarrow & \widetilde{\mathcal{M}}_{g,m} \\ \downarrow & & \downarrow \\ \mathcal{M}(J) & \longrightarrow & \mathcal{M}_{g,m} \end{array} \qquad (*)$$

This definition behaves well: the parametrised moduli space projects to the unparametrised space. In particular, our bookkeeping choices do not lose information.

**Lemma 3.29.** *The canonical projection $\widetilde{\mathcal{M}}(J) \to \mathcal{M}(J)$ is surjective.*

*Proof.* This follows from Lemma 3.22: for $[(\Sigma', j', \theta', u')] \in \mathcal{M}_{g,m}(C, J)$, the domain $(\Sigma', j', \theta')$ has a reparametrisation $(\Sigma, j, \theta) \in \widetilde{\mathcal{M}}_{g,m}$. □

Finally, we define parametrised versions of the pre-strata $\mathcal{M}^A(J)$.

**Definition 3.30.** *For a group $A \leqslant \mathrm{Diff}(\Sigma, \theta)$, consider the* parametrised pre-stratum

$$\widetilde{\mathcal{M}}^A(J) := \{(\Sigma, j, \theta, u) \in \widetilde{\mathcal{M}}(J) \mid \mathrm{Aut}(\Sigma, j, \theta) = A\}.$$

**Observation 3.31.** The projection $\widetilde{\mathcal{M}}(J) \to \mathcal{M}(J)$ restricts to a map $\Phi\colon \widetilde{\mathcal{M}}^A(J) \to \mathcal{M}^A(J)$; the commutative diagram $(*)$ restricts to a commutative diagram. $\square$

$$\begin{array}{ccc} \widetilde{\mathcal{M}}^A(J) & \xrightarrow{\tilde{\pi}} & \widetilde{\mathcal{M}}^A_{g,m} \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \phi} \\ \mathcal{M}^A(J) & \xrightarrow{\pi} & \mathcal{M}^A_{g,m} \end{array}$$

Lemma 3.23 implies the projection $\Phi$ is surjective, i.e. $\widetilde{\mathcal{M}}^A(J)$ captures the pre-stratum $\mathcal{M}^A(J)$.

### 3.3.3. Correct definition of iso-symmetric strata

Let us now give the correct definition of iso-symmetric strata. The full definition features several components: we always stratify by the automorphism group $A := \mathrm{Aut}(\Sigma, j, \theta)$ of the domain $(\Sigma, j, \theta)$ of the curve $u$, the stabiliser $(A \times G)_u$ of $u$ under the $A \times G$-action given by $\phi \cdot u = u \circ \phi^{-1}$ and the number and orders of the critical points of $u$. The strata also differ for simple and multiply covered curves. For each open subset $U \subset M$, we consider strata of simple curves having an injective point which is mapped to $U$. For a multiply covered curve $u = v \circ \phi$, we use, in addition to all the above information on the underlying simple curve $v$, the degree of $\phi$ as well as the number and orders of $\phi$'s branched points.

Phew, this is quite a mouthful. It is perfectly fine to be a bit intimidated by this list. Take a deep breath, get a glass of water and bear with the author — we will examine these features in turn.

The first aspect, the group $A$, is built into our construction of the strata: each iso-symmetric stratum refines the corresponding pre-stratum. For the stabilisers under the $A \times G$-action, the previous subsection is comes in. The advantage of the parametrised pre-strata is that each $\widetilde{\mathcal{M}}^A(J)$ enjoys a genuine $A$-action: $A$ acts from the left by $\phi \cdot u = u \circ \phi^{-1}$; the parametrised pre-stratum is $A$-invariant because of Lemma 3.24.

Observe that $G$ acts on $\widetilde{\mathcal{M}}(J)$ by $g \cdot u := \psi_g \circ u$. This descends to $\mathcal{M}(J)$, and agrees with the $G$-action defined in Lemma 3.13. Each parametrised pre-stratum $\widetilde{\mathcal{M}}^A(J)$ is vacuously $G$-invariant. The $A$- and the $G$-actions on $\widetilde{\mathcal{M}}^A(J)$ commute, hence induce a left $A \times G$-action on each $\widetilde{\mathcal{M}}^A(J)$. Let us consider the orbit types with respect to this action.

**Definition 3.32.** *For closed subgroups $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$ and $H \leqslant A \times G$, we consider the subsets*

$$\widetilde{\mathcal{M}}^{A,H}(J) := \{(\Sigma, j, \theta, u) \in \widetilde{\mathcal{M}}^A(J) \mid (A \times G)_u \cong H\},$$

*where $\cong$ denotes conjugate subgroups of $A \times G$, and*

$$\mathcal{M}^{A,H}(J) := \Phi(\widetilde{\mathcal{M}}^{A,H}(J)) \subset \mathcal{M}^A(J),$$

*the image of the projection $\Phi\colon \widetilde{\mathcal{M}}^A(J) \to \mathcal{M}^A(J)$.*

**Remark 3.33.** Two remarks about this definition are in order. Firstly, by definition, the unparametrised sets $\mathcal{M}^{A,H}(J)$ only depend on the conjugacy class of $A$ and $H$, not on the groups themselves. Secondly, if $A \times G$ is finite, we could have equivalently asked for *isomorphic* instead of conjugate groups. In fact, locally, these sets are characterised by an *equality* of subgroups $(A \times G)_u = H$. In general, the situation is more subtle: for instance, defining the parametrised strata using an equality of sub-groups $(A \times G)_u = H$ would usually yield uncountably many distinct strata.

**Lemma 3.34.** *Each set $\mathcal{M}^{A,H}(J)$ is $G$-invariant.*

*Proof.* Observe that $\widetilde{\mathcal{M}}^{A,H}(J)$ is $G$-invariant: the stabilisers of curves $u$ and $g \cdot u$ are conjugate by $g$. □

We have now covered the hardest part of defining iso-symmetric strata. Let us proceed to the remaining aspects. The first extra data are required to define suitable walls in Chapter 5: each holomorphic curve $u$ has an associated *restricted normal Cauchy–Riemann operator* $D_u^{N,\mathrm{restr}}$ (see Definition 5.12); this is a Cauchy–Riemann type operator defined on sections of the *restricted generalised normal bundle* $N_{u^K}$ of $u$. (We will define this in Section 5.1.[4]) We would like $N_{u^K}$ to vary smoothly within each stratum: if $u = v \circ \phi$ for a simple curve $v$ and a branched cover $\phi$, the topology of $N_{u^K}$ depends on the branch points of $\phi$ and critical points of $v$, and their orders. Thus, we need to include these data in the definition of the stratum. A second condition is only relevant if $M$ is not compact: we require curves to have an injective point mapped to a particular open subset. We will explain the particular reason for this additional condition on page 65.

Finally, the automorphism group of a multiply covered curve adds additional symmetries, which our definition does not capture yet. We merely propose a candidate definition at this stage. For simple curves, the are no branch points to consider and prescribing the critical points and their orders is sufficient.

**Definition 3.35** (Iso-symmetric strata of simple curves)**.** *Let $U \subset M$ be an open set. For closed subgroups $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$ and $H \leqslant A \times G$ and an integer $0 \le k \le m$, consider the set*

$$\widetilde{\mathcal{M}}_U^{A,H}(J) := \{(\Sigma, j, \theta, u) \in \widetilde{\mathcal{M}}^{A,H}(J) \mid v \text{ has an injective point mapped to } U\}$$

*and denote by $\mathcal{M}_U^{A,H}(J) := \{[(\Sigma, j, \theta, u)] \mid (\Sigma, j, \theta, u) \in \widetilde{\mathcal{M}}_U^{A,H}(J)\}$ its space of equivalence classes.*

*For $k$-tuple $\boldsymbol{l} = (l_1, \ldots, l_k)$ of positive integers, the* iso-symmetric stratum $\mathcal{M}_{U,\boldsymbol{l}}^{A,H}(J)$ *corresponding to $A$, $H$, $U$, $\boldsymbol{l}$ and $J$ consists of all equivalence classes of simple curves $(\Sigma, j, \theta, v) \in \widetilde{\mathcal{M}}_U^{A,H}(J)$ such that for all $i = 1, \ldots, k$, the marked point $\zeta_i$ is a critical point of $v$ of order $l_i$.*

[4] For the experts, yes we can already confirm that $D_u^{N,\mathrm{restr}}$ is indeed related to the normal Cauchy–Riemann operator of $u$.

For multiply covered curves, we combine Wendl's stratification [Wen23d] with our setup.[5] We add additional data describing the branch points and their orders. These are called the *branching data* of a branched cover.

**Definition 3.36** (Branching data [Wen23d, Section 2.2.3])**.** *Let $d \geq 1$ and $r \geq 0$ be integers.* Branching data of degree $d$ with $r$ critical values *are an $r$-tuple $\boldsymbol{b} = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_r)$, where each $\boldsymbol{b}_i$ is a non-empty finite ordered set $(\boldsymbol{b}_i = b_{i,1}, \dots, b_{i,q_i})$ of natural numbers such that $b_{i,1} + \cdots + b_{i,q_i} = d$ and at least one of $b_{i,1}, \dots, b_{i,q_i}$ is at least two.*

The space of all degree $d$ branched covers of $\Sigma$ decomposes into strata $\mathcal{M}_{\mathbf{b},d}(j)$. As in the previous definitions, we will use parametrised representatives.

**Definition 3.37** ([Wen23d, Section 2.2.3])**.** *Let a complex structure $j$ on $\Sigma$, integers $d \geq 1$ and $r \geq 0$ and branching data $\boldsymbol{b}$ of degree $d$ with $r$ critical values be given. We denote by $\widetilde{\mathcal{M}}_{\boldsymbol{b},d}(j)$ the moduli space of all closed connected unparametrised curves $\phi$ of degree $d$ mapping into $(\Sigma, j)$ with $q_1 + \cdots + q_r$ marked points $\zeta_1^1, \dots, \zeta_1^{q_1}, \dots, \zeta_r^1, \dots, \zeta_r^{q_r}$ such that*

- *there are distinct points $w_1, \dots, w_r \in \Sigma$ such that $\phi^{-1}(w_i) = \{\zeta_i^1, \dots, \zeta_i^{q_i}\}$ for each $i = 1, \dots, r$*
- *for each $i = 1, \dots, r$ and $j = 1, \dots, q_i$, $\phi$ is $b_i^j$-to-one on a punctured neighbourhood of $\zeta_i^j$;*
- *$\phi$ has no critical points outside of the marked points.*

*Note that $d$ and $\boldsymbol{b}$ determine the genus $h$ of $\phi$ via the Riemann–Hurwitz formula. Define $\mathcal{M}_{\boldsymbol{b},d}(j) \subset \mathcal{M}_{h,0}(d[\Sigma], j)$ as the image of $\widetilde{\mathcal{M}}_{\boldsymbol{b},d}(j)$ modulo reparametrisations and forgetting the marked points. (If $h$ is negative, the space $\mathcal{M}_{\boldsymbol{b},d}(j)$ is simply empty.)*

Finally, for a multiply covered curve $u = v \circ \psi$, where $v$ is a simple curve and $\psi$ a holomorphic branched cover, the automorphism group of $\psi$ should be relevant. It turns out that this is not quite right, as $\psi$ need not be regular: $\psi$ could have degree larger than one, but trivial automorphism group — hence, considering just automorphisms of $\psi$ would not show the full extent of symmetries present. (The operators defining walls in Chapter 5 should be equivariant under this group: considering just the automorphism group discards useful and important information.) One solution is to consider the *generalised automorphism group* (see e.g. [Wen23d, Definition 2.6.]) instead. It is a finite group of cardinality $\deg(\psi)$; if $\psi$ is regular, the generalised automorphism group and the automorphism group coincide. As we do not study multiply covered curves yet, we omit further details.
Altogether, we arrive at the following candidate definition for multiply covered curves.

**Candidate Definition 3.38** (Iso-symmetric strata for multiple covers)**.** *Given*

[5] An alternative definition could be given using the language of local systems, following the approach of Doan and Walpuski [DW23].

- *closed subgroups* $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$ *and* $H \leqslant A \times G$,
- *an open subset* $U \subset M$,
- *integers* $d \geq 1$, $r \geq 0$ *and branching data* $\boldsymbol{b}$ *of degree* $d$ *with* $r$ *critical values,*
- *a group* $K$ *of order at most* $d$,
- *an integer* $0 \leq k \leq m$ *and a* $k$*-tuple* $\boldsymbol{l} = (l_1, \ldots, l_k)$ *of positive integers,*

*the corresponding iso-symmetric stratum* $\mathcal{M}^{A,H;\,K}_{U,\mathbf{l};\,\mathbf{b},d}(J)$ *consists of all curves* $[u] \in \mathcal{M}^{A,H}(J)$ *which have representatives factorising as* $u = v \circ \phi$, *where*

- $[v] \in \mathcal{M}^{A,H}_{U,\boldsymbol{l}}(J)$ *is a simple curve* (*intersecting* $U$) *whose critical points have prescribed orders* $\boldsymbol{l}$,
- $\phi \in \widetilde{\mathcal{M}}_{\boldsymbol{b},d}(j)$ *for some* $j \in \mathcal{J}(\Sigma)$, *and*
- $\phi$ *has generalised automorphism group* $K$.

Whenever the iso-symmetric stratum $\mathcal{M}^{A,H}_{\mathcal{U},\mathbf{l}}(J)$ of simple curves is smooth, the set $\mathcal{M}^{A,H;\,K}_{U,\mathbf{l};\,\mathbf{b},d}(J)$ is also a smooth manifold, of dimension $\dim \mathcal{M}^{A,H;\,K}_{U,\mathbf{l};\,\mathbf{b},d}(J) = 2r + \dim \mathcal{M}^{A,H}_{\mathcal{U},\mathbf{l}}(J)$. This is a well-known fact: the core idea is that each multiply covered curve has a *unique* parametrization as a product $u = v \circ \psi$ (see Lemma 3.39)— up to reparametrisations, which we quotient by. The space of branched covers $\mathcal{M}_{\mathbf{b},d}(j)$ is locally parametrised by the location of the critical values, hence a smooth manifold of real dimension $2r$.

Suppose $u = v \circ \psi \in \mathcal{M}^{A,H;\,K}_{U,\mathbf{l};\,\mathbf{b},d}(J)$. Playing devil's advocate, we could ask if the above definition is really the correct one: could the curve $u$ have larger stabiliser than $v$ w.r.t. the $A \times G$-action? If so, this should impose another constraint, which should be incorporated into the stratification. We will see that the answer is no: there is no need to worry about this, and the current definition looks sensible.

As a warm-up, let us compare the stabilisers of $u$ and $v$ under just the $G$-action: every stabiliser of $v$ clearly stabilises $u$, so $G_v \subset G_u$. There is also a partial converse: $g \in G_u$ implies $g \circ v$ is a reparametrisation of $v$: observe

$$v \circ \psi = u = g \cdot u = (g \circ v) \circ \psi,$$

and — most importantly — that $g \circ v$ is a simple curve. Thus, $v \circ \psi$ and $(g \cdot v) \circ \psi$ are two different decompositions of $u$ into a simple curve and a holomorphic branched cover. The following standard fact shows that $v$ and $g \cdot$ are reparametrisations of each other.

**Lemma 3.39** (e.g. [Wen15, Theorem 2.15.2])**.** *If* $u$ *is multiply covered,* $u$ *decomposes as* $u = v \circ \psi$ *for a simple holomorphic curve* $v$ *and* $\psi$ *a holomorphic branched cover. Moreover,* $v$ *is unique up to reparametrisation.* □

Let us now compare $(A \times G)_u$ and $(A \times G)_v$. Suppose $g$ acts by a reparametrisation of $u = v \circ \psi$, i.e. $g \circ u = u \circ \phi$ for some $(\phi, g) \in A \times G$. Then we have

$$v \circ \psi = u = g \cdot u \circ \phi^{-1} = (g \cdot v) \circ (\psi \circ \phi^{-1}),$$

so $g \cdot v$ and $v$ are reparametrisations of each other. In other words, if $g$ acts by a reparametrisation of $u$, then so does $v$. This does not imply a direct relationship between $(A \times G)_u$ and $(A \times G)_v$, but a relation between their projections to $G$. This should be sufficient for all later arguments. To summarize, the author is optimistic that this candidate definition will be suitable for proving smoothness of the walls derived from it. The discussion of Chapter 5 has not been adapted to multiple covers yet; doing so would give conclusive evidence about the definition being suitable.

To close this discussion of definitions, note that our definitions for simple and multiply covered curves are consistent.

**Observation 3.40.** We have $\mathcal{M}^{A,H}_{U,1}(J) = \mathcal{M}^{A,H;\,\langle e \rangle}_{U,1;\,\emptyset,1}(J)$.

*Proof.* By convention, empty branching data means a covering is unbranched. An unbranched degree one holomorphic covering is biholomorphic, hence a reparametrisation, and has trivial automorphism group. □

In the remainder of this chapter, we will prove a few basic properties of the isosymmetric strata: most importantly, the number of disjoint non-empty strata is countable. An important ingredient of the proof is that the group $H$ is a compact Lie group whenever $u$ is a stable curve; this is proven via properness of the $A \times G$-action on $\widetilde{\mathcal{M}}^A(J)$. Hence, it behoves us to explain stable and unstable curves first.

## 3.4. Unstable curves and their decomposition

Holomorphic curves can be classified as stable or unstable. For many intents and purposes, unstable curves are an exceptional case: they must be treated separately, but their analysis is often much simpler. Since the argument of the next section requires stability, let us analyse unstable curves first. Since connected unstable curves are constant, they are never somewhere injective — but we can describe them via the sets $\mathcal{M}^{A,H}(J)$.

**Definition 3.41.** *A pointed closed surface $(\Sigma, \theta)$ is called* stable *if and only if $\chi(\Sigma \setminus \theta) < 0$. In other words, if $\Sigma$ has genus $g$ and $m := |\theta|$, $(\Sigma, \theta)$ is stable if and only if $(2 - 2g) - m < 0 \Leftrightarrow 2g + m \geq 3$.*

**Definition 3.42.** *A holomorphic curve $(\Sigma, j, \theta, u)$ is called* stable *if and only if each connected component of $(\Sigma, \theta)$ on which $u$ is constant is a stable pointed Riemann surface. Otherwise, $u$ is called* unstable.

**Remark 3.43.** Remember that in this thesis, we assume $\Sigma$ to be connected; we chose the phrasing above since it is also correct for disconnected curves. In our setting, Definition 3.42 boils down to the following: if $2g+m \geq 3$, every curve in $\mathcal{M}_{g,m}(C, J)$ is stable; for $2g + m < 3$, a curve in $\mathcal{M}_{g,m}(C, J)$ is stable if and only if it is non-constant.

Stability is useful for several reasons. The second property will be proven as Proposition 3.56 in the next section.

**Lemma 3.44** ([Wen15, Corollary 4.2.7])**.** *The automorphism group* $\mathrm{Aut}(\Sigma, j, \theta)$ *of a stable Riemann surface is finite.* □

**Lemma 3.45.** *For each closed subgroup* $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$*, the* $A$*-action on* $\widetilde{\mathcal{M}}^A(J)$ *is proper at each stable curve.*

As a corollary, we will obtain the following.

**Proposition 3.46.** *A (possibly constant) holomorphic curve* $[(\Sigma, j, \theta, u)]$ *is stable if and only if its automorphism group* $\mathrm{Aut}(u)$ *is finite.*

In our setting of connected curves, constant-ness (hence stability) is determined by the homology class: since $J$ is tamed, a curve in $\mathcal{M}_{g,m}(C, J)$ is constant if and only if $C = 0$. Thus, unstable curves are precisely the curves in $\mathcal{M}_{g,m}(0, J)$ for $2g + m < 3$. In these cases, the moduli space $\mathcal{M}_{g,m}(0, J)$ is identified with $M$ via

$$\mathcal{M}_{g,m}(0, J) \ni [(\Sigma, j, \theta, u \equiv p)] \mapsto p \in M, \tag{3.1}$$

and trivially a $2n$-dimensional smooth manifold.

Stability of a curve $u$ is also reflected by which set $\mathcal{M}^{A,H}(J)$ $u$ belongs to: this will imply that the unstable curves are partitioned into countably many distinct sets.

**Lemma 3.47.** *Let* $M$ *and* $N$ *be smooth manifolds; suppose* $A \leqslant \mathrm{Diff}(M)$ *acts transitively on* $N$*. Then a smooth map* $f\colon M \to N$ *is* $A$*-invariant (w.r.t. the action* $\phi \cdot u := u \circ \phi^{-1}$*) if and only if* $u$ *is constant.*

*Proof.* Direction "⇐" is immediate. For direction "⇒", suppose $f\colon M \to N$ is $A$-invariant. Fix $p \in M$ and consider $q := f(p)$. For each $p' \in M$, choose a diffeomorphism $\phi \in A$ mapping $p$ to $p'$; then $f(p') = f(\phi(p)) = f(p) = q$ by invariance. □

**Observation 3.48.** A constant curve $u \in \widetilde{\mathcal{M}}^A(J)$ with $u \equiv p \in M$ has stabiliser $(A \times G)_u = A \times G_p$. □

If $2g + m < 3$, the automorphism group of $(\Sigma, j, \theta)$ always acts transitively. Altogether, we obtain the following.

**Corollary 3.49.** *All unstable curves in* $\widetilde{\mathcal{M}}(J)$ *necessarily have* $C = 0$.
*Each unstable curve* $[(\Sigma, j, \theta, u)]$ *belongs to some set* $\mathcal{M}^{A,A\times K}(J)$*, where* $A = \mathrm{Aut}(\Sigma, j, \theta)$ *for some complex structure* $j \in \mathcal{J}(\Sigma)$ *and* $K \leqslant G$ *is some orbit type of* $M$*. Conversely, the set* $u \in \mathcal{M}^{A,A\times G_p}(J)$ *consists of constant curves* $u \equiv q$ *for* $q \in g \cdot p$ *for some* $g \in G$*.* □

**Remark 3.50.** The same result holds for constant curves on a stable domain; so which set $\mathcal{M}^{A,H}(J)$ an unstable curve belongs to also determines constant-ness.

**Corollary 3.51.** *For each non-empty set $\mathcal{M}^{A,H}(J)$, either all curves in $u \in \mathcal{M}^{A,H}(J)$ are stable or all $u \in \mathcal{M}^{A,H}(J)$ are unstable.* □

All sets $\mathcal{M}^{A,A\times H}(J)$ of unstable curves are smooth, since they correspond to the orbit types of $M$ under the identification (3.1) above.

**Observation 3.52.** For each orbit type $H = G_p \leqslant G$, the set $\mathcal{M}^{A,A\times H}(J)$ is a smooth manifold of dimension $\dim G_p$. □

The dimension of the orbit types $G_p$ is a classical result; we omit the details. If $G$ is finite, locally $G_p$ is the fixed point set $\mathrm{Fix}(G_p)$; the dimension of $\mathrm{Fix}(G_p)$ is easy to compute using representation theory.

## 3.5. Compactness of stabilisers

In this section, we show that all stable curves $u \in \mathcal{M}_{g,m}(C, J)$ have compact stabilisers. This is required for proving that the number of iso-symmetric strata is always countable. It also simplifies later analysis as the representation theory of compact Lie groups is much better behaved.

To this end, we prove that $A \times G$ acts properly at each stable curve in the prestratum $\widetilde{\mathcal{M}}^A(J)$. A Gromov compactness argument shows that $A$ acts properly on the stable curves in $\mathcal{M}^A(J)$; this arguments in fact extends to the action of $\mathrm{Diff}_+(\Sigma, \theta)$. By hypothesis, $G$ acts properly on $M$, which easily implies the induced action on $\mathcal{M}(J)$ (hence on $\mathcal{M}^A(J)$) is also proper, and properness of the product action follows quickly. Taken together, these results imply the following.

**Proposition 3.53.** *If the set $\mathcal{M}^{A,H}(J)$ consists of stable curves,*[6] *$H$ is a compact Lie group.*

### 3.5.1. $A$ acts properly

As first step of the proof, we show that $A$ acts properly at each the stable curve in $\widetilde{\mathcal{M}}^A(J)$. For the convenience of a non-expert reader, let us recall the definition of a proper action from Section 2.3.

**Recall.** Let $H$ be a topological group acting on a topological space $X$. The $H$-action is called *proper* if the map $H \times X \to X \times X, (h, x) \mapsto (h \cdot x, x)$ is proper. We say $H$ acts *properly at* $x \in X$ if and only if for every sequence $(x_n)$ in $X$ and $(h_n)$ in $H$ such that $\lim_n x_n = x_0$ and $\lim_n h_n \cdot x_n = x_0$, there exists a subsequence $n = n(k)$ such that $h_{n(k)}$ converges in $H$ as $k \to \infty$. If $H$ acts properly, it acts properly at every point $x \in X$.

[6] By Corollary 3.51, either all curves in $\mathcal{M}^{A,H}(J)$ are stable or all curves are unstable.

**Lemma 3.54.** *If $H$ acts properly at $x$, the stabiliser $H_x$ is compact.*

*Proof.* Consider the constant sequence $(x_n) \equiv x$. □

Recall that stable curves of genus $g > 1$ are non-constant; this is used in the following technical lemma.

**Lemma 3.55.** *If $u$ is non-constant and $u_n \to u$ in $C^0(\Sigma, M)$, for all $w \in \Sigma$ there exist $N \in \mathbb{N}$ and a non-empty open subset $V \subset M$ such that $U := u^{-1}(V) \cup \bigcup_{n \geq N} u_n^{-1}(V) \subset \Sigma$ contains $w$ as an interior point, while $\Sigma \setminus U$ has non-empty interior.*

*Proof.* Choose $z, z' \in \Sigma$ such that $u(z)$ is distinct from $u(z')$ and $u(w)$. (This is possible since $u$ is continuous and non-constant.) Choose disjoint open subsets $V, \tilde{V} \subset M$ containing $u(z)$ and $\{u(z'), u(w)\}$, respectively. Then $U := u^{-1}(V)$ and $U' := u^{-1}(\tilde{V})$ are disjoint non-empty open subsets of $\Sigma$.

By construction, $u(\Sigma \setminus U) \subset M \setminus V$. Choose an open subset $V' \subset M$ with $u(z) \in V' \subset \overline{V'} \subset V$. Now $u_n \to u$ implies $u^{-1}(V') \cup \bigcup_{n \geq N} u_n^{-1}(V') \subset U$ for $N$ sufficiently large: by construction, $u(\Sigma \setminus U) \subset M \setminus \overline{V'}$; for $N$ sufficiently large, $u_n(\Sigma \setminus U) \subset M \setminus \overline{V'}$ for all $n \geq N$, which implies $u_n^{-1}(V') \subset U$.

Thus, we may choose $V := V'$: since each $u_i$ is continuous, $u^{-1}(V') \cup \bigcup_{n \geq N} u_n^{-1}(V') \subset U$ is open; by construction $U$ and $U'$ are disjoint, $U' \subset \Sigma$ is open and contains $w$. □

**Proposition 3.56.** *The $A$-action on $\widetilde{\mathcal{M}}^A(J)$ is proper at each stable curve $u$.*

*Proof by contradiction.* Let $(\Sigma, j, \theta, u) \in \widetilde{\mathcal{M}}^A(J)$ be stable, in particular non-constant. Suppose the action is not proper at $u$: i.e., there exist a convergent sequence $u_n \to u$ of curves $(\Sigma, j_n, \theta, u_n) \in \widetilde{\mathcal{M}}^A(J)$ and a sequence $(\phi_n)$ in $A$ such that $u_n \circ \phi_n \to u$, but $(\phi_n)$ admits no convergent subsequence.

We regard the $\phi_k$ as holomorphic curves $(\Sigma, j) \to (\Sigma, j)$ in $\mathcal{M}_{g,0}([\Sigma], j)$. (Each curve is an orientation-preserving diffeomorphism, hence $[\phi_k] = (\phi_k)_*[\Sigma] = [\Sigma]$ for all $k$.) The complex structure $j$ is compatible with any area form on $\Sigma$ inducing the same orientation.[7] Thus, by Gromov's compactness theorem, after passing to a subsequence, $(\phi_n)$ converges to some $[(S, j', \Delta, \phi_0)] \in \overline{\mathcal{M}_{g,0}([\Sigma], j)}$. By hypothesis, $\phi_0$ is nodal.

Since the complex structure $j$ is fixed, the surfaces $(\Sigma, j_n)$ converge to a smooth element of Deligne–Mumford space. For $g \geq 1$, no bubbling happens because $\pi_2(\Sigma) = 0$: hence $\phi_0$ is smooth, contradicting our hypothesis, and properness of the action follows.

If bubbling happens for $g = 0$, as $[\Sigma]$ is primitive, there is at most one bubble and $\phi_0$ is constant outside of the bubble. Thus, there exist $w, w' \in \mathbb{S}^2$ such that $\phi_n|_{\mathbb{S}^2 \setminus \{w\}}$ converges to the constant map $z \mapsto w'$ in $C^\infty_{\text{loc}}$. This yields a different contradiction. Since $u$ is non-constant, there exists some $z \in \mathbb{S}^2 \setminus \{w, w'\}$ with $u(z)$ distinct from $u(w)$ and $u(w')$. Broadly speaking, we will argue that $\text{im}(\phi_n)$ will move further and

[7]This is an easy exercise: if $(\Sigma, j)$ is a Riemann surface, $j$ is compatible with any area form on $\Sigma$ that induces the same orientation as $j$.

further away from $z$, hence $\mathrm{im}(u_n \circ \phi_n)$ must avoid a neighbourhood of $u(z)$, contradicting $u_n \circ \phi_n \to u$.

Using Lemma 3.55, choose disjoint open subsets $V, V' \subset M$ containing $u(z)$ and $u(w')$, respectively, such that for some $N \in \mathbb{N}$, the complement of $U := u^{-1}(V) \cup \bigcup_{n \geq N} u_n^{-1}(V) \subset \Sigma$ contains an open subset $U'' \subset \Sigma$ with $w' \in U''$. Then $u \circ \phi_n(z) \in V$ and $\phi_n(z) \in U$ for all $n \geq N$ sufficiently large: by hypothesis, $u(z) \in V$; now $u \circ \phi_n \to u$ implies the first statement; the second statement follows from $\phi_n(z) = u_n^{-1}((u_n \circ \phi_n)(z)) \in u_n^{-1}(V) \subset U$.

Next, we deduce a contradiction to the convergence $\phi_n|_{\mathbb{S}^2 \setminus \{w\}} \to (z \mapsto w')$. Choose a compact subset $K' \subset \Sigma$ such that $u(K') \subset V'$ and $u^{-1}(u(w)) \in U' \subset K'$ for some open subset $U' \subset \Sigma$. (By construction, $u(w') \in V'$, hence $u^{-1}(u(w')) \in u^{-1}(V')$. Since $u^{-1}(V')$ is open, we can choose $K'$ accordingly.) Then $u_n \circ \phi_n(K') \subset V'$ for $n$ sufficiently large, since $u_n \circ \phi_n \to u$.

Choose another compact subset $K'' \subset \Sigma$ such that $w' \in K''$, $w \notin K''$ and $K' \cup K'' = \Sigma$. (This is possible since $w$ has an interior point of $K'$.) By construction, there exists an open subset $U'' \subset \Sigma$ containing $w'$ which is disjoint from $U$. Then $\phi_n(K'') \subset U''$ for $n$ sufficiently large: the constant function $c := (z \mapsto w')$ satisfies $c(K'') \subset U''$, hence $\phi_n|_{K''} \to c|_{K''}$ (using $w \notin K''$) implies $\phi_n(K'') \subset U''$ for $n$ sufficiently large.

Now we have a contradiction: $z \in K'$ implies $u_n \circ \phi_n(z) \in V'$ for $n$ sufficiently large, but we also have $u_n \circ \phi_n(z) \in V$ for $n$ sufficiently large and $V \cap V' = \emptyset$. On the other hand, $z \in K''$ implies $\phi_n(z) \in U''$, while $\phi_n(z) \in U$ for $n$ sufficiently large; by construction $U$ and $U''$ are disjoint. □

**Remark 3.57.** In fact, Proposition 3.56 generalises in two ways. Firstly, the same proof shows that the $A$-action on $C^0(\Sigma, M)$ by $\phi \cdot u = u \circ \phi^{-1}$ is proper at each non-constant curve. So, if $k, p \in \mathbb{N}$ with $kp > 2$, the $A$-action on the Banach manifold $\mathcal{B} = W^{k,p}(\Sigma, M)$ of maps is also proper at each non-constant curve: by the Sobolev embedding theorem, there exists a continuous inclusion $\mathcal{B} \hookrightarrow C^0(\Sigma, M)$, which is equivariant.

Secondly, we could also drop the pre-strata and consider the full action by $\mathrm{Diff}_+(\Sigma, \theta)$. Consider $\widetilde{\mathcal{N}}(J) := \{(j, u) \in \mathcal{J}(\Sigma) \times \widetilde{\mathcal{M}}(J) \mid (\Sigma, j, \theta, u) \text{ is stable}\}$, endowed with the product topology. Then $\mathcal{D} := \mathrm{Diff}_+(\Sigma, \theta)$ acts on $\widetilde{\mathcal{N}}(J)$ by $\phi \cdot (j, u) := (\phi_* j, u \circ \phi^{-1})$.

**Observation 3.58.** The action of $\mathcal{D}$ on $\widetilde{\mathcal{N}}(J)$ is proper, and the stabiliser of any point $(j, u) \in \widetilde{\mathcal{N}}(J)$ is the finite group $\mathrm{Aut}(u)$.

*Proof sketch.* Determining the stabiliser is an easy computation. For properness, we follow the same argument as for Proposition 3.56. Suppose the action is not proper: then, for some $(j, u) \in \widetilde{\mathcal{N}}(J)$ and $(j', u') \in \widetilde{\mathcal{N}}(J)$, there exist sequences $(j_n, u_n)$ in $\widetilde{\mathcal{N}}(J)$ and $(\phi_n)$ in $\mathcal{D}$ such that $(j_n, u_n) \to (j, u)$ and $\phi \cdot (j_n, u_n) \to (j', u')$, but $(\phi_n)$ has no convergent subsequence.

The $\phi_k$ are holomorphic curves $(\Sigma, j_k) \to (\Sigma, j)$ in $\mathcal{M}_{g,0}([\Sigma], j_k)$. (Since each curve is an orientation-preserving diffeomorphism, $[\phi_k] = (\phi_k)_*[\Sigma] = [\Sigma]$ for all $k$.) By Gromov's compactness theorem, some subsequence of $(\phi_n)$ converges to a nodal curve $[(S, j,' \delta, \phi_0)] \in \overline{\mathcal{M}_{g,0}([\Sigma], j)}$. Since $j_n \to j$ by hypothesis, the surfaces $(\Sigma, j_n)$ converge to a smooth element in Deligne–Mumford space. By the exact same argument as for Proposition 3.56, stability of $u$ excludes bubbling, so $\phi_0$ is smooth, contradiction. □

To close the loop, here comes a proof of Proposition 3.46.

*Proof of Proposition 3.46.* "⇐": Suppose $\mathrm{Aut}(u)$ is finite. If $2g + m \geq 3$, the curve $u$ is automatically stable since its domain is. Otherwise, $\mathrm{Aut}(\Sigma, j, \theta)$ is explicitly known and infinite[Wen15, Section 4.2]: up to isomorphism, we have

$$\mathrm{Aut}(\Sigma, j, \theta) \cong \begin{cases} \mathrm{PSL}(2, \mathbb{C}) & \text{if } (g, m) = 0 \\ \mathrm{Aut}(\mathbb{C}, i) & \text{if } (g, m) = (0, 1) \\ \mathbb{C}^* & \text{if } (g, m) = (0, 2) \\ \mathbb{T}^2 \rtimes H \text{ for some finite group } H & \text{if } (g, m) = (1, 0) \end{cases}.$$

If $u$ is constant, $\mathrm{Aut}(u) = \mathrm{Aut}(\Sigma, j, \theta)$; since $\mathrm{Aut}(u)$ is finite by hypothesis, $u$ must be non-constant.

"⇒": Suppose $(\Sigma, j, \theta, u)$ is stable. If $(\Sigma, j, \theta)$ is stable, $\mathrm{Aut}(\Sigma, j, \theta)$ is a finite group (Lemma 3.44), hence so is $\mathrm{Aut}(u)$. Suppose $(\Sigma, j, \theta)$ is unstable, hence $u$ is non-constant. Write $u = v \circ \phi$ for some simple curve $v$ and a holomorphic degree $d \geq 1$ branched cover $\phi$. Since $v$ is simple, it is embedded (except at finitely many points). Hence, any automorphism of $u$ is an automorphism of $\phi$, so $|\mathrm{Aut}(u)| \leq d$. □

### 3.5.2. $A \times G$ acts properly

By hypothesis, $G$ acts properly on $M$: this implies a proper action on $\widetilde{\mathcal{M}}(J)$.

**Observation 3.59.** If $G$ acts properly on $M$, it acts properly on $\widetilde{\mathcal{M}}(J)$.

*Proof.* Let $u \in \widetilde{\mathcal{M}}(J)$ be arbitrary and suppose $(g_n)$ and $(u_n)$ are sequences in $G$ and $\widetilde{\mathcal{M}}(J)$, respectively, such that $u_n \to u$ and $g_n \cdot u_n \to v$ in $\widetilde{\mathcal{M}}(J)$. We need to find a convergent subsequence of $(g_n)$. Let $z \in \Sigma$ be arbitrary. Since convergence in $\widetilde{\mathcal{M}}(J)$ implies point-wise convergence, $u_n(z) \to u(z)$ and $g_n \cdot u_n(z) \to v(z)$. Now, properness of the $G$-action on $M$ implies that $(g_n)$ has a convergent subsequence. □

**Remark 3.60.** Again, the same proof shows that $G$ acts properly on $\mathcal{B} = W^{k,p}(\Sigma, M)$.

It remains to deduce properness of the $A \times G$-action from properness of the $A$ and $G$-actions. In general, a product of two proper actions need not be proper (unless e.g. one factor is compact). The $A \times G$-action in our setting, however, always is.

**Proposition 3.61.** *The $(A \times G)$-action on $\widetilde{\mathcal{M}}^A(J)$ is proper at each stable curve $u$.*

*Proof.* Firstly, observe that the set of stable curves is $A \times G$-invariant: hence, the $A \times G$-action on $\widetilde{\mathcal{M}}^A(J)$ descends to an action on the subset of stable curves.
Suppose $(j_n, u_n) \to (j, u)$ and $(\phi_n, g_n) \cdot (j_n, u_n) = (\phi_n{}^* j_n, g_n \cdot (u_n \circ \phi_n)) \to (j', u')$. Pick a point $z \in \Sigma$. Since $\Sigma$ is compact, by passing to a subsequence we may assume $\phi_n(z)$ converges to some point $z' \in \Sigma$. Then $u_n(\phi_n(z)) \to z'$ and $g_n \cdot (u_n(\phi_n(z))) \to u'(z)$ follow, so properness of the $G$-action (Observation 3.59) implies that, after passing to another subsequence, the sequence $(g_n)$ converges to some $g' \in G$. Now, continuity of the action implies $(\phi_n^* j_n, u_n \circ \phi_n) \to (j', g'^{-1} \cdot u')$, and properness of the $\mathcal{D}$-action (by Observation 3.58) implies that $(\phi_n)$ also has a convergent subsequence. □

**Remark 3.62.** Looking at the argument closely, the same proof shows that the $\mathcal{D} \times G$-action on $\widetilde{\mathcal{M}}^A(J)$ is proper at each stable curve $u$.

*Proof of Proposition 3.53.* Combine Lemma 3.54 with Proposition 3.61. □

## 3.6. Countable number of iso-symmetric strata

In this section, we prove that the number of distinct non-empty iso-symmetric strata is countable. This is important for a subtle reason: our proof of smoothness (in Chapter 4) will consider each stratum $\mathcal{M}^{A,H}_{U,\mathbf{l}}(J)$ separately, and exhibits for each stratum a co-meagre set of equivariant $J$ such that $\mathcal{M}^{A,H}_{U,\mathbf{l}}(J)$ is smooth. If there are countably many strata, the intersection of these co-meagre sets yields a co-meagre set, concluding the existence of a co-meagre subset of $\mathcal{J}^G(M, \omega)$ for which all iso-symmetric strata are smooth simultaneously.

**Proposition 3.63.** *Fix an open subset $U \subset M$. Then, the number of distinct non-empty iso-symmetric strata $\mathcal{M}^{A,H}_{U,\mathbf{l}}(J)$ of simple curves is countable. The number of distinct non-empty iso-symmetric strata $\mathcal{M}^{A,H;K}_{U,\mathbf{l};\mathbf{b},d}(J)$ of multiply covered curves is also countable.*

**Remark 3.64.** This proposition depends on our chosen definition of iso-symmetric strata: if we had defined the sets $\mathcal{M}^{A,H}(J)$ (and hence the iso-symmetric strata) using equality of subgroups instead, the number of strata is generally uncountable.

On the other hand, merely demanding isomorphic groups $(A \times G)_u \cong H$ makes this much easier: up to isomorphism, there are only countably many compact Lie groups (as they are classified by combinatorial data which are countable).

The proof of Proposition 3.63 splits in three steps: we first show that the number of non-empty pre-strata $\mathcal{M}^A(J)$ is countable. Then, we argue that for given $A$, there are countably many non-empty sets $\mathcal{M}^{A,H}(J)$ refining $\mathcal{M}^A(J)$. Finally, the set of possible data $d$, $\mathbf{b}$ and $\mathbf{l}$ describing critical points and branch points is countable.

### 3.6.1. Countability of pre-strata

The first step is showing that the number of pre-strata $\mathcal{M}^A(J)$ is countable. In fact, for later use let us also note that they are locally finite.

**Lemma 3.65.** *There are countably many conjugacy classes of closed subgroups $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$ such that the corresponding pre-stratum $\mathcal{M}^A(J)$ is non-empty.*

**Lemma 3.66.** *The collection $\{\mathcal{M}^A(J)\}$ of pre-strata is locally finite.* □

The proof is surprisingly hands-on: by the definition of pre-strata, each group $A$ with whose pre-stratum is non-empty is the automorphism group of a pointed closed Riemann surface; we can use the classification of closed Riemann surfaces to handle these explicitly. We treat the cases of spheres, tori with at most one point and other stable surfaces separately: each case requires different methods.

*Proof of Lemmas 3.65 and 3.66.* As mentioned, we consider three different cases, depending on the genus $g$.

#### Case 1: spheres

Suppose $g = 0$, i.e. $\Sigma$ is diffeomorphic to a sphere.

**Claim 1.** *For each $m$, there the automorphism group is unique up to conjugation.*

*Proof of claim.* Let $j \in \mathcal{J}(\mathbb{S}^2)$ be arbitrary. By the uniformisation theorem (e.g. [Mil06; FK92]), $(\mathbb{S}^2, j)$ is biholomorphic to $(\mathbb{S}^2, i)$. Hence, conjugating by such a diffeomorphism, we may assume $(\mathbb{S}^2, j, \theta) = (\mathbb{S}^2, i, \theta')$ for some $\theta' \subset \mathbb{S}^2$. Conjugating by a suitable automorphism of $(\mathbb{S}^2, i)$, we may further assume that

$$\theta' = \begin{cases} \emptyset & \text{if } m = 0 \\ \{\infty\} & \text{if } m = 1 \\ \{0, \infty\} & \text{if } m = 2 \\ \{0, 1, \infty, \zeta_4, \ldots, \zeta_m\} & \text{if } m \geq 3 \end{cases};$$

this uses that $\mathrm{Aut}(\mathbb{S}^2, i) = \mathrm{PSL}(2, \mathbb{C})$ acts triply transitively on $\mathbb{S}^2$. In particular, for $m \geq 3$, the automorphism group is always trivial; for $m < 3$ the automorphism group is uniquely determined up to conjugation. △

In particular, exactly one pre-stratum is non-empty.

#### Case 2: tori with $m \leq 1$

Suppose $g = 1$ and $m \leq 1$, i.e. $\Sigma$ is diffeomorphic to a torus. In this case, the uniformisation theorem does not help much, but we can compute the automorphism groups rather explicitly.

**Notation.** Let us fix an explicit model surface. Since a torus has universal cover $\mathbb{C}$, its given as a quotient $\mathbb{C}/\Lambda$ by some lattice $\Lambda \subset \mathbb{C}$. Without loss of generality, assume $\Lambda = \mathbb{Z} \oplus \lambda\mathbb{Z}$ for some $\lambda \in \mathbb{H}$. Choosing a real-linear map which fixes 1 and sends $\lambda$ to $i$, we identify $(\mathbb{C}/\Lambda, i)$ with $(\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}), j_\lambda)$, where $j_\lambda$ is the translation-invariant complex structure on $\mathbb{C}$ which maps 1 to $\lambda$ (and $\lambda$ to $-1$). Every translation-invariant complex structure is of this form; conversely, every translation-invariant complex structure descends to a complex structure on the torus [Wen15, p. 160].

**Reduction.** We reduce our analysis to the complex structures $\mathcal{T} := \{j_\lambda\}_{\lambda\in\mathbb{H}}$ on the torus given above. Each complex structure $j \in \mathcal{J}(\mathbb{T}^2)$ is conjugate in $\mathrm{Diff}_0(\mathbb{T}^2, \theta)$ to some $j_\lambda$. This result is classical: the collection $\mathcal{T}$ forms a so-called *Teichmüller slice* for the *Teichmüller space* $\mathcal{T}(\mathbb{T}^2, \theta) = \mathcal{J}(\mathbb{T}^2)/\mathrm{Diff}_0(\mathbb{T}^2, \theta)$.[8] For each Teichmüller slice (in particular, for $\mathcal{T}$), the quotient projection $\mathcal{T} \to \mathcal{T}(\mathbb{T}^2, \theta)$ is a diffeomorphism (e.g. [Wen15, Theorem 4.2.14]). In particular, each complex structure $j \in \mathcal{J}(\mathbb{T}^2)$ is conjugate in $\mathrm{Diff}_0(\mathbb{T}^2, \theta)$ to some $j_\lambda$.

Thus, after a suitable conjugation, we need to consider only the automorphism groups $\mathrm{Aut}(\mathbb{T}^2, j_\lambda, \theta)$. By conjugating further with a suitable translation, we may assume the marked point (if any) is 0.[9]

For unmarked tori, the automorphism groups are known quite explicitly: they are semi-direct products $\mathrm{Aut}(\mathbb{T}^2, j_\lambda) \cong \mathbb{T}^2 \rtimes G_\lambda$ of all translations on the torus with a finite group $G_\lambda$ [Wen15, p. 160]. Moreover, $G_\lambda = \mathrm{Aut}(\mathbb{T}^2, j_\lambda, \{0\})$, i.e. $G_\lambda$ is the automorphism group of a once-marked torus.

Therefore, it suffices to show that there are only countably many groups $G_\lambda$, and this collection is locally finite (w.r.t. the parameter $\lambda$). To keep the flow of ideas moving, we postpone the proof to the end of this section.

**Lemma 3.67.** *The collection $\{G_\lambda\}_{\lambda\in\mathbb{H}}$ is countable and locally finite.*

### Case 3: stable surfaces

Suppose none of the cases above holds, then $g = 1$ and $m \geq 2$ or $g \geq 2$. In both cases, we have a stable surface. Our proof is similar in spirit to case 2, except that many details simplify. By reparametrisation, we may restrict attention to our fixed surface $(\Sigma, \theta)$.

**Lemma 3.68.** *Let $(\Sigma, \theta)$ be a stable closed pointed surface. Then the collection of conjugacy classes $\{[\mathrm{Aut}(\Sigma, j, \theta)]\}_{j\in\mathcal{J}(\Sigma)}$ is countable and locally finite.*

The starting point of the proof is the following observation.

**Observation 3.69.** The space of smooth almost complex structures $\mathcal{J}(\Sigma)$ is Lindelöf, i.e. every open cover has a countable sub-cover.

[8] We will encounter this definition in more detail in Section 4.2.

[9] For the pedantic: very precisely, we assume the marked point, if any, is the image of $0 \in \mathbb{C}$ under the projection $\mathbb{C} \to \mathbb{C}/(\mathbb{Z} + \lambda\mathbb{Z}) \cong \mathbb{T}^2$.

*Proof.* Recall that $\mathcal{J}(\Sigma)$ is endowed with the $C^\infty_{\mathrm{loc}}$-topology, which is second countable and metrisable by Proposition 2.6. In particular, it is Lindelöf. $\square$

*Proof of Lemma 3.68.* By stability, $\mathrm{Aut}(\Sigma, j, \theta)$ is a finite group [Wen15, Corollary 4.2.7]. Hence, it suffices to consider the cases of no $(g \geq 2)$ resp. one universal $(g = 1)$ marked point $p$: if there exists a countable family of finite groups $\{A_n\}$ such that $\mathrm{Aut}(\Sigma, j, \emptyset)$ resp. $\mathrm{Aut}(\mathbb{T}^2, j, \{p\})$ is conjugate to one of the $A_i$, then $\mathrm{Aut}(\Sigma, j, \theta)$ is a conjugate to a subgroup of one of the $A_i$. However, each $A_i$ has only finitely many subgroups (as it is finite itself).

Suppose $\theta$ is empty (if $g \geq 2$) resp. a singleton set (if $g = 1$). By Lemma 4.28 (proven in Section 4.2), the automorphism group $\mathrm{Aut}(\Sigma, j, \theta)$ decreases locally: each $j \in \mathcal{J}(\Sigma)$ has a neighbourhood $U$ such that $\mathrm{Aut}(\Sigma, j', \theta)$ is conjugate to a subgroup of $\mathrm{Aut}(\Sigma, j, \theta)$ for all $j' \in U$. In particular, the collection of conjugacy classes $\{[\mathrm{Aut}(\Sigma, j', \theta)]\}_{j' \in U}$ is finite. This proves local finiteness. Since $\mathcal{J}(\Sigma)$ is Lindelöf, we can cover it by countably many such neighbourhoods; altogether there are only countably many automorphism groups occurring. $\square$

This completes the proof of Lemma 3.65, up to proving Lemma 3.67. $\square$

**Remark 3.70.** In fact, for $g \geq 2$ the possible number of pre-strata is even finite: by Lemma 3.72 below, each group $\mathrm{Aut}(\Sigma, j, \theta)$ induces a unique finite subgroup of the mapping class group $M(\Sigma, \theta)$. For $g \geq 2$, there are finitely many conjugacy classes of finite subgroups in $M(\Sigma, \theta)$ [FM12, Theorem 7.14].

Finally, we come to the postponed proof of Lemma 3.67: the collection $\{G_\lambda\}_{\lambda \in \mathbb{H}}$ is countable. This is based on Wendl's proof that each group $G_\lambda$ is finite [Wen15, Proposition 4.2.19]: Wendl shows that $G_\lambda$ is both discrete and compact. Our observation is that the argument can be applied "uniformly" for all $\lambda$, giving a simultaneous description of all $G_\lambda$; analysing this description shows that there are only countably many distinct groups $G_\lambda$.

Discreteness of the groups $G_\lambda$ follows by relating them to the homology of $\mathbb{T}^2$.

**Lemma 3.71.** *For each $\lambda \in \mathbb{H}$, the map $f \colon G_\lambda \to \mathrm{SL}(2, \mathbb{Z}), \phi \mapsto \phi_* \in \mathrm{End}(H_1(\mathbb{T}^2)) \cong \mathrm{End}(\mathbb{Z}^2) \cong \mathrm{SL}(2, \mathbb{Z})$ is injective.*

*Proof.* The map $f$ is the composition of the maps $G_\lambda \to M(\mathbb{T}^2)$ and $M(\mathbb{T}^2) \to \mathrm{SL}(2, \mathbb{Z})$ below. By Lemmas 3.72 and 3.73 below, both maps are well-defined and injective, hence so is $f$. $\square$

**Recall.** The *mapping class group* of a pointed smooth surface $(\Sigma, \theta)$ is

$$M(\Sigma, \theta) = \mathrm{Diff}_+(\Sigma, \theta) / \mathrm{Diff}_0(\Sigma, \theta),$$

where $\mathrm{Diff}_0(\Sigma, \theta) = \{\phi \in \mathrm{Diff}(\Sigma, \theta) \mid \phi \sim_h \mathrm{id}\}$.

**Lemma 3.72.** *If $(\Sigma, j, \theta)$ is a stable pointed Riemann surface, the natural map $\mathrm{Aut}(\Sigma, j, \theta) \hookrightarrow \mathrm{Diff}_+(\Sigma, \theta) \twoheadrightarrow M(\Sigma, \theta)$ is injective.*

*Proof.* We show that this group homomorphism has trivial kernel. A standard argument using Lefschetz' fixed point theorem (e.g. [Wen15, Lemma 4.2.5]) implies that $\phi \in \mathrm{Aut}(\Sigma, j, \theta)$ being homotopic to the identity implies $\phi = \mathrm{id}$. □

**Lemma 3.73** ([FM12, Theorem 2.5])**.** *The map*

$$M(\mathbb{T}^2) \to \mathrm{SL}(2,\mathbb{Z}), [\phi] \mapsto \phi_* \in \mathrm{End}(H_1(\mathbb{T}^2)) \cong \mathrm{End}(\mathbb{Z}^2) \cong \mathrm{SL}(2,\mathbb{Z})$$

*is a group isomorphism.* □

**Corollary 3.74.** *Each group $G_\lambda$ for $\lambda \in \mathbb{H}$ is discrete.*

*Proof.* The inclusion $G_\lambda \to M(\mathbb{T}^2)$ is continuous by definition; since the mapping class group $M(\mathbb{T}^2)$ is discrete, so is $G_\lambda$. □

A crucial detail is that map $G_\lambda \to M(\mathbb{T}^2)$ is independent of the choice of $\lambda$: we can apply this to all groups $G_\lambda$ at once. Using this slight abuse of notation, we regard each $G_\lambda$ as a subgroup of $\mathrm{SL}(2,\mathbb{Z})$.

**Fact** ([Wen15, p. 160])**.** Under this identification, $G_\lambda = \{A \in \mathrm{SL}(2,\mathbb{Z})\colon A^* j_\lambda = j_\lambda\}$. □

Compactness of each $G_\lambda$ uses a different argument: we embed $G_\lambda \subset \mathrm{SL}(2,\mathbb{Z})$ into a compact 1-manifold. Again, this argument can be run "for all parameters $\lambda$ at once", yielding the following.

**Proposition 3.75.** *There exists a smooth family $\{C_\lambda\}_{\lambda\in\mathbb{H}}$ of compact submanifolds $C_\lambda \subset \mathrm{GL}(2,\mathbb{R})$ such that $G_\lambda \subset C_\lambda$. In particular, each $G_\lambda$ is compact.*

*Proof.* Given $\lambda \in \mathbb{H}$, choose a basis $B_\lambda := (e_{\lambda,1}, e_{\lambda,2})$ of $\mathbb{R}^2$ such that $e_{\lambda,1} \in \mathbb{R}e_1$, $e_{\lambda,2} = j_\lambda e_{\lambda 1}$ and $\mathrm{area}(e_{\lambda,1}, e_{\lambda,2}) = 1$. In other words, we have $B_\lambda = (c_\lambda e_1, c_\lambda \lambda)$ for some constant $c_\lambda > 0$. In fact, $c_\lambda$ is not hard to compute.

**Claim 1.** $c_\lambda = 1/\sqrt{\mathrm{Im}\,\lambda}$

*Proof.* We compute $\mathrm{area}(c_\lambda e_1, c_\lambda \lambda) = c_\lambda^2 \,\mathrm{Im}(\lambda)$; since $c_\lambda > 0$, this implies the claim. By definition, $j_\lambda$ is the conjugate of the standard complex structure by the linear map $\phi_\lambda^{-1}\colon \mathbb{C} \to \mathbb{C}, 1 \mapsto 1, i \mapsto \lambda$. Hence, $j_\lambda$ is described by $j_\lambda(1) = \lambda$ and $j_\lambda(\lambda) = -1$. Since $\lambda = \mathrm{Re}\,\lambda + \mathrm{Im}\,\lambda i$, we have $i = \frac{\lambda - \mathrm{Re}\,\lambda}{\mathrm{Im}(\lambda)} = \frac{1}{\mathrm{Im}(\lambda)}\lambda - \frac{\mathrm{Re}\,\lambda}{\mathrm{Im}\,\lambda}1$ and

$$j_\lambda(i) = \frac{1}{\mathrm{Im}\,\lambda}(-1) - \frac{\mathrm{Re}\,\lambda}{\mathrm{Im}\,\lambda}\lambda = -\frac{1}{\mathrm{Im}\,\lambda} - \frac{\mathrm{Re}\,\lambda^2}{\mathrm{Im}\,\lambda} - \mathrm{Re}(\lambda)i = -\frac{1 + \mathrm{Re}(\lambda)^2}{\mathrm{Im}\,\lambda} - \mathrm{Re}(\lambda)i.$$

Hence, $j_\lambda$ has coefficient matrix $J_\lambda := \begin{pmatrix} \mathrm{Re}\,\lambda & -\frac{1+\mathrm{Re}(\lambda)^2}{\mathrm{Im}\,\lambda} \\ \mathrm{Im}\,\lambda & -\mathrm{Re}\,\lambda \end{pmatrix}$ w.r.t. the standard basis. Plugging in, we conclude $j_\lambda e_1 = \begin{pmatrix} \mathrm{Re}\,\lambda & -\frac{1+\mathrm{Re}(\lambda)^2}{\mathrm{Im}\,\lambda} \\ \mathrm{Im}\,\lambda & -\mathrm{Re}\,\lambda \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\,\lambda \\ \mathrm{Im}\,\lambda \end{pmatrix}$, and deduce $(e_{\lambda,1}, e_{\lambda,2}) = (c_\lambda e_1, c_\lambda \lambda)$, where $c_\lambda \in \mathbb{R}$ is determined by

$$1 = \mathrm{area}(e_{\lambda,1}, e_{\lambda,2}) = \det(c_\lambda \begin{pmatrix} 1 & \mathrm{Re}\,\lambda \\ 0 & \mathrm{Im}\,\lambda \end{pmatrix}) = c_\lambda^2 \,\mathrm{Im}(\lambda). \qquad \triangle$$

Observe that w.r.t. the basis $B_\lambda$, $A \in G_\lambda$ lies in both $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{GL}(1,\mathbb{C})$: the former because $\det(A) = 1$ is basis-invariant; the latter since $j_\lambda$ corresponds to multiplication by $i$ in the basis $B_\lambda$ and $A^* j_\lambda = j_\lambda$. Note that $\mathrm{SL}(2,\mathbb{R}) \cap \mathrm{GL}(1,\mathbb{C}) = \mathrm{U}(1)$ is compact.

In other words, the coefficient matrix $M_\lambda = c_\lambda \begin{pmatrix} 1 & \operatorname{Re}\lambda \\ 0 & \operatorname{Im}\lambda \end{pmatrix}$ of $B_\lambda$ w.r.t. the standard basis $(e_1, e_2)$ of $\mathbb{R}^2$ satisfies

$$M_\lambda G_\lambda M_\lambda^{-1} \subset \mathrm{U}(1) \cong \mathrm{SO}(2),$$

i.e. $G_\lambda \subset M_\lambda^{-1}\,\mathrm{SO}(2) M_\lambda =: C_\lambda$. Each $C_\lambda$ is a smooth 1-dimensional submanifold of $\mathbb{R}^{2\times 2}$; since $M_\lambda$ depends smoothly on $\lambda$, the family $\{C_\lambda\}_{\lambda\in\mathbb{H}}$ is a smooth family. Since $\mathrm{SL}(2,\mathbb{Z})$ and the condition $A^* j_\lambda = j_\lambda$ are closed, $G_\lambda$ is closed, hence compact. □

Now, countability of the family $\{G_\lambda\}_{\lambda\in\mathbb{H}}$ follows from the following.

**Lemma 3.76.** *Let $P$ and $M$ be a smooth manifolds. Suppose $S \subset M$ is a closed discrete set and $(C_p)_{p\in P}$ a smooth family of compact submanifolds of $M$. For $p \in P$, denote $G_p := S \cap C_p$. Then the set of possible $G_p$ is countable.*

*Proof of Lemma 3.76.* Each $G_p$ is a discrete set (as a subset of $S$) and compact, hence finite. Since $M$ is second countable, $S$ is countable, so there are only countably many finite subsets of $S$. □

Perhaps the reader found the above proof a bit underwhelming: after all, the smoothness of the family $C_p$ was not used at all. We *do* use it, however, for proving local finiteness: this follows by looking at the structure of the sets $G_p$ a little more carefully.

**Lemma 3.77.** *Let $P$, $M$, $S$ and $\{C_p\}_{p\in P}$ be as in Lemma 3.76. Then the collection $\{G_p\}$ is locally finite.*

*Proof.* Let $p \in M$ be arbitrary. Choose an open subset $\tilde{C}$ of $M$ containing $C_p$ such that $\tilde{C} \cap S$ is still finite. By the subsequent lemma, for $q$ sufficiently close to $p$, the set $C_q$ is still contained in $\tilde{C}$: then $G_q \subset \tilde{C} \cap S$ for each such $q$, and such sets $G_q$ take only finitely many possible values. □

The proof becomes complete by showing the following.

**Lemma 3.78.** *Let $M$ be a smooth manifold, and $S, K \subset M$ be a closed discrete resp. compact subset. There exists an open subset $U \subset M$ such that $K \subset U$ and $U \cap S$ is finite.*

*Proof.* By hypotheses, $K \cap S$ is a finite set. Write $K \cap S = \{x_1, \dots, x_n\}$ for suitable $x_1, \dots, x_n \in M$. For each $x_i$, choose an open set $U_i \subset M$ containing $x_i$ such that $U_i \cap S = \{x_i\}$. Let $\mathcal{U}_0 := U_1 \cup \dots U_n$: this is the first component of the desired set $U$. By construction, $\mathcal{U}_0$ is open and $K \setminus \mathcal{U}_0$ is compact. For each $x \in K \setminus \mathcal{U}_0$, we have $x \notin S$ be construction. Choose an open set $\mathcal{U}_x \subset M$ containing $x$ such that $\mathcal{U}_x \cap S = \emptyset$.

Now $\mathcal{U} := \mathcal{U}_0 \cup \bigcup_{x \in K \setminus \mathcal{U}_0} \mathcal{U}_x \subset M$ has the desired properties: by construction, it is open, contains $K$ (since $K \subset \mathcal{U}_0$) and we have

$$\mathcal{U} \cap S = \bigcup_{i=1}^{n} (U_i \cap S) \cup \bigcup_{x \in K \setminus \mathcal{U}_0} (\mathcal{U}_x \cap S) = \bigcup_{i=1}^{n} \{x_i\} \cup \bigcup_{x \in K \setminus \mathcal{U}_0} \emptyset = \{x_1, \dots, x_n\} = K \cap S. \quad \square$$

To close the loop, here is how to specialise this discussion to Lemma 3.67.

*Proof of Lemma 3.67.* Apply Lemma 3.76 to the collection $\{C_\lambda\}_{\lambda \in \mathbb{H}}$, with $P = \mathbb{H}$, $S = \mathrm{SL}(2, \mathbb{Z})$ and $M = \mathbb{R}^{2 \times 2}$. Observe that $G_\lambda = \mathrm{SL}(2, \mathbb{Z}) \cap C_\lambda$; the sets $C_\lambda$ consist of those matrices which satisfy $A^* j_\lambda = j_\lambda$. □

### 3.6.2. Countability of iso-symmetric strata

The second step is showing that for each pre-stratum, at most countably many of the sets $\mathcal{M}^{A,H}(J)$ refining it are non-empty.

**Lemma 3.79.** *For a closed subgroup $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$, there are countably many conjugacy classes of closed subgroups $H \leqslant A \times G$ such that the set $\mathcal{M}^{A,H}(J)$ refining the pre-stratum $\mathcal{M}^A(J)$ is non-empty.*

An unstable curve $u \equiv p$ always has stabiliser $(A \times G)_u = A \times G_p$, where $G_p$ is the stabiliser of $p \in M$. This is a compact subgroup of $G$ since $G$ acts properly on $M$. For stable curves, the set $\mathcal{M}^{A,H}(J)$ consists (by definition) of those curves $[u] \in \mathcal{M}^A(J)$ with $(A \times G)_u$ conjugate to $H$. The stabiliser $(A \times G)_u$ is always a compact Lie group (Proposition 3.53). Hence, Lemma 3.79 follows from the following.

**Theorem 3.80** ([Kha21, Corollary 3.9; AAV12, Theorem 3.1])**.** *A Lie group has countably many conjugacy classes of compact subgroups.*[10] □

For compact Lie groups, this result is classical. While it was known before the end of the second world war [Kha20], the first quotable reference is due to Palais [Pal60, Theorem 1.7.27]. Palais' proof uses the Peter–Weyl theorem and Yang's theorem about local finiteness of orbit types. Khan [Kha21] provides a proof explaining the classical reasoning, which we sketch here since it is fairly short.

A key building block is the neighbouring subgroups theorem, which we will also use later. Montgomery and Zippin gave a first proof (for compact Lie groups), using facts about geodesics in convex spheres and result of Cartan that the space of cosets $G/K$ admits a Riemannian metric on which $G$ acts by isometries. Palais [Pal60] has an alternative proof avoiding differential geometry. Khan [Kha20] has generalised the result to non-compact groups.

[10] All manifolds and Lie groups in this text are tacitly assumed to be Hausdorff and second countable. This convention is used here: the theorem requires that $G$ have countably many connected components; otherwise the result is trivially false.

**Theorem 3.81** (Neighbouring Subgroups Theorem [MZ42; Kha21]). *Any compact subgroup $K$ of any Lie group $G$ admits an open neighbourhood $U \subset G$ such that every closed subgroup contained in $U$ is conjugate (in $G$) to a closed subgroup of $K$.* □

*Proof sketch of Theorem 3.80 for compact groups, classical argument.* Let $K$ be a compact Lie group. Choose a bi-invariant Riemannian metric on $K$, making $K$ into a metric space. Consider the collections $\mathrm{Cpt}(K)$ and $\mathrm{CptSgp}(K)$ of compact subsets (resp. closed, hence compact, subgroups) of $K$, endowed with the Hausdorff metric. Since $K$ is compact, so is $\mathrm{Cpt}(K)$. By continuity of the group operations, $\mathrm{CptSgp}(K) \subset \mathrm{Cpt}(K)$ is closed in the Hausdorff metric, hence compact as well.

Observe that a proper closed subgroup $L \lneq K$ has positive Hausdorff distance of $K$. (Since $K$ and $L$ are Lie groups, $L$ has either lower dimension or fewer connected components than $K$.) Thus, we obtain a decomposition $\mathrm{CptSgp}(K) = \{K\} \cup \bigcup_{n=1}^{\infty} X_n$, where $X_n := K \setminus B(K, \frac{1}{n})$ is the complement of an open ball in the Hausdorff topology.

We prove the result by double induction: the trivial group is the base case. Assume it holds for all closed subgroups which have smaller dimension or fewer connected components. (As we just argued, this includes all proper Lie subgroups of $K$.)

**Claim 1.** *Each $X_n$ has only countably many conjugacy classes of elements.*

*Proof.* By the neighbouring subgroups theorem, each compact subgroup $H \in \mathrm{CptSgp}(K)$ admits an $\epsilon_H > 0$ such that all subgroups contained in $B(\epsilon_H, H)$ are conjugate to a subgroup of $H$. Since $X_n$ is compact, it has a finite sub-cover by balls $B(\epsilon_i, H_i)$. Therefore, any closed subgroup $H \in X_n$ is conjugate to a subgroup of some $H_i$. Each $H_i$ is compact and satisfies the inductive hypothesis, hence there are only countably many conjugacy classes within $H_i$. △

This completes the proof: $K$ is the only subgroup of $K$ conjugate to itself. □

*Proof of Proposition 3.63.* Combining Lemma 3.65 and Lemma 3.79 above shows that the number of distinct non-empty sets $\mathcal{M}^{A,H}(J)$ is countable. To conclude countability for strata of simple curves, we observe that the set of possible tuples $\mathbf{l}$ is countable. For multiple covers, the argument is similar: for given degree $d \in \mathbb{N}$, the overall number of degree $d$ branching data $\mathbf{b}$ is countable, and there are only finitely many groups of order at most $d$ (up to isomorphism). Theorem 3.80 also implies there are only countably many possible values of $H'$ (up to conjugation), completing the proof. □

## 3.7. First properties of iso-symmetric strata

Let us collect a few direct properties of the iso-symmetric strata which we just proved.

**Proposition 3.82** (Properties of iso-symmetric strata). *Suppose $G$ acts properly and symplectically. Let $U \subset M$ be an open subset.*

(1) *The iso-symmetric strata* $\mathcal{M}^{A,H;K}_{U,\mathbf{l};\mathbf{b},d}(J)$ *partition the set of stable simple curves in the moduli space* $\mathcal{M}_{g,m}(C,J)$ *of unparametrised curves.*

(2) *If* $U$ *is* $G$*-invariant, each iso-symmetric stratum* $\mathcal{M}^{A,H;K}_{U,\mathbf{l};\mathbf{b},d}(J)$ *is* $G$*-invariant.*

(3) $u \in \widetilde{\mathcal{M}}^{A,H}(J)$ *is unstable if and only if* $(A \times G)_u = A \times K$ *for some orbit type* $K$ *of* $M$.

(4) *If* $[u] \in \mathcal{M}^{A,H}(J)$ *is stable,* $H$ *is a compact Lie group.*

(5) *The number of distinct non-empty iso-symmetric strata is countable.*

(6) *The collection* $\{\mathcal{M}^{A,H}(J)\}_{A \leqslant \mathrm{Diff}_+(\Sigma,\theta) \text{ closed}, H \leqslant A \times G \text{ closed}}$ *is locally finite, as is the collection* $\{\mathcal{M}^{A,H}_U(J)\}$.

(7) *Every stable curve* $u \in \widetilde{\mathcal{M}}^{A,H}(J)$ *has a neighbourhood* $V \subset \widetilde{\mathcal{M}}(J)$ *such that all* $v \in V$ *have stabiliser* $(A \times G)_v$ *conjugate to a subgroup of* $(A \times G)_u$.

*Proof.* Item (1) is true by construction. Items (3), (4) and (5) have been proven in the previous sections (in Corollary 3.49, Proposition 3.53 and Proposition 3.63, respectively). Item (6) follows from Lemma 3.66 and Proposition 3.5(i). For Item (2), we already noted in Lemma 3.34 that $\widetilde{\mathcal{M}}^{A,H}(J)$ and $\mathcal{M}^{A,H}(J)$ are $G$-invariant. The degree $d$ and critical orders $\mathbf{l}$ are $G$-invariant since $A$ acts by diffeomorphisms. It remains to show that the branching data are $G$-invariant. This is apparent since the $G$-action on $u = v \circ \phi$ only acts on $v$, but not $\phi$.

The final item (7) uses the neighbouring subgroups theorem (Theorem 3.81). By Proposition 3.53, the stabiliser $(A \times G)_u$ is a compact Lie subgroup of $A \times G$. By the neighbouring subgroup theorem, there exists an open set $U \subset A \times G$ containing $(A \times G)_u$ such that each subgroup of $A \times G$ contained in $U$ is conjugate to a subgroup of $(A \times G)_u$. Now Lemma 3.85 below shows that $(A \times G)_v \subset U$ for all curves $v$ in a suitable neighbourhood $V \subset \widetilde{\mathcal{M}}^A(J)$ of $u$. □

**Remark 3.83.** The author suspects that the collections of iso-symmetric strata $\mathcal{M}^{A,H}_{\mathcal{U},\mathbf{l}}(J)$ resp. $\mathcal{M}^{A,H;K}_{U,\mathbf{l};\mathbf{b},d}(J)$ are also locally finite: it remains to prove that the degree $d$, the branching data $\mathbf{b}$ and the critical orders $\mathbf{l}$ are locally finite.

**Remark 3.84.** Properties (6) and (7) are analogues of Propositions 3.5(i) and 3.6. Currently, showing that the iso-symmetric strata form a smooth stratification is out of reach, as describing the boundary of the iso-symmetric strata is more involved.[11] However, the consequences (6) and (7) are within reach today.
For unstable curves, item (7) can *almost* be proven, by arguing with the orbit types $G_u$ instead. One detail is missing: suppose $G_u$ has a neighbourhood $U$ in $G$ such that each compact subgroup $H \leqslant G$ contained in $U$ is conjugate to a subgroup of $G_u$. Does it follow that every subgroup $H \leqslant A \times G$ of $A \times U$ is conjugate to a subgroup of $A \times G_u$?

[11] Section 4.5 contains the first steps of this investigation, as they are required to apply Taubes' trick.

**Lemma 3.85.** *Let $u \in \widetilde{\mathcal{M}}^A(J)$ be a stable curve; suppose $V \subset A \times G$ is an open neighbourhood of $(A \times G)_u$. Then $u$ admits a neighbourhood $U \subset \widetilde{\mathcal{M}}^A(J)$ such that $(A \times G)_{u'}$ is contained in $V$ for all $u' \in U$.*

A key ingredient in proving Lemma 3.85 holds in greater generality: to make the proof more transparent and notionally cleaner, let us present that first.

**Lemma 3.86.** *Let $G$ be a topological group acting continuously on a topological space $X$. Suppose $G$ acts properly at each $y \in X$ in an open neighbourhood of $x \in X$. Suppose $G_x$ is contained in some open subset $U \subset G$. Then, $x$ has a neighbourhood $V \subset X$ such that $G_y \subset U$ for all $y \in V$.*

Let us illustrate the main idea by proving a simpler version of this lemma first. Choose an invariant metric on $G$ and consider the induced *Hausdorff distance* $d_H$ on compact subsets of $G$.

**Lemma 3.87.** *Let $G$ be a topological group acting properly and continuously on a topological space $X$. Let $x \in X$ and suppose $G_x$ is contained in some open subset $U \subset G$. Then, $x$ has a neighbourhood $V \subset X$ such that $G_y \subset U$ for all $y \in V$.*

*Proof.* By properness, each stabiliser $G_x$ is compact. Since $G_x$ is contained in $U$, by compactness the $\epsilon$-collar $B_\epsilon(G_x) = \bigcup_{g \in G_x} B_\epsilon(g) \subset G$ of $G_x$ is contained in $U$, for some $\epsilon > 0$: for each $g \in G_x$, some ball $B_{\epsilon_g}(g)$ is contained in $U$. These cover $U$; since $G_x$ is compact, finitely many of these balls $B_{\epsilon_1}(g_1), \dots, B_{\epsilon_n}(g_n)$ cover $G_x$, so we may choose $\epsilon = \min \epsilon_i$.

Suppose the conclusion fails. Pick a sequence $y_n$ in $X$ such that $d(x, y_n) \leq \frac{1}{n}$ while $d_H(G_{y_n}, G_x) > \epsilon$.[12] Choose $g_n \in G_{y_n}$ such that $d_H(g_n, G_x) > \epsilon$. Observe $y_n \to x$ in $X$ by choice of $(y_n)$ and $g_n \cdot y_n = y_n$ (since $y \in G_{y_n}$).

Properness of the action implies $\bigcup_n G_{y_n}$ is contained in a compact set $K$: the set $L := \{y_n\}_{n \in \mathbb{N}} \cup \{x\}$ is compact, hence so is $L \times L \subset X \times X$. Properness of the $G$-action implies $K := \pi_1(\phi^{-1}(L \times L))$ is compact. Since $K$ contains all the stabilisers $G_{y_n}$ by construction, after passing to a subsequence, we may assume $g_n \to g \in G$. Now, continuity of the $G$-action implies

$$x = \lim_{n\to\infty} y_n = \lim_{n\to\infty} g_n \cdot y_n = \lim_{n\to\infty} g_n \cdot \lim_{n\to\infty} y_n = g \cdot x,$$

hence $g \in G_x$, contradicting the relation $d_H(g_n, G_x) \geq \epsilon$ for all $n$. □

*Proof of Lemma 3.86.* Since $G$ acts properly at $x$, the stabiliser $G_x$ is compact. As in the proof of Lemma 3.87, $U$ contains the $\epsilon$-collar $B_\epsilon(G_x)$ of the stabiliser $G_x$, for some $\epsilon > 0$. We argue by contradiction: suppose there exists a sequence $(y_n)$ in $X$ such that $d(x, y_n) \leq \frac{1}{n}$ while $d_H(G_{y_n}, G_x) > \epsilon$. Ignoring finitely many terms, assume all $G$ acts properly at each $y_n$.

[12] By hypothesis, no ball $B(x, \frac{1}{n})$ satisfies $G_y \subset B_\epsilon(G_x)$: that would imply $G_y \subset U$. Pick $y_n$ witnessing this.

**Claim 1.** *Both* $K := \bigcup_{n\in\mathbb{N}} G_{y_n} \cup G_x$ *and* $L := (\bigcup_{n\in\mathbb{N}} G_{y_n} \times \{y_n\}) \cup G_x \times \{x\}$ *are compact.*

*Proof.* Since $K = \pi_2(L)$ for the coordinate projection $\pi_2 \colon G \times X \to X$, it suffices to prove compactness of $L$. Let $z_n = (v_n, w_n)$ be any sequence in $L$. Consider the sequence $w_n$ first. Since $\{y_n\}_{n\in\mathbb{N}} \cup \{x\}$ is a compact set, $w_n$ has a convergent subsequence — either a constant subsequence (i.e. $w_n \equiv y_i$ for some $i$) or a subsequence converging to $x$. Passing to a subsequence, assume that either $w_n \equiv y_i$ for some $i$ or $w_n \to x$. In the first case, $z_n \subset G_{y_i} \times \{y_i\}$; properness at $y_i$ implies this is a compact set, so a further subsequence is convergent. In the second case, properness at $x$ applies: $w_n \to x$ is a convergent sequence in $X$, $v_n$ is a sequence in $G$ with $v_n \cdot w_n = w_n \to x$, hence properness at $x$ implies a convergent subsequence $v_n \to v$. Altogether, the corresponding subsequence converges to $(v, x)$. △

The remaining proof works as above: by construction, $K$ contains all the stabilisers $G_{y_n}$, hence $(g_n)$ has a subsequence converging to some $g \in G$. Now, continuity of the $G$-action implies

$$x = \lim_{n\to\infty} y_n = \lim_{n\to\infty} g_n \cdot y_n = \lim_{n\to\infty} g_n \cdot \lim_{n\to\infty} y_n = g \cdot x,$$

hence $g \in G_x$, contradicting the relation $d_H(g_n, G_x) \geq \epsilon$ for all $n$. □

*Proof of Lemma 3.85.* Suppose $u \in \widetilde{\mathcal{M}}^A(J)$ is stable. By Proposition 3.61, the $(A \times G)$-action on $\widetilde{\mathcal{M}}^A(J)$ is proper at every stable curve. Since $\Sigma$ is connected, stable curves are precisely the non-constant ones; these form an open subset of $\mathcal{M}^A(J)$. Hence, Lemma 3.86 implies the claim. □

# 4. Smoothness of the iso-symmetric strata

In this chapter, we prove that for generic equivariant $J$, all iso-symmetric strata $\mathcal{M}^{A,H}_{U,\mathbf{l}}(J)$ defined in the previous chapter are smooth manifolds. For technical reasons explained in Section 4.7, we need to assume $A$ and $G$ are finite. There are good reasons to expect a proof in greater generality, but some technical details need to be modified substantially. We will expand on this in Section 4.7 at the end.

The precise statement is easiest to state if $M$ is compact: in this case, the result reads as follows. (The operator $D_u$ is the *linearised Cauchy–Riemann operator* of $u$; it is defined in Section 4.4).

**Theorem 4.1** (Smoothness of iso-symmetric strata, compact case)**.** *Suppose $M$ is closed and $2g + m \geq 3$ and $G$ is finite. There exists a co-meagre subset $\mathcal{J}_{reg}$ of $\mathcal{J}^G(M,\omega)$ resp. $\mathcal{J}^G_\tau(M,\omega)$ such that for all $J \in \mathcal{J}_{reg}$, all closed subgroups $A \leqslant \mathrm{Diff}_+(\Sigma,\theta)$ and $H \leqslant A \times G$ and all $k$-tuples $\boldsymbol{l} = (l_1,\dots,l_k)$ of positive integers with $k \leq m$, the iso-symmetric stratum $\mathcal{M}^{A,H}_{M,\boldsymbol{l}}(J)$ is a smooth finite-dimensional manifold, whose dimension near $u \in \mathcal{M}^{A,H}_{M,\boldsymbol{l}}(J)$ is given by*

$$\dim \mathcal{M}^{A,H}_{M,\boldsymbol{l}}(J) = m^A_1(\mathrm{coker}\, D_{(j,\theta)}) + m^H_1(\ker D_u) - 2\sum_{i=1}^{k}(nl_i - 1).$$

If $M$ is not compact, we need to modify the statement, by adding a bit of bookkeeping. Instead of allowing $J$ to vary within *all* tame resp. compatible $G$-equivariant almost complex structures, we fix one equivariant almost complex structure $J_{\mathrm{fix}}$ and only consider $J$ which match $J_{\mathrm{fix}}$ away from a compact set.

**Definition 4.2.** *Let $U \subset M$ be open and $J_{fix} \in \mathcal{J}^G_\tau(M,\omega)$ resp. $J_{fix} \in \mathcal{J}^G(M,\omega)$ be given. We consider the spaces*

$$\mathcal{J}^G_\tau(M,\omega;U,J_{fix}) := \{J \in \mathcal{J}^G_\tau(M,\omega) \mid J = J_{fix} \text{ on } M \setminus U\}$$

*and*

$$\mathcal{J}^G(M,\omega;U,J_{fix}) := \{J \in \mathcal{J}^G(M,\omega) \mid J = J_{fix} \text{ on } M \setminus U\},$$

*both of which are complete metric spaces.*

Following the argument of Section 2.3 mutatis mutandis shows that $\mathcal{J}^G_\tau(M,\omega;U,J_{\mathrm{fix}})$ and $\mathcal{J}^G(M,\omega;U,J_{\mathrm{fix}})$ are contractible whenever $\mathcal{J}^G_\tau(M,\omega)$ resp. $\mathcal{J}^G(M,\omega)$ are. If $M$ is compact, we may (and often do) take $U = M$; in this case, the choice of $J_{\mathrm{fix}}$ is of course immaterial.

This additional constraint is necessary for technical reasons. To apply the Baire category theorem later, we must work in a complete metric space. The spaces $\mathcal{J}_\tau^G(M,\omega)$ and $\mathcal{J}^G(M,\omega)$ may not be complete; even defining their topology is subtle in the non-compact case. Endowing them with the subspace topology from $C^\infty(M,TM)$ yields a complete space, but this is not the topology we want:[1] to prove that the inclusion $C_\epsilon(\overline{\mathrm{End}}_{\mathbb{C}}(TM,J)) \hookrightarrow \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(TM,J))$ of the $C_\epsilon$-space $C_\epsilon(\overline{\mathrm{End}}_{\mathbb{C}}(TM,J))$ (which we will define in Section 4.1) is continuous, we implicitly require the topology to be defined using the $C^k$-norm in local trivialisations. This is only equivalent to the other definition if $M$ is compact. (Compactness of $M$ also ensures these norms are independent of some auxiliary choices in their definition, which is also desirable.)

The full statement of this chapter's main result is the following.

**Theorem 4.3** (Smoothness of iso-symmetric strata). *Suppose $2g+m \geq 3$ and $G$ is finite. For every open subset $\mathcal{U} \subset M$ with compact closure, there exist co-meagre subsets $\mathcal{J}_{reg,comp} \subset \mathcal{J}^G(M,\omega;\mathcal{U},J_{fix})$ and $\mathcal{J}_{reg,tame} \subset \mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{fix})$ such that for all $J \in \mathcal{J}_{reg,comp}$ resp. $J \in \mathcal{J}_{reg,tame}$, for all closed subgroups $A \leqslant \mathrm{Diff}_+(\Sigma,\theta)$ and $H \leqslant A \times G$ and all $k$-tuples $\mathbf{l} = (l_1,\dots,l_k)$ of positive integers with $k \leq m$, the iso-symmetric stratum $\mathcal{M}_{\mathcal{U},\mathbf{l}}^{A,H}(J)$ is a smooth finite-dimensional manifold. Its dimension near $u \in \mathcal{M}_{\mathcal{U},\mathbf{l}}^{A,H}(J)$ is given as*

$$\dim \mathcal{M}_{\mathcal{U},\mathbf{l}}^{A,H}(J) = m_1^A(\operatorname{coker} D_{(j,\theta)}) + m_1^H(\ker D_u) - 2n\sum_{i=1}^k l_i.$$

Our proof uses the countability of the iso-symmetric strata: given fixed $A$, $G$, $\mathcal{U}$ and $\mathbf{l}$, we find a co-meagre subset $\mathcal{J}_{\mathrm{reg}}$ (of $\mathcal{J}^G(M,\omega;\mathcal{U},J_{\mathrm{fix}})$ resp. $\mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{\mathrm{fix}})$) such that $\mathcal{M}_{\mathcal{U},\mathbf{l}}^{A,H}(J)$ is a smooth manifold for all $J \in \mathcal{J}_{\mathrm{reg}}$. Moreover, we neglect the constraint $\mathbf{l}$ on the critical points at first; adding this in the end only requires standard adjustments (e.g. [Wen23d, Appendix A]). To summarize, the bulk of this chapter is devoted to proving the following result.

**Theorem 4.4.** *Let $A \leqslant \mathrm{Diff}_+(\Sigma,\theta)$ and $H \leqslant A \times G$ be closed subgroups, suppose $A$ and $G$ are finite. For each open subset $\mathcal{U} \subset M$ with compact closure, there exists a co-meagre subset $\mathcal{J}_{reg}$ (of $\mathcal{J}^G(M,\omega;\mathcal{U},J_{fix})$ resp. $\mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{fix})$) such that for all $J \in \mathcal{J}_{reg}$, the set $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$ is a smooth finite-dimensional manifold, of dimension (near $u$)*

$$\dim \mathcal{M}_{\mathcal{U},\mathbf{l}}^{A,H}(J) = m_1^A(\operatorname{coker} D_{(j,\theta)}) + m_1^H(\ker D_u).$$

The proof of Theorem 4.4 follows a standard framework. The main idea is to allow $J$ to vary in a sufficiently large space of perturbations: the resulting universal moduli space of holomorphic curves can be shown to be a smooth Banach manifold, and applying the Sard–Smale theorem yields a co-meagre set $\mathcal{J}_{\mathrm{reg}}$ as desired.

[1] in addition to this being an odd construction, as this completely ignores the additional structure on $TM$ as a vector bundle over $M$

Thus, we begin by constructing, for each fixed $J_{\text{ref}} \in \mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{\text{fix}})$ (resp. $J_{\text{ref}} \in \mathcal{J}^G(M,\omega;\mathcal{U},J_{\text{fix}})$), a suitable space $\mathcal{J}_\epsilon$ of perturbations of $J_{\text{ref}}$, through $G$-equivariant tame resp. compatible almost complex structures. We use an equivariant version of the *Floer $C_\epsilon$-space*; this is detailed in Section 4.1.

Next, we prove that the corresponding *universal moduli space* $\mathcal{U}^*(\mathcal{J}_\epsilon) := \{(u,J) \mid u \in \widetilde{\mathcal{M}}_{\mathcal{U}}^{A,H}(J), J \in \mathcal{J}_\epsilon\}$ is a smooth separable and metrisable Banach manifold. This is the hardest and most technical part of the proof. We present the sets $\widetilde{\mathcal{M}}_{\mathcal{U}}^{A,H}(J)$ locally as the zero-set of a suitable section $\overline{\partial}_J \colon \mathcal{B} \to \mathcal{E}$ of a Banach space vector bundle (which we will define in Section 4.3). We begin by finding such a local model for the parametrised pre-strata $\widetilde{\mathcal{M}}^A(J)$: one technical detail arising in this process is handling the non-continuous dependence of $\mathrm{Aut}(\Sigma,j,\theta)$ on $j$. We do so via *adapted Teichmüller slices* in Section 4.2. In Section 4.3, we extend this to local models for $\widetilde{\mathcal{M}}^A(J)$ and deduce smoothness of the universal moduli space in Section 4.4.

Finally, in Section 4.5 we apply the Sard–Smale theorem to conclude the existence of a co-meagre subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}_\epsilon$. As the topology on $\mathcal{J}_\epsilon$ differs from the $C^\infty_{\text{loc}}$-topology, a co-meagre subset of $\mathcal{J}_\epsilon$ is not obviously co-meagre in $\mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{\text{fix}})$. Showing it is *dense* in $\mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{\text{fix}})$ (resp. $\mathcal{J}^G(M,\omega;\mathcal{U},J_{\text{fix}})$) is easy; an argument due to Taubes [Tau96, Section 5] allows upgrading this to a co-meagre subset. This completes the proof of Theorem 4.4. We end the chapter by deducing Theorem 4.3 from Theorem 4.4 (Section 4.6).

For the rest of this chapter, we shall make the following convention.

**Convention.** Fix an open subset $\mathcal{U} \subset M$ with compact closure and $J_{\text{fix}} \in \mathcal{J}_\tau^G(M,\omega)$ resp. $J_{\text{fix}} \in \mathcal{J}^G(M,\omega)$. In addition, recall the set-up we made in the previous chapter: in particular, we fixed a closed connected genus $g$ surface $\Sigma$ with an ordered subset $\theta$ of $m$ points.

## 4.1. Equivariant $C_\epsilon$-space

Fix a $G$-equivariant tame resp. compatible almost complex structure $J_{\text{ref}} \in \mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{\text{fix}})$ resp. $J_{\text{ref}} \in \mathcal{J}^G(M,\omega;\mathcal{U},J_{\text{fix}})$. The aim of this section is to construct a space $\mathcal{J}_\epsilon$ of perturbations of $J_{\text{ref}}$ — which is suitably large to allow proving smoothness of the universal moduli space.

A priori, the easiest option is to consider $\mathcal{J}^G(M,\omega;\mathcal{U},J_{\text{fix}})$ (resp. $\mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{\text{fix}})$) with its natural $C^\infty_{\text{loc}}$-topology: however, these spaces are not Banach manifolds as they are not complete (for the same reasons that the space of smooth functions between two $C^k$ manifolds is not complete). Presumably, they still form a smooth Fréchet manifold,[2] but this will not matter: we prefer working in a Banach space so we can apply the implicit function theorem.

[2]Without equivariance, this is a statement commonly made in the field, usually without elaborating on the details (what is a Fréchet manifold? what is a smooth map between Fréchet manifolds? why is this space a Fréchet manifold; why are the transition maps smooth? These details are more subtle than one might think [Wen23c]) nor providing a reference. In fact, the author is not aware of any reference proving this, even in the classical case. In our particular setting, being a Fréchet

There are two common strategies in the literature for creating a Banach manifold setup. The first (employed e.g. in McDuff and Salamon's textbook [MS12]) is to work with almost complex structures of class $C^k$, i.e. replacing $\mathcal{J}^G(M,\omega;\mathcal{U},J_{\text{fix}})$ with its completion in the $C^k$ topology: this means the moduli space $\mathcal{M}_{g,m}(C,J)$ and our strata are (at best) $C^k$-manifolds for large $k$, but not smooth. Carrying this out entails having to count derivatives at each step to ensure sufficient smoothness, which we would like to avoid.

Hence, we will follow a different strategy and replace $\mathcal{J}_\epsilon$ by a smaller space, consisting of smooth objects only, with a finer topology making it a Banach manifold. We use the *Floer $C_\epsilon$-topology*, introduced by Andreas Floer [Flo88]: this depends on an auxiliary parameter $\epsilon$. As $\epsilon$ is not canonical at all, some effort is required to remove the mention of the $C_\epsilon$-space in final theorem statements.

Let us recall the standard set-up before adapting it to $G$-equivariant almost complex structures. Our presentation follows the standard approach (e.g. [Wen23d, Section 5.4] or the less densely written [Wen21] or [Bar24]).

Let $E \to M$ be a smooth vector bundle of finite rank. For each $k$, denote by $C^k(E)$ the space of $C^k$-sections of $E$; we will abbreviate $\Gamma(E) := C^\infty(E)$. If $U \subset M$ is open, we write $C^k(E;U)$ resp. $\Gamma(E;U)$ for all $C^k$ (resp. smooth) sections vanishing outside of $U$.

**Recall.** Let $E \to M$ be a smooth vector bundle of finite rank; suppose $M$ is compact. Each choice of a connection on $E$ and bundle metrics on $E$ and $TM$ induces a norm on $C^k(E)$. Since $M$ is compact, different choices yield equivalent norms.

Write $\boldsymbol{\mathcal{E}} := \{\epsilon = (\epsilon_n)_{n\in\mathbb{N}} \mid \epsilon_i > 0, \epsilon_n \to 0\}$ for the space of positive real sequences converging to zero.[3]

**Definition 4.5** ($C_\epsilon$-space, compact case)**.** *Let $E \to M$ be a smooth vector bundle of finite rank, over a compact manifold $M$. For each $\epsilon \in \boldsymbol{\mathcal{E}}$, the corresponding $C_\epsilon$-*norm *on $\Gamma(E)$ is defined as*

$$\|\eta\|_{C_\epsilon} := \sum_k^\infty \epsilon_k \|\eta\|_{C^k(E)};$$

*the* Floer $C_\epsilon$-space *is given as*

$$C_\epsilon(E) := \{\eta \in \Gamma(E) \mid \|\eta\|_{C_\epsilon} < \infty\}.$$

If $M$ is not compact, we can use a small trick to still define the $C_\epsilon$-space. For any open subset $U \subset M$, observe that $C_\epsilon(E;U) := \{\eta \in C_\epsilon(E) \mid \operatorname{supp}\eta \subset U\}$ is a closed subspace of $C_\epsilon(E)$.

manifold is easy to define (the topology on the model space is still generated by a single norm), but that does not prove smoothness of transition maps. In either case, we cannot use the implicit function anyway.

[3]Beware: in later sections, $\mathcal{E}$ (which is not bold) will denote a certain Banach space bundle. In most sections, we use at most one of these symbols.

**Definition 4.6** ($C_\epsilon$-space, non-compact case)**.** *Let $\mathcal{U} \subset M$ be an open subset with compact closure; we do not assume $M$ to be compact. For each $\epsilon \in \boldsymbol{\mathcal{E}}$, we define*

$$C_\epsilon(E;\mathcal{U}) := \{\eta \in \Gamma(E) \mid \operatorname{supp}\eta \subset \mathcal{U}, \eta|_{M_0} \in C_\epsilon(E|_{M_0})\},$$

*where $M_0 \subset M$ is any compact manifold (possibly with boundary) such that $\overline{\mathcal{U}} \subset M_0$.*

Let us emphasize that the sequence $\epsilon = (\epsilon_n)$ is an auxiliary object, as is the $C_\epsilon$-space itself. As advertised, $C_\epsilon(E;\mathcal{U})$ is a separable Banach space.

**Lemma 4.7** ([Wen20, Theorems B.2 and B.5])**.** *Let $E \to M$ be a smooth vector bundle of finite rank, and $\mathcal{U} \subset M$ be an open subset with compact closure. Then $C_\epsilon(E;\mathcal{U})$ is a separable Banach space.* □

Which particular sections belong to $C_\epsilon(E;\mathcal{U})$ is a bit mysterious: however, in any case, $C_\epsilon(E;\mathcal{U})$ is a large space of perturbations (in particular, infinite-dimensional): for each point $p \in \mathcal{U}$, for well-chosen $\epsilon$, it contains bump functions with arbitrarily small support around $p$.

**Proposition 4.8** ([Wen20, Theorem B.6])**.** *Let $E \to M$ be a smooth vector bundle of finite rank, and $\mathcal{U} \subset M$ be an open subset with compact closure. The sequence $\epsilon \in \boldsymbol{\mathcal{E}}$ can be chosen such that*

(1) *$C_\epsilon(E;\mathcal{U})$ is dense in the space of continuous sections vanishing outside $\mathcal{U}$.*

(2) *Given any point $p \in \mathcal{U}$, a neighbourhood $\mathcal{N}_p \subset \mathcal{U}$ of $p$, a $\delta > 0$ and a continuous section $\eta_0$ of $E$, there exist a section $\eta \in \Gamma(E)$ and a smooth compactly supported function $\beta\colon \mathcal{N}_P \to [0,1]$ such that*

$$\beta\eta \in C_\epsilon(E;\mathcal{U}), \qquad \beta(p)\eta(p) = \eta_0(p) \qquad \text{and} \qquad \|\eta - \eta_0\|_{C^0(E)} < \delta. \qquad \square$$

It is not obvious whether a given smooth section $\eta \in \Gamma(E)$ lies in $C_\epsilon(E;\mathcal{U})$. However, this is the wrong question to ask [Wen21]: instead of wondering whether a given $\eta \in \Gamma(E)$ lies in the $C_\epsilon$-space, we should *choose* $\epsilon$ to ensure $\eta \in C_\epsilon(E;\mathcal{U})$. Later in this chapter, we would like to ensure a given *family* of sections $\mathcal{Q} \subset \Gamma(E)$ lies in $C_\epsilon(E;\mathcal{U})$. For countable families, this is possible by a diagonal argument. This will suffice for our purposes, since all spaces of sections we are dealing with are separable.

The starting point of this analysis is defining a pre-order on $\boldsymbol{\mathcal{E}}$, according to which sequence "converges to 0 faster".

**Lemma/Definition 4.9** ([Wen21])**.** *For $\epsilon, \epsilon' \in \boldsymbol{\mathcal{E}}$, we write $\epsilon \prec \epsilon'$ if and only if $\limsup_n \frac{\epsilon_n}{\epsilon'_n} < \infty$. This defines a pre-order on $\boldsymbol{\mathcal{E}}$.* □

Clearly, $\epsilon \prec \epsilon'$ implies $C_{\epsilon'}(E;\mathcal{U}) \subset C_\epsilon(E;\mathcal{U})$: makes $\epsilon$ converge to zero faster *enlarges* the $C_\epsilon$-space. Slightly stronger, the following holds.

**Lemma 4.10.** *The natural inclusion $C_\epsilon(E;\mathcal{U}) \hookrightarrow \Gamma(E)$ is continuous.*[4]
*If $\epsilon \prec \epsilon'$, the natural inclusion $C_{\epsilon'}((;E);\mathcal{U}) \hookrightarrow C_\epsilon(E;\mathcal{U})$ is continuous.*

*Proof.* The first item follows since $\epsilon = (\epsilon_n) \to 0$ by definition. For the second item, if $\epsilon \prec \epsilon'$ there exists a constant $C > 0$ such that $\epsilon_n \leq C\epsilon'_n$ for all sufficiently large $n$. This implies continuity of the inclusion $C_{\epsilon'}(E;\mathcal{U}) \hookrightarrow C_\epsilon(E;\mathcal{U})$. □

The following simple lemma powers the observation we need.

**Lemma 4.11** ([Wen23d, Lemma 5.29])**.** *Every countable subset of $\boldsymbol{\mathcal{E}}$ has a lower bound w.r.t. the pre-order $\prec$.*

*Proof.* This follows by a diagonal argument: if $S = \{\epsilon^{(k)}\}$ is a countable subset of $\boldsymbol{\mathcal{E}}$, the sequence $\epsilon = (\epsilon_n)$ defined by $\epsilon_l := \min(\epsilon_l^{(1)}, \epsilon_l^{(2)}, \dots, \epsilon_l^{(l)})$ is a lower bound for $S$: for each $k \in \mathbb{N}$, by construction we have $\limsup_n \frac{\epsilon_n^{(k)}}{\epsilon_n} < 1$. □

**Lemma 4.12** ([Wen23d, Lemma 5.29; Bar24, Proposition 2.2.14])**.**
*One has $\Gamma(E;\mathcal{U}) = \bigcup_{\epsilon \in \boldsymbol{\mathcal{E}}} C_\epsilon(E)$. Moreover, given a countable set $\mathcal{Q} \subset \Gamma(E;\mathcal{U})$ of smooth sections, there exists a sequence $\epsilon \in \boldsymbol{\mathcal{E}}$ decaying sufficiently fast so $\mathcal{Q} \subset C_\epsilon(E;\mathcal{U})$.*

*Proof.* Each smooth section $\eta \in \mathcal{Q}$ belongs to $C_{\epsilon_\eta}(E;\mathcal{U})$ for some $\epsilon_\eta$: for instance, we may choose $\epsilon_\eta := 2^{-n}\|\eta\|_{C^k(E)}$. By Lemma 4.11, $\{\epsilon_\eta\}_{\eta \in \mathcal{Q}}$ has some lower bound $\eta \in \boldsymbol{\mathcal{E}}$; for any such lower bound $\eta$, indeed $\mathcal{Q} \subset C_\epsilon(E;\mathcal{U})$. □

This completes our review of the classical proof. Adapting this to $G$-equivariant almost complex structures is straightforward. Let $E \to M$ be a smooth vector bundle of finite rank, and $\mathcal{U} \subset M$ be an open subset with compact closure. Suppose a group $G$ acts smoothly on $E$, by bundle isomorphisms $A_g \colon E \to E$ over diffeomorphisms $\phi_g \colon M \to M$. Denote the space of $G$-equivariant sections by $\Gamma^G(E)$.

**Definition 4.13** (Equivariant $C_\epsilon$-space)**.** *For $\epsilon \in \boldsymbol{\mathcal{E}}$, the* equivariant $C_\epsilon$-space $C_\epsilon^G(E;\mathcal{U})$ *is defined as*

$$\begin{aligned} C_\epsilon^G(E;\mathcal{U}) &:= C_\epsilon(E;\mathcal{U}) \cap \Gamma^G(E) \\ &= \{\eta \in C_\epsilon(E;\mathcal{U}) \mid \eta \text{ is } G\text{-equivariant: } \forall g \in G, A_g \circ \eta = \eta \circ \phi_g\}. \end{aligned}$$

The equivariant $C_\epsilon$-space has analogous properties to its classical counterpart.

**Proposition 4.14** (Properties of $C_\epsilon^G(E;\mathcal{U})$)**.** *Let $\epsilon \in \boldsymbol{\mathcal{E}}$ be arbitrary.*

(1) *$C_\epsilon^G(E;\mathcal{U})$ is a separable Banach space, and the natural inclusion $C_\epsilon^G(E;\mathcal{U}) \hookrightarrow \Gamma^G(E)$ is continuous.*

(2) *If $\epsilon \prec \epsilon'$, the natural inclusion $C_{\epsilon'}^G(E;\mathcal{U}) \hookrightarrow C_\epsilon^G(E;\mathcal{U})$ is continuous.*

[4] In this statement, we implicitly use that the topology on $\Gamma(E)$ is equivalently given by bundle norms constructed from local trivialisations.

(3) *Given a countable set* $\mathcal{Q} \subset \Gamma^G(E;\mathcal{U})$ *of smooth* $G$*-equivariant sections, there exists a sequence* $\epsilon \in \boldsymbol{\mathcal{E}}$ *decaying sufficiently fast so* $\mathcal{Q} \subset C^G_\epsilon(E;\mathcal{U})$.

(4) *If* $G$ *is compact,* $C^G_\epsilon(E;\mathcal{U})$ *contains bump functions with arbitrarily small support around each point in* $\mathcal{U}$*: there exists a*[5] *sequence* $\epsilon \in \boldsymbol{\mathcal{E}}$ *such that*

- $C^G_\epsilon(E;\mathcal{U})$ *is dense in* $(C^0)^G(E;\mathcal{U})$ *and*
- *for all* $p \in \mathcal{U}$, $\delta > 0$, $\eta_0 \in (C^0)^G(E)$ *and any* $G$*-invariant neighbourhood* $\mathcal{N}_p \subset \mathcal{U}$ *of* $p$*, there exist a smooth equivariant section* $\eta \in \Gamma^G(E)$ *and a smooth* $G$*-invariant compactly supported function* $\beta\colon \mathcal{N}_p \to [0,1]$ *such that*

$$\beta\eta \in C^G_\epsilon(E;\mathcal{U}) \qquad \text{and} \qquad \beta\eta(p) = \eta_0(p) \qquad \text{and} \qquad \|\eta - \eta_0\|_{C^0} < \delta.$$

*Proof.* Using the previous propositions, many proofs are straightforward. Items (2) and (3) follow from Lemmas 4.10 and 4.12 by restriction. For Item (1), by Lemma 4.7 it suffices to show that $C^G_\epsilon(E;\mathcal{U}) \subset C_\epsilon(E;\mathcal{U})$ is a closed subspace. Let $\eta_j$ be a sequence in $C^G_\epsilon(E;\mathcal{U})$ such that $\eta_j \to \eta_\infty$ in $C_\epsilon(E;\mathcal{U})$. For any $g \in G$ and $p \in M$, we compute

$$\eta_\infty(\phi_g(p)) = \lim_{j\to\infty} \eta_j(\phi_g(p)) = \lim_{j\to\infty} A_g \circ \eta_j(p) = A_g \lim_{j\to\infty} \eta_j(p) = A_g\eta_\infty(p),$$

using that $\eta_j \to \eta_\infty$ in $C_\epsilon(E;\mathcal{U})$ also implies $C^\infty_{\mathrm{loc}}$-convergence. Hence, $\eta_\infty \in C^G_\epsilon(E;\mathcal{U})$.

Item (4) needs some technical work. Let $p \in \mathcal{U}$, $\delta > 0$, a $G$-invariant neighbourhood $\mathcal{N}_p \subset \mathcal{U}$ of $p$ and a $G$-equivariant section $\eta_0$ be given. Choose a $G$-invariant compactly supported smooth function $\beta\colon \mathcal{N}_p \to [0,1]$ with $\beta(p) = 1$ and $\operatorname{supp}\beta \subset \operatorname{im}(\phi)$.[6]

It remains to find a smooth $G$-equivariant section $\eta$ sufficiently close to $\eta_0$. Choose a smooth compactly supported function $\alpha\colon \mathring{\mathbb{D}}^m \to [0,1]$ with $\alpha(0) = 1$ and an embedding $\phi\colon \mathring{\mathbb{D}}^m \to \mathcal{N}_p$ which maps $0$ to $p$. This embedding will allow us to rescale $\alpha$ around $p$, producing functions with arbitrarily small support.
More precisely, for each $r \in (0,1)$, consider the "rescaled" function

$$\alpha_r\colon \mathcal{N}_p \to [0,1], \alpha_r(q) := \begin{cases} \alpha(r\phi^{-1}(q)) & \text{if } q \in \operatorname{im}\phi \\ 0 & \text{otherwise} \end{cases}.$$

Observe that $(\alpha_r)$ converges to the function $f \equiv 1$ as $r \to 0$.

The functions $\alpha_r$ need not be $G$-invariant, but we can turn them into $G$-invariant functions. Since $G$ is compact, we can "average" smooth functions $f\colon \mathcal{N}_p \to \mathbb{R}$ using the Haar measure on $G$: the average of $f$ is the smooth function $x \mapsto \int_G f(g\cdot x)\,\mathrm{d}g$,

[5] This sequence is not special: any $\epsilon' \prec \epsilon$ has the same property.

[6] The existence of $\beta$ is a standard fact: for instance, extend $\{\operatorname{im}\phi\}$ to a $G$-invariant locally finite open cover of $M$ (where $\phi$ is defined below). Choose a $G$-invariant partition of unity subordinate to that cover, and pick the function corresponding to $\operatorname{im}\phi$. A $G$-invariant partition of unity exists since $G$ acts smoothly and properly on $M$ [DK00, Lemma 2.5.1].

where $\mathrm{d}g$ is the Haar measure on $G$. Clearly, averaging is continuous and each averaged function is $G$-invariant. Let $\beta_r$ denote the function obtained by averaging $\alpha_r$ over $G$ and rescaling so that $\beta_r(0) = 1$. Since $f \equiv 1$ is $G$-invariant, we deduce $\beta_r \to f \equiv 1$ as $r \to 0$.

For each $r \in (0, 1)$, consider the $G$-equivariant section $\eta_r$ defined by

$$\eta_r(q) = \begin{cases} \beta_r(q)\ \eta_0 & q \in \mathrm{i}\mathring{\mathrm{m}}(\phi) \\ 0 & \text{otherwise;} \end{cases}$$

we have $\eta_r(p) = \eta_0(p)$ by construction. For each $r > 0$, we can choose $\epsilon \in \boldsymbol{\mathcal{E}}$ so $\eta_r \in C_\epsilon(E; U)$. Observe that $\eta_r \to \eta_0$ in $C^0$ as $r \to 0$. Hence, for $r > 0$ sufficiently large, we have $\|\eta_r - \eta_0\|_{C^0} < \delta$, and we may choose $\eta := \eta_r$ for one such $r$. $\square$

Finally, let us describe the actual construction of $\mathcal{J}_\epsilon$ from the $C_\epsilon$-space: this uses the exponential map

$$\begin{aligned} \Gamma^G(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}})) &\to \mathcal{J}^G(M, \omega; \mathcal{U}, J_{\mathrm{fix}}), \\ Y &\mapsto J_Y := (\mathrm{id} + \frac{1}{2} J_0 Y) J_0 (\mathrm{id} + \frac{1}{2} J_0 Y)^{-1} \end{aligned} \tag{4.1}$$

from Chapter 2 on the bundle $E := (TM, J_{\mathrm{ref}})$. We restrict the domain to ensure the exponential map is a homeomorphism. The definition will be differ in the tame and compatible cases.

**Lemma 4.15.** *The map* (4.1) *is a homeomorphism from a $C^0$-small neighbourhood of* $0$ *in* $\Gamma^G(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{ref}); \mathcal{U})$ *to a neighbourhood of $J_{ref}$ in* $\mathcal{J}^G(M, \omega; \mathcal{U}, J_{fix})$. $\square$

Fix $\delta > 0$ sufficiently small so (4.1) restricted to all $Y$ with $\|Y\|_{C^0} < \delta$ is a homeomorphism to its image. Then, our space $\mathcal{J}_\epsilon$ is (in the tame case) given as

$$\mathcal{J}_\epsilon := \mathcal{J}_\epsilon^{\mathcal{U},G} := \{J_Y \mid Y \in C_\epsilon^G(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U}), \|Y\|_{C^0} < \delta\}.$$

For compatible almost complex structures, we need to additionally demand that $Y$ be symmetric w.r.t. $\omega(\cdot, J_0 \cdot)$, and instead define

$$\mathcal{J}_\epsilon := \mathcal{J}_\epsilon^{\mathcal{U},G} := \{J_Y \mid Y \in C_\epsilon^G(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U}), \|Y\|_{C^0} < \delta; \omega(Y\cdot, \cdot) + \omega(\cdot, Y\cdot) = 0\}.$$

**Lemma 4.16.** *The space $\mathcal{J}_\epsilon$ is a smooth, separable metrizable Banach manifold* (*with one chart*) *which embeds continuously into* $\mathcal{J}^G(M, \omega; \mathcal{U}, J_{fix})$ *resp.* $\mathcal{J}_\tau^G(M, \omega; \mathcal{U}, J_{fix})$.

*Proof.* $\mathcal{J}_\epsilon$ has a single Banach manifold chart, the restriction of (2.7). Since the space $C_\epsilon^G(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U})$ (resp. $\{Y \in C_\epsilon^G(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U}) \mid \omega(Y\cdot, \cdot) + \omega(\cdot, Y\cdot) = 0\}$ in the compatible case) is a separable metrizable Banach manifold, so is $\mathcal{J}_\epsilon$. Continuous inclusion into $\mathcal{J}^G(M, \omega; \mathcal{U}, J_{\mathrm{fix}})$ resp. $\mathcal{J}_\tau^G(M, \omega; \mathcal{U}, J_{\mathrm{fix}})$ follows from Lemma 4.10. $\square$

The tangent spaces of $\mathcal{J}_\epsilon$ are tricky to describe in general, but the tangent space at $J_{\mathrm{ref}}$ is straightforward.

**Observation 4.17.** The tangent space $T_{J_{\mathrm{ref}}} \mathcal{J}_\epsilon$ is $C_\epsilon^G(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U})$ in the tame case resp. $\{Y \in C_\epsilon^G(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U}) \mid \omega(\cdot, Y\cdot) + \omega(Y\cdot, \cdot)\}$ in the compatible case. $\square$

## 4.2. Adapted Teichmüller slices

As the next step for setting up the universal moduli space, we describe the variation of $j$ among parametrised holomorphic curves $(\Sigma, j, \theta, u)$: this is an important ingredient for the local model of $\widetilde{\mathcal{M}}^{A,H}(J)$. In the classical setting (just for curves in $\widetilde{\mathcal{M}}(J)$), this is captured by so-called *Teichmüller slices*. We review this construction and adapt it to the pre-strata $\widetilde{\mathcal{M}}^{A}(J)$.

### 4.2.1. Review: Teichmüller slices

Let us ignore the group $A$ for a moment, and consider how $j$ varies locally within the moduli space $\mathcal{M}(J)$. The variation can be described using a *Teichmüller slice*. This is standard; all proofs in this section can be found in [Wen15, §4.2]. The word "Teichmüller" comes from a relation to *Teichmüller space*; let us explain that first.

**Fact.** $\mathcal{M}_{g,m}$ is homeomorphic to the quotient $\mathcal{M}(\Sigma,\theta) := \mathcal{J}(\Sigma,\theta)/\operatorname{Diff}_+(\Sigma,\theta)$, where $\mathcal{J}(\Sigma)$ is the space of smooth complex structures on $\Sigma$ and $\operatorname{Diff}_+(\Sigma,\theta)$ consists of all orientation-preserving diffeomorphisms $\phi$ of $\Sigma$ with $\phi(\theta) = \theta$ as ordered sets. □

**Recall.** The *Teichmüller space* $\mathcal{T}(\Sigma,\theta)$ of $(\Sigma,\theta)$ is the quotient $\mathcal{J}(\Sigma)/\operatorname{Diff}_0(\Sigma,\theta)$, where

$$\operatorname{Diff}_0(\Sigma,\theta) = \{\phi \in \operatorname{Diff}_+(\Sigma,\theta) \mid \phi \text{ is homotopic to } \mathrm{id}\}.$$

The *mapping class group* of $(\Sigma,\theta)$ is $M(\Sigma,\theta) = \operatorname{Diff}_+(\Sigma,\theta)/\operatorname{Diff}_0(\Sigma,\theta)$. In particular, we have $\mathcal{M}(\Sigma,\theta) = \mathcal{T}(\Sigma,\theta)/M(\Sigma,\theta)$.

In fact, this description is used to show $\mathcal{M}_{g,m}$ is a smooth orbifold: the Teichmüller space is a smooth finite-dimensional manifold, the mapping class group is a discrete group, and (in the stable case) acts properly with finite stabiliser. We are less interested in the fact that Teichmüller space is a manifold than we are in its local charts: these are provided by Teichmüller *slices*, which we will adapt for our purposes.

To construct Teichmüller slices, we exhibit $\operatorname{Aut}(\Sigma, j, \theta)$ locally as the zero set of a suitable section of a Banach space bundle.

**Recall** (e.g. [Wen15, §4.2])**.** Fix an integer $p > 2$ and $j \in \mathcal{J}(\Sigma)$. Consider the Banach manifold $\mathcal{B}_\theta = \{\phi \in W^{1,p}(\Sigma,\Sigma) \mid \phi|_\theta = \mathrm{id}\}$, and the Banach space bundle $\mathcal{E} \to \mathcal{B}_\theta$ with fibres $\mathcal{E}_\phi = L^p(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, \phi^*T\Sigma))$. The non-linear operator

$$\overline{\partial}_j \colon \mathcal{B}_\theta \to \mathcal{E}, \phi \mapsto d\phi + j \circ d\phi \circ j$$

is a smooth section of $\mathcal{E}$.

Observe that zeroes of $\overline{\partial}_j$ are holomorphic maps on $(\Sigma, j)$ which fix $\theta$: in particular, a neighbourhood of id in $\overline{\partial}_j^{-1}(0)$ gives a local description of $\operatorname{Aut}(\Sigma, j, \theta)$. An elliptic regularity argument[7] shows that the linearisation $D_{(j,\theta)} := D\overline{\partial}_j(\mathrm{id}) \colon W^{1,p}_\theta(T\Sigma) \to$

[7] which is standard, or, to quote an old common saying, "well-known to those who know it well"

$L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ of $\bar\partial_j$ at id is a Fredholm operator[8], in particular has finite-dimensional co-kernel.

**Definition 4.18.** *A* Teichmüller slice *through $j \in \mathcal{J}(\Sigma)$ is an injective smooth map $\mathcal{O} \to \mathcal{J}(\Sigma)$, $\tau \mapsto j_\tau$ such that $j_0 = j$, where $\mathcal{O}$ is a neighbourhood of $0$ in some finite-dimensional Euclidean space and $\operatorname{im} D_{(j,\theta)} \oplus T_j\mathcal{T} = L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$, where*

$$T_j\mathcal{T} = \{\partial_t j_{\tau(t)}|_{t=0} \;:\; \tau(t) \in \mathcal{O} \text{ smooth path through } \tau(0) = 0\}$$

*is the "tangent space" to the image $\mathcal{T} = \{j_\tau\}_{\tau\in\mathcal{O}}$. In addition, we ask that $\mathcal{O}$ and $T_j\mathcal{T}$ have the same dimension.*

**Remark 4.19.** The words "smooth map" deserve explanation: after all, we saw in Section 4.1 that $\mathcal{J}(\Sigma)$ is *not* a Banach manifold, so the map $\mathcal{O} \to \mathcal{J}(\Sigma)$ is not a smooth map between Banach manifolds. Instead, it is a smooth map in the sense of smooth families, i.e. $\mathcal{T}$ defines a family of endomorphisms $\{j_{(\tau,z)} \in \mathrm{End}(T_z\Sigma)\}_{(\tau,z)\in\mathcal{O}\times\Sigma}$ which depends smoothly on $(\tau, z)$ (as measured in a smooth atlas of $\Sigma$). In other words, $\mathcal{T}$ defines a smooth section of the bundle $E = T\Sigma \to \mathcal{O} \times \Sigma$ with fibres $E_{(\tau,z)} = T_z\Sigma$.

We generally identify a Teichmüller slice with its image $\mathcal{T} = \{j_\tau\}_{\tau\in\mathcal{O}} \subset \mathcal{J}(\Sigma)$. Intuitively, we may consider $\mathcal{T}$ as a finite-dimensional smooth embedded submanifold of $\mathcal{J}(\Sigma)$. The co-dimension of the image of $D_{(j,\theta)}$ is independent of $p$; again, this is a standard instance of elliptic regularity results.

Teichmüller slices exist in abundance: they can be explicitly constructed by hand, and even make them $\mathrm{Aut}(\Sigma, j, \theta)$-invariant (as a set).

**Lemma 4.20** (e.g. [Wen15, Lemma 4.3.4])**.** *For any $j \in \mathcal{J}(\Sigma)$, there exists an $\mathrm{Aut}(\Sigma, j, \theta)$-invariant (as a set) Teichmüller slice $\mathcal{T}$ through $j$.* □

Let us sketch the main idea of the proof, as adapted Teichmüller slices use the same argument. We use an exponential map very similar to Equation (2.7) in Section 2.3.

*Sketch of proof.* Let $j \in \mathcal{J}(\Sigma)$ be arbitrary. Since the linearisation $D_{(j,\theta)} \colon W^{1,p}_\theta(T\Sigma) \to L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ is a Fredholm operator, we may choose a finite-dimensional complement $C \subset L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ of $\operatorname{im} D_{(j,\theta)}$. By approximation, we may assume $C$ consists of smooth sections. Given a neighbourhood $\mathcal{O}$ of $0$ in $C$, consider the map

$$\mathcal{O} \to \mathcal{J}(\Sigma), y \mapsto j_y := (1 + \frac{1}{2}jy)j(1 + \frac{1}{2}jy)^{-1}. \tag{4.2}$$

Shrinking $\mathcal{O}$, we can ensure this map is injective; then $\mathcal{T} := \{j_\tau\}_{\tau\in\mathcal{O}}$ is a Teichmüller slice through $j$. To ensure $\mathcal{T}$ is $\mathrm{Aut}(\Sigma, j, \theta)$-invariant, simply choose a complement $C$ which is $\mathrm{Aut}(\Sigma, j, \theta)$-invariant. □

[8] The definition of Fredholm operators is recalled in Section 4.5.

**Definition 4.21.** *A Teichmüller slice $\mathcal{T}$ is called* good *if it obtained as in the above proof: for some neighbourhood of* $0$ *in a finite-dimensional subspace $C \subset L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$, we have $\mathcal{T} = \{j_\tau\}_{\tau \in \mathcal{O}}$, with $j_\tau$ being determined by* (4.2).

Teichmüller slices give local charts for $\mathcal{T}(\Sigma, \theta)$, via the quotient projection $\pi_\theta \colon \mathcal{J}(\Sigma) \to \mathcal{T}(\Sigma, \theta), j \mapsto [j]$ to the Teichmüller space of $(\Sigma, \theta)$.

**Lemma 4.22** ([Wen15, Theorem 4.2.14])**.** *If $\mathcal{T}$ is a Teichmüller slice through $j \in \mathcal{J}(\Sigma)$, the quotient projection $\pi_\theta|_{\mathcal{T}} \colon \mathcal{T} \to \mathcal{T}(\Sigma, \theta)$ is a local diffeomorphism near $j$.* □

This is closely related to a property we need: Teichmüller slices capture the local variation of the complex structure $j$ within $\mathcal{M}(J)$. In other words, locally we can restrict attention to complex structures varying within a Teichmüller slice.

**Proposition 4.23** ([Wen15, pp. 162+163])**.** *Let $j \in \mathcal{J}(\Sigma)$ and $\mathcal{T}$ be an* $\mathrm{Aut}(\Sigma, j, \theta)$*-invariant Teichmüller slice through $j_0$. Any holomorphic curve $[(j_0, u_0)] \in \mathcal{M}_{g,m}(C, J)$ has a neighbourhood $U \subset \mathcal{M}_{g,m}(C, J)$ such that each $[(\Sigma, j, \theta, u)] \in U$ has a reparametrisation $(\Sigma, \phi^* j, \theta, u \circ \phi)$ with $\phi^* j \in \mathcal{T}$.* □

### 4.2.2. Adapted Teichmüller slices

When we consider holomorphic curves in the parametrised pre-stratum $\widetilde{\mathcal{M}}^A(J)$, all complex structures on the domain have automorphism group conjugate to $A$. Classical Teichmüller slices do not capture this feature. Instead, we should take a Teichmüller slice consisting only of complex structures with that automorphism group.

**Definition 4.24.** *Let $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$ be a closed subgroup and $j \in \mathcal{J}(\Sigma)$ such that* $\mathrm{Aut}(\Sigma, j, \theta) = A$*. An $A$-*adapted Teichmüller slice *through $j$ is an injective smooth map $\mathcal{O} \to \mathcal{J}(\Sigma), \tau \mapsto j_\tau$, where $\mathcal{O}$ is a neighbourhood of* $0$ *in some finite-dimensional Euclidean space, such that $j_0 = j$,* $\mathrm{Aut}(\Sigma, j_\tau, \theta) = A$ *for all $\tau$ and $D_{(j,\theta)}(W^{1,p}_{A,\theta}) \oplus T_j\mathcal{T} = L^p_A$, where*

$$L^p_A := \{\eta \in L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma)) \mid \phi^*\eta = \eta \text{ a.e. for all } \phi \in A\}$$

*is the set of $A$-invariant sections, and $W^{1,p}_{A,\theta}(T\Sigma) \subset W^{1,p}_\theta(T\Sigma)$ is the set of all $A$-invariant vector fields. The space $T_j\mathcal{T}$ is defined as in Definition 4.18. Moreover, we ask that $\mathcal{O}$ and $T_j\mathcal{T}$ have the same dimension.*

We generally identify an adapted Teichmüller slice with its image $\mathcal{T} = \{j_\tau\}_{\tau \in \mathcal{O}} \subset \mathcal{J}(\Sigma)$. For the purposes of intuition only, we may think of $\mathcal{T}$ as a finite-dimensional embedded submanifold of $\mathcal{J}(\Sigma)$. Note that if $A = \mathrm{Aut}(\Sigma, j, \theta)$ is trivial, an $A$-adapted Teichmüller slice through $j$ is precisely an ordinary Teichmüller slice.

The complement condition merits some scrutiny: is it well-defined and independent of $p$? Because of smoothness, $T_j\mathcal{T} \subset L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ for all $p$; the complement condition is independent of the choice of $p$, for the same reasons as for ordinary Teichmüller slices. It remains to verify that the tangent space $T_j\mathcal{T}$ and the image of the linearisation $D_{(j,\theta)}$ consist of $A$-invariant sections.

**Lemma 4.25.** *If $\mathcal{T} = \{j_\tau\}_{\tau \in \mathcal{O}} \subset \mathcal{J}(\Sigma)$ is a smooth family with $A \subset \mathrm{Aut}(\Sigma, j_\tau, \theta)$ for all $\tau$, we have $T_j\mathcal{T} \subset L^p_A$.*

*Proof.* This is a simple computation; we omit the details. □

**Lemma 4.26.** *Suppose $j \in \mathcal{J}(\Sigma)$ is $A$-invariant. Then $D_{(j,\theta)}$ is $A$-equivariant; in particular, $D_{(j,\theta)}(W^{1,p}_{A,\theta}(T\Sigma)) \subset L^p_A$ is $A$-invariant.*

*Proof.* It suffices to show the $A$-equivariance. This is also a simple computation, whose details we omit for now. □

$A$-adapted Teichmüller slices exist in abundance. Intuitively, their construction is very easy: choose any $A$-invariant Teichmüller slice and consider the fixed point set of $A$. For good Teichmüller slices, this can be made rigorous.

**Proposition 4.27.** *For any $j \in \mathcal{J}(\Sigma)$ with $\mathrm{Aut}(\Sigma, j, \theta) = A$, there exists an $A$-adapted Teichmüller slice through $j$.*

*Proof.* Let $j \in \mathcal{J}(\Sigma)$ with $\mathrm{Aut}(\Sigma, j, \theta) = A$ be arbitrary. Using Lemma 4.20, choose a good $A$-invariant Teichmüller slice $\mathcal{T}$ through $j$. We show that the fixed point set $\mathrm{Fix}(A) \subset \mathcal{T}$ is an $A$-adapted Teichmüller slice through $j$. By definition, $C := T_j\mathcal{T}$ is a complement of $\mathrm{im}(D_{(j,\theta)})$ in $L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$, consisting of smooth sections (as $\mathcal{T}$ consists of smooth complex structures). As $\mathcal{T}$ is good, it is parametrised by a neighbourhood $\mathcal{O} \subset C$ of 0. Observe that $C^A := \{y \in C \mid y \text{ is } A\text{-invariant}\}$ is an $A$-invariant complement of $D_{(j,\theta)}(W^{1,p}_A)$ in $L^p_A$.
Now, restricting the exponential map (4.2) to $\mathcal{O}^A := \mathcal{O} \cap C^A$ defines an $A$-adapted Teichmüller slice through $j$. By construction, each $j_\tau$ for $\tau \in \mathcal{O}^A$ is $A$-invariant. Shrinking $\mathcal{O}$ further if necessary, we can assume $\mathrm{Aut}(\Sigma, j_\tau, \theta)$ is conjugate to a subgroup of $A$ for all $j_\tau$ (by Corollary 4.29 below), yielding $\mathrm{Aut}(\Sigma, j_\tau, \theta) = A$. This completes the proof. □

In the above proof, we have used the following result.

**Lemma 4.28.** *Every $j_0 \in \mathcal{J}(\Sigma)$ has a neighbourhood $U \subset \mathcal{J}(\Sigma)$ such that for all $j \in U$, the automorphism group $\mathrm{Aut}(\Sigma, j, \theta)$ is conjugate to a subgroup of $\mathrm{Aut}(\Sigma, j_0, \theta)$.*

Lemma 4.28 follows from a more general fact: we use that $\mathcal{J}(\Sigma)$ projects into the moduli space $\mathcal{M}_{g,m}$ of pointed Riemann surface, which is a global quotient orbifold.

*Proof of Lemma 4.28.* Recall (e.g. [Wen15, p. 155, Theorem 4.2.10]) that the moduli space $\mathcal{M}_{g,m}$ is homeomorphic to the quotient orbifold $\mathcal{M}(\Sigma, \theta) = \mathcal{J}(\Sigma)/\mathrm{Diff}_+(\Sigma, \theta)$, with isotropy group $G_{[j]} = \mathrm{Aut}(\Sigma, j, \theta)$.
Consider the canonical projection $\pi\colon \mathcal{J}(\Sigma) \to \mathcal{M}_{g,m}$. By Lemma 3.16, $[j_0]$ has a neighbourhood $\mathcal{U}$ in $\mathcal{M}_{g,m}$ such that $G_{[j]}$ is conjugate to a subgroup of $G_{[j_0]}$ for all $[j] \in \mathcal{U}$. Since $G_{[j]} = \mathrm{Aut}(\Sigma, j, \theta)$, by continuity the neighbourhood $U = \pi^{-1}(\mathcal{U})$ of $j$ has the desired property. □

**Corollary 4.29.** *Let $P$ be a manifold and $\gamma\colon P \to \mathcal{J}(\Sigma)$ continuous. Each $p_0 \in P$ has a neighbourhood $\mathcal{O}$ such that for all $p \in \mathcal{O}$ the automorphism group $\mathrm{Aut}(\Sigma, j_p, \theta)$ is conjugate to a subgroup of $\mathrm{Aut}(\Sigma, j_{p_0}, \theta)$.* □

We conclude this subsection with a formula for the dimension of $A$-adapted Teichmüller slices — which is one ingredient for computing the dimension of the isosymmetric strata.

Observe that the kernel and cokernel of $D_{(j,\theta)}\colon W^{1,p}_\theta(T\Sigma) \to L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ are $A$-invariant. Since $D_{(j,\theta)}$ is a Fredholm operator, its kernel and cokernel define finite-dimensional $A$-representations.

**Lemma 4.30.** *The dimension of an $A$-adapted Teichmüller slice through $j \in \mathcal{J}(\Sigma)^A$ is $\dim T_j\mathcal{T} = m_1^A(\operatorname{coker} D_{(j,\theta)})$, the multiplicity of the trivial $A$-representation in $\operatorname{coker} D_{(j,\theta)}$.*

*Proof.* Since $D_{(j,\theta)}$ is $A$-equivariant, it maps $W^{1,p}_{A,\theta}(T\Sigma)$ to $L^A_p$, thus restricts (and corestricts) to a bounded linear operator $D^A := D^A_{(j,\theta)}\colon W^{1,p}_{A,\theta}(T\Sigma) \to L^p_A$. An easy computation[9] shows $\ker D^A = \ker D_{(j,\theta)} \cap W^{1,p}_{A,\theta}$ and $\operatorname{coker} D^A = \operatorname{coker} D_{(j,\theta)} \cap L^p_A$. In particular, $D^A$ is also Fredholm. By definition, $\operatorname{im} D^A \oplus T_j\mathcal{T} = L^p_A$, so

$$\dim T_j\mathcal{T} = \dim \operatorname{coker} D^A = \dim(\operatorname{coker} D_{(j,\theta)} \cap L^p_A).$$

The right hand side consists precisely of all $A$-invariant elements in $\operatorname{coker} D_{(j,\theta)}$, i.e. is the fixed subspace under the $A$-action. But the dimension of that fixed subspace is precisely the multiplicity of the trivial $A$-representation on $\operatorname{coker} D_{(j,\theta)}$. □

### 4.2.3. Adapted Teichmüller slices and Teichmüller space

At the end of this section, we want to prove that adapted Teichmüller slices capture the variation of the complex structures on $\Sigma$ within $\widetilde{\mathcal{M}}^A(J)$. The precise statement is the following.

**Proposition 4.31.** *Let $(\Sigma, j_0, \theta) \in \widetilde{\mathcal{M}}^A_{g,m}$; let $\mathcal{T}$ be an $A$-adapted Teichmüller slice through $j_0$. Any holomorphic curve $(j_0, u_0) \in \widetilde{\mathcal{M}}^A(J)$ has a neighbourhood $U \subset \widetilde{\mathcal{M}}(J)$ such that each $(\Sigma, j, \theta, u) \in U \cap \widetilde{\mathcal{M}}^A(J)$ has a reparametrisation $(\Sigma, \phi^*j, \theta_0, u \circ \phi)$ with $\phi^*j \in \mathcal{T}$.*

The astute reader will notice the similarities to Proposition 4.23, and may wonder if there is a more direct proof, using the construction of adapted Teichmüller slices. Unfortunately, the situation is not as simple. If $j \in \mathcal{J}(\Sigma)$ satisfies $\mathrm{Aut}(\Sigma, j, \theta) = A$ and $\mathcal{T}$ is a good Teichmüller slice through $j$, consider the corresponding $A$-adapted Teichmüller slice $\mathcal{T}_A = \mathrm{Fix}(A) \subset \mathcal{T}$. For $j' \in \widetilde{\mathcal{M}}^A_{g,m}$ sufficiently close to $j$, we need to show that $\phi^*j' \in \mathcal{T}^A$ for some $\phi \in \mathrm{Diff}_+(\Sigma, \theta)$. By definition of $\widetilde{\mathcal{M}}^A_{g,m}$, we know

[9] Those reader who wish to see more detail may consult the proof of Lemma 4.82, which explains this step more thoroughly.

$\mathrm{Aut}(\Sigma, \phi^* j', \theta) = A$ for some $\phi \in \mathrm{Diff}_+(\Sigma, \theta)$. By Proposition 4.23, we also know $\tilde{\phi}^* j' \in \mathcal{T}$ for some $\phi \in \mathrm{Diff}_+(\Sigma, \theta)$. If $\tilde{\phi} = \phi$, we could deduce $\phi^* j' \in \mathrm{Fix}(A) \cap \mathcal{T} = \mathcal{T}_a$ — but in general, there is no reason this should hold.

Instead, we repeat the classical proof mutatis mutandis. The first step is to observe that adapted Teichmüller slices provide charts for corresponding strata of Teichmüller space, and the projection of the adapted slice to the corresponding stratum is a local diffeomorphism. To begin, observe that Teichmüller space has a stratification corresponding to adapted Teichmüller strata.

**Definition 4.32.** *For a closed subgroup $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$, the corresponding* Teichmüller space stratum *is the orbit type $\mathcal{T}(\Sigma, \theta)^A$ of $\mathcal{T}(\Sigma)^\theta$ under the $\mathrm{Diff}_0(\Sigma, \theta)$-action. More explicitly,*

$$\mathcal{T}(\Sigma, \theta)^A := \{[j] \in \mathcal{T}(\Sigma, \theta) \mid \mathrm{Aut}(\Sigma, j, \theta) \sim A\} \subset \mathcal{T}(\Sigma, \theta),$$

*where $A \sim A'$ denotes $A$ and $A'$ being conjugate by an element of $\mathrm{Diff}_0(\Sigma, \theta)$.*

These strata are well-defined.

**Lemma 4.33.** *Let $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$ and $j \in \mathcal{J}(\Sigma)$ be arbitrary. We have $[j] \in \mathcal{T}(\Sigma, \theta)^A$ if and only if $\mathrm{Aut}(\Sigma, \phi^* j, \theta) = A$ for some $\phi \in \mathrm{Diff}_0(\Sigma, \theta)$.*

*Proof.* This is an easy computation, using Lemma 3.24 from the previous chapter. □

Consider the quotient projection $\pi_\theta \colon \mathcal{J}(\Sigma) \to \mathcal{T}(\Sigma, \theta), j \mapsto [j]$. We can equivalently describe the Teichmüller stratum via this projection map: for each closed subgroup $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$, consider the corresponding orbit type $\mathcal{J}(\Sigma)^A \subset \mathcal{J}(\Sigma)$ of complex structures (w.r.t. the $\mathrm{Diff}_0(\Sigma, \theta)$-action by $\phi \cdot j := \phi^* j$). In particular, $\mathcal{J}(\Sigma)^A$ is $\mathrm{Diff}_0(\Sigma, \theta)$-invariant.

**Corollary 4.34.** *The Teichmüller stratum $\mathcal{T}(\Sigma, \theta)^A$ is the image of $\mathcal{J}(\Sigma)^A$ under the quotient projection $\pi_\theta$.*

*Proof.* The inclusion "$\subseteq$" is obvious: $j \in \mathcal{J}(\Sigma)^A$ implies $[j] \in \mathcal{T}(\Sigma, \theta)^A$.
"$\supseteq$" If $[j] \in \mathcal{T}(\Sigma, \theta)^A$, Lemma 4.33 implies $\mathrm{Aut}(\Sigma, \phi^* j, \theta) = A$ for some $\phi \in \mathrm{Diff}_0(\Sigma, \theta)$. Thus, $\phi^* j \in \mathcal{J}(\Sigma)^A$ by definition. Since $[j] = [\phi^* j]$ we deduce $[j] \in \mathrm{im}(\pi_\theta)$. □

Recall that the transversality property of Teichmüller slices has an important consequence for the quotient projection: if $\mathcal{T}$ is any Teichmüller slice, the restriction $\pi_\theta|_{\mathcal{T}} \colon \mathcal{T} \to \mathcal{T}(\Sigma, \theta)$ of the quotient projection is a local diffeomorphism. Adapted Teichmüller slices enjoy an analogous property, which features Teichmüller strata.

**Proposition 4.35.** *Let $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$ be arbitrary. Then $\mathcal{T}(\Sigma, \theta)^A$ is a submanifold of $\mathcal{T}(\Sigma, \theta)$. Moreover, for any $j \in \mathcal{J}(\Sigma)^A$ and any $A$-adapted Teichmüller slice $\mathcal{T}$ through $j$, the restricted quotient projection $\pi_\theta|_{\mathcal{T}} \colon \mathcal{T} \to \mathcal{T}(\Sigma, \theta)^A$ is a local diffeomorphism near $j$.*

*Proof of Proposition 4.35.* Observe that $\pi_\theta$ is well-defined: by definition, $\mathcal{T} \subset \mathcal{J}(\Sigma)^A$, hence Corollary 4.34 implies $\operatorname{im}(\pi_\theta) \subset \mathcal{T}(\Sigma, \theta)^A$. To prove that $\pi_\theta$ is a local diffeomorphism near $j$, we consider the cases of spheres with few marked points, an unmarked torus and stable surfaces separately.

For spheres with $m \leq 3$ marked points, the uniformisation theorem implies $\mathcal{T}(\mathbb{S}^2, \theta) = \{\mathrm{pt}\}$ and $\mathcal{M}_{g,m} = \{\mathrm{pt}\}$. A Teichmüller slice is just a point: $\mathcal{J}(\Sigma) = \mathrm{Diff}_+(\Sigma, \theta)$ means $D_{(j,\theta)} = L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$, so $\mathcal{T} = \{i\}$ has the desired properties. This slice is $A$-adapted as *every* $j$ has the same automorphism group. In this case, $\pi_\theta$ is a map of singleton sets, hence vacuously a diffeomorphism.

In the stable case, the classical proof applies mutatis mutandis: intuitively, the tangent space to a $\mathrm{Diff}_0(\Sigma, \theta)$-orbit is the image $\operatorname{im} D_{(j,\theta)}$; hence the transversality condition for the tangent space $T_j\mathcal{T}$ implies the local diffeomorphism property. Making this intuition rigorous works exactly as in the classical case, restricting to $L^p_A$ and $\mathcal{J}(\Sigma)^A$. More precisely, the argument is the following. We sketch the adapted details; all omitted parts are exactly as in the classical case (e.g. [Wen15, pp. 162+163]).

- For $k \in \mathbb{N}_{\geq 1}$ and $p > 2$, we consider the spaces $\mathcal{J}^{k,p}(\Sigma)$ of $W^{k,p}$-smooth complex structures on $\Sigma$.[10] For $k \geq 2$, consider the space $\mathcal{D}^{k,p}_\theta \subset W^{k,p}_\theta(\Sigma, \Sigma)$ of maps $\phi \in W^{k,p}(\Sigma, \Sigma)$ which are $C^1$-smooth diffeomorphisms and fix $\theta$.

- For any closed subgroup $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$, consider the subspace $\mathcal{J}^{k,p}_A(\Sigma) := \{j \in \mathcal{J}^{k,p}(\Sigma) : \mathrm{Aut}(\Sigma, j, \theta) \sim A\}$, where $\sim$ denotes conjugation by a $C^1$-smooth diffeomorphism.

- Choose $j_0 \in \mathcal{J}(\Sigma)$ with $\mathrm{Aut}(\Sigma, j, \theta) = A$ and an $A$-adapted Teichmüller slice $\mathcal{T} \subset \mathcal{J}(\Sigma)$ through $j_0$. Then the tangent space $T_{j_0}\mathcal{T} \subset \Gamma^A(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ is complementary to the image of $D_{(j_0,\theta)} \colon W^{k,p}_\theta(T\Sigma) \to W^{k-1,p}_A(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ for each $k$.

- Since every $j \in \mathcal{T}$ is smooth, each $j$-orbit under the $\mathcal{D}^{k+1,p}_\theta$-action is in $\mathcal{J}^{k,p}_A(\Sigma)$.

- Hence, $F \colon \mathcal{D}^{k+1,p}_\theta \times \mathcal{T} \to \mathcal{J}^{k,p}_A(\Sigma), (\phi, j) \mapsto \phi^* j$ is a well-defined map; $F$ is smooth with derivative
$$dF(\mathrm{id}, j_0) \colon W^{k+1,p}_\theta(T\Sigma) \oplus T_{j_0}\mathcal{T} \to W^{k,p}_A(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma)), (X, y) \mapsto j_0 D_{(j_0,\theta)} X + y.$$

- $dF$ is an isomorphism: it is linear by definition, surjective by the transversality condition, and injective by the transversality condition and injectivity of $D_{(j_0,\theta)}$ (as we are in the stable case).

- Hence, by the implicit function theorem, $F$ is a smooth diffeomorphism between neighbourhoods of $(\mathrm{id}, j_0) \in \mathcal{D}^{k+1,p}_\theta \times \mathcal{T}$ and $j_0 \in \mathcal{J}^{k,p}_A(\Sigma)$.

[10] $W^{k,p}$ is a *Sobolev space*, of bundle sections with $k$ weak derivatives in $L^p$. We refer the reader to e.g. [Wen15, §3.1] for a brief treatment sufficient for our purposes, and to e.g. [AF03] for a more detailed discussion from an analyst's point of view.

- $\pi_\theta$ is surjective near $j_0$: every $j \in \mathcal{J}(\Sigma)^A$ in some neighbourhood of $j_0$ lies in $\mathrm{im}(F)$.
- injectivity of $\pi_\theta$ proceeds as in the classical case: we use properness of the $\mathcal{J}(\Sigma)$-action
- smoothness of transition maps also proceeds as in the classical case

The only remaining case is $(g, m) = (1, 0)$, a torus without marked points. This case will follow from Lemmas 4.36 and 4.38 below. □

**Lemma 4.36.** *Let $(\Sigma, \theta)$ be a pointed Riemann surface, $A$ a compact Lie group and $j_0 \in \mathcal{J}(\Sigma)^A$. Suppose $\mathcal{T}_{std}$ is an $A$-adapted Teichmüller slice through $j_0$ such that the quotient projection $\phi\colon \mathcal{T}_{std} \to \mathcal{T}^A(\Sigma, \theta)$ is a local diffeomorphism near $j_0$; let $\mathcal{T}$ be any $A$-adapted Teichmüller slice through $j_0$. Then the quotient projection $\mathcal{T} \to \mathcal{T}^A(\Sigma, \theta)$ is also a local diffeomorphism near $j_0$.*

*Proof.* It suffices to show that the composition $\Phi\colon \mathcal{T} \to \mathcal{T}(\Sigma, \theta) \overset{\phi^{-1}}{\to} \mathcal{T}_{\text{std}}$ is a local diffeomorphism near $j_0$. The map $\Phi$ is the composition of two smooth maps, hence smooth. Note that $\Phi(j_0) = j_0$.

**Claim 1.** *The linearisation $D\Phi(j_0)\colon T_{j_0}\mathcal{T} \to T_{j_0}\mathcal{T}_{std}$ is an isomorphism.*

*Proof.* By definition, we have $T_{j_0}\mathcal{T} \oplus \mathrm{im}\, D_{(j,\theta)} = L^p_A$ and $T_{j_0}\mathcal{T}_{\text{std}} \oplus \mathrm{im}\, D_{(j,\theta)} = L^p_A$, hence the quotient map $j \to [j]$ provides isomorphisms $T_{j_0}\mathcal{T} \to \mathrm{coker}\, D_{j,\theta}$ and $T_{j_0}\mathcal{T}_{\text{std}} \to \mathrm{coker}\, D_{(j,\theta)}$. We show that $\mathrm{coker}\, D_{(j,\theta)} \cong T_{j_0}\mathcal{T} \overset{D\Phi(j_0)}{\to} T_{j_0}\mathcal{T}_{\text{std}} \cong \mathrm{coker}\, D_{(j,\theta)}$ is the identity.

Let $y = \partial_t j_{\tau(t)}|_{t=0} \in T_{j_0}\mathcal{T}$ be arbitrary. For each $t$, we have $\Phi(j_{\tau(t)}) = j_{\tilde\tau(t)}$, where $\in \mathcal{T}_{\text{std}}$ such that $[j_{\tau(t)}] = [j_{\tilde\tau(t)}]$ in $\mathcal{T}(\Sigma, \theta)$. By definition of $\mathcal{T}(\Sigma, \theta)$, that means $j_{\tilde\tau(t)} = \phi_t^* j_{\tau(t)}$ for suitable $\phi_t \in \mathrm{Diff}_0(\Sigma, \theta)$ with $\phi_0 = \mathrm{id}$. Plugging in, we obtain

$$D\Phi(j_0)y = \partial_t \Phi(j_{\tau(t)})|_{t=0} = \partial_t j_{\tilde\tau(t)}|_{t=0} = \partial_t \phi_t^* j_{\tau(t)}|_{t=0}.$$

We need to show $[y] = [D\Phi(j_0)y]$ in $\mathrm{coker}\, D_{(j,\theta)}$, i.e. $D\Phi(j_0)y - y \in \mathrm{im}\, D_{(j,\theta)}$. Indeed, consider the map $F\colon \mathcal{D}^{k+1,p}_\theta \times \mathcal{T} \to \mathcal{J}^{k,p}_A(\Sigma), (\phi, j) \mapsto \phi^* j$ from the stable case (see there for details).[11] $F$ is smooth with derivative

$$dF(\mathrm{id}, j_0)\colon W^{k+1,p}_\theta(T\Sigma) \oplus T_{j_0}\mathcal{T} \to W^{k,p}_A(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma)), (X, y) \mapsto j_0 D_{(j_0,\theta)}X + y.$$

In particular, we have $D\Phi(j_0)y = \partial_t F(\phi_t, j_{\tau(t)})|_{t=0} = dF(\mathrm{id}, j_0)(X, y) = j_0 D_{(j_0,\theta)}X + y$, where we denote $X := \partial_t \phi_t|_{t=0}$. As $D_{(j_0,\theta)}$ is complex-linear, its image is invariant under multiplication with $j_0$ and we have $D\Phi(j_0)y - y = j_0 D_{(j_0,\theta)}X \in \mathrm{im}\, D_{(j_0,\theta)}$ as desired. △

By the implicit function theorem, $\Phi$ is a local diffeomorphism near $j_0$. □

[11] Note that defining or differentiating $F$ does not use stability.

**Remark 4.37.** The same result holds for classical Teichmüller slices; the exact same proof works mutatis mutandis.

**Lemma 4.38.** *For all* $A \leqslant \mathrm{Aut}(\mathbb{T}^2, j)$ *and all* $j \in \mathcal{J}(\mathbb{T}^2)^A$*, there exists an* $A$*-adapted Teichmüller slice* $\mathcal{T}$ *through* $j$ *such that the quotient projection* $\mathcal{T} \to \mathcal{T}(\Sigma, \theta)^A$ *is a local diffeomorphism near* $j$*.*

On an unmarked torus, the operator $D_{(j,\theta)}$ is no longer injective, nor can we rely on the uniformisation theorem. However, we can compute the operator explicitly, and construct an adapted Teichmüller slice by hand. To that end, let us recall some facts about complex structures on the unmarked torus.

**Recall** ([Wen15, §4.2.3])**.** The Teichmüller space on an unmarked torus is $\mathcal{T}(\mathbb{T}^2, ) = \{[j_\lambda] \colon \lambda \in \mathbb{H}\} \cong \mathbb{H}$, where $j_\lambda$ is (the projection of) the unique translation-invariant complex structure on $\mathbb{C}$ which sends $1 \to \lambda$ (and $\lambda$ to $-1$).
Consider the natural Cauchy–Riemann operator $D_j \colon W^{1,p}(T\mathbb{T}^2) \to L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\mathbb{T}^2))$ on $(T\mathbb{T}^2, j)$. The natural identification of $(T\mathbb{T}^2, j)$ with $\mathbb{T}^2 \times \mathbb{C}$ yields a complex trivialisation of $(T\mathbb{T}^2, j)$; under this identification, $D_j$ corresponds to $\overline{\partial} = \partial_s + i\partial_t \colon W^{1,p}(\mathbb{T}^2, \mathbb{C}) \to L^p(\mathbb{T}^2, \mathbb{C})$. Its formal adjoint is $\partial = \partial_s - i\partial_t \colon W^{1,p}(\mathbb{T}^2, \mathbb{C}) \to L^p(\mathbb{T}^2, \mathbb{C})$.

In our setting, constructing an adapted Teichmüller slice is simplified greatly as there are only two possibilities for the co-kernel. As above, we identify $\mathrm{coker}\, D_j$ with a subspace of $L^p(T^2, \mathbb{C})$.

**Lemma 4.39.** *Either* $\mathrm{coker}\, D_j$ *is trivial, or* $\mathrm{coker}\, D_j \cong \mathbb{C}$*.*
*In both cases,* $\mathrm{coker}\, D_j \cong \{f \in L^p_A(\mathbb{T}^2, \mathbb{C}) \colon f = const.\}$*, where* $A$ *acts on* $L^p(\mathbb{T}^2, \mathbb{C})$ *by* $\phi \cdot f = \xi(\phi)^{-1} f(\phi)\xi$*, where* $\xi \colon \mathbb{T}^2 \to \mathbb{C}^*$ *is the principal part of* $d\phi$*.*

*Proof.* Because $j$ is $A$-invariant, the operator $D_j$ maps to $A$-invariant sections, i.e. into $L^p_A(\overline{\mathrm{End}}_{\mathbb{C}}(T\mathbb{T}^2, j))$. Under the identification above, $D_j$ corresponds to the map $\overline{\partial} \colon W^{1,p}(\mathbb{T}^2, \mathbb{C}) \to L^p_A(\mathbb{T}^2, \mathbb{C})$. Here $A$ acts on $L^p(\mathbb{T}^2, \mathbb{C})$ by $\phi \cdot f = \xi(\phi)^{-1} f(\phi)\xi$, where $\xi \colon \mathbb{T}^2 \to \mathbb{C}^*$ is the principal part of $d\phi$. As $\phi \in A$ is holomorphic by definition, its differential $d\phi$ is $\mathbb{C}$-linear. As each tangent space $T_z\mathbb{T}^2$ is complex 1-dimensional, the above is just multiplication of $\mathbb{C}$-valued functions.[12]

The operator $\overline{\partial}$ still has formal adjoint $\partial$ (we simply need to check fewer conditions); the formal adjoint is now defined on $W^{1,p}_A(\mathbb{T}^2, \mathbb{C})$. This identification and the standard fact $\mathrm{coker}\, D_j \cong \ker D_j^*$ imply $\mathrm{coker}\, D_j \cong \ker \partial$. Observe that $\partial f = 0$ if and only if $f$ is constant, thus $\mathrm{coker}\, D_j$ corresponds to the space of $A$-invariant constant functions.

Recall that $f \in L^p(\mathbb{T}^2, \mathbb{C})$ is $A$-invariant if and only if for all $\phi \in A$ (with $\xi \colon \mathbb{T}^2 \to \mathbb{C}^*$ being the principal part of $d\phi$), we have $\xi(\phi)^{-1} f(\phi)\xi = f$. If $f$ is constant, this is independent of the value of $f$: the zero function $f \equiv 0$ is always $A$-invariant; for

[12]This is a subtle detail: if $\phi$ were any smooth map, the linearisation $d\phi$ were only real linear: hence, $A$-invariance of $f$ would be an equation of the form $AvB = v$, for real $2 \times 2$-matrices $A$ and $B$ and $v$ being the components of $f$ in a real basis.

$f \equiv c \in \mathbb{C}^*$ we have $\xi(\phi)^{-1} f(\phi)\xi = f \Leftrightarrow \xi(\phi)^{-1}\xi c = c \Leftrightarrow \xi(\phi)^{-1}\xi = 1$. Thus, we either have $\ker \partial = \{f \equiv 0\}$ or $\ker \partial = \{f \in L^p(\mathbb{T}^2, \mathbb{C}) : f = \text{const.}\} \cong \mathbb{C}$. □

To summarise: a priori, the subspace of $A$-invariant constant functions could also have real dimension one. In our case, this cannot happen: complex linearity implies the real dimension is even, hence zero or two.

*Proof of Lemma 4.38.* Let $A \leqslant \mathrm{Aut}(\mathbb{T}^2, j)$ and $j \in \mathcal{J}(\mathbb{T}^2)^A$ be arbitrary. By Lemma 4.39, it suffices to consider two cases. In the first case, $\operatorname{coker} D_j$ is trivial. This implies the trivial slice $\{j\}$ is $A$-adapted. It is parametrised by the 0-dimensional Euclidean space $\{0\}$, its tangent space is trivial and $\operatorname{im} D_j = L^p_A$. The quotient projection is a map of singleton sets, hence vacuously a diffeomorphism.

Assume $\operatorname{coker} D_j$ is non-trivial, then Lemma 4.39 shows $\operatorname{coker} D_j \cong \mathbb{C}$. This implies $L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\mathbb{T}^2, j)) = L^p_A$, i.e., every section in $L^p(\overline{\mathrm{End}}_{\mathbb{C}}(\mathbb{T}^2, j))$ is in fact $A$-invariant: we show that $D_j \colon W^{1,p}(\overline{\mathrm{End}}_{\mathbb{C}}(\mathbb{T}^2, j)) \to L^p(\overline{\mathrm{End}}_{\mathbb{C}}(\mathbb{T}^2, j))$ and its co-restriction to $L^p_A$ have the same co-kernel. Write $C := \operatorname{coker} D_j \subset L^p(\overline{\mathrm{End}}_{\mathbb{C}}(\mathbb{T}^2, j))$ and $C' := \operatorname{coker} D_j \subset L^p_A$. We always have $C' \subset C$. By hypothesis, we also have $C' \cong \mathbb{C}$; since always $C \cong \mathbb{C}$, we conclude that $C = C'$.

Choose an $A$-adapted Teichmüller slice $\mathcal{T}$ through $j$ (using Proposition 4.27). Then, $\operatorname{im} D_j \oplus T_j\mathcal{T} = L^p_A = L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\mathbb{T}^2, j))$, so $\mathcal{T}$ is in fact a Teichmüller slice. Thus, by Lemma 4.22, the projection $\mathcal{T} \to \mathcal{T}(\mathbb{T}^2)$ is a local diffeomorphism near $j$. Since $\mathcal{T}$ is $A$-adapted, we obtain $\operatorname{im} \mathcal{T} \subset \mathcal{T}(\mathbb{T}^2)^A$ and the claim follows. □

### 4.2.4. Local variation property

Teichmüller slices capture the variation of the complex structure $j$ as $u$ varies within the moduli space $\mathcal{M}(J)$. As announced, $A$-adapted Teichmüller slices enjoy an analogous variation property within the pre-strata $\mathcal{M}^A(J)$.

**Proposition 4.40.** *Let $(\Sigma, j_0, \theta) \in \widetilde{\mathcal{M}}^A_{g,m}$; let $\mathcal{T}$ be an $A$-adapted Teichmüller slice through $j_0$. Any holomorphic curve $(j_0, u_0) \in \widetilde{\mathcal{M}}^A(J)$ has a neighbourhood $U \subset \widetilde{\mathcal{M}}(J)$ such that each $(\Sigma, j, \theta, u) \in U \cap \widetilde{\mathcal{M}}^A(J)$ has a reparametrisation $(\Sigma, \phi^* j, \theta_0, u \circ \phi)$ with $\phi^* j \in \mathcal{T}$.*

The proof is essentially the same as in the classical case, with one small exception. In the classical case, the proof requires constructing an $\mathrm{Aut}(\Sigma, j, \theta)$-invariant Teichmüller slice. For adapted Teichmüller slices, this invariance is automatic.

**Lemma 4.41.** *Suppose $j \in \mathcal{J}(\Sigma)$ with $\mathrm{Aut}(\Sigma, j, \theta) = A$. Then any $A$-adapted Teichmüller slice $\mathcal{T}$ through $j$ is $A$-invariant, both as a set and point-wise.*

*Proof.* Let $\mathcal{T}$ be any $A$-adapted Teichmüller slice through $j$; let $j_\tau \in \mathcal{T}$ be arbitrary. Then $j_\tau$ is $A$-invariant by definition of $\mathcal{T}$. In particular, $\mathcal{T}$ is $A$-invariant as a set. □

We need one final observation for the proof.

**Lemma 4.42.** *If $g \geq 1$ or $2g + 3 \geq 3$, the action of $\mathrm{Diff}_0(\Sigma, \theta_0)$ on $\mathcal{J}(\Sigma)$ is free.*

*Proof sketch.* If $(\Sigma, \theta)$ is stable, this is a short argument using the Lefschetz fixed point theorem (e.g. [Wen15, Lemma 4.2.5]). For the unmarked torus, the argument is contained in e.g. [Wen15, Proposition 4.2.17]: if $\phi^* j = j$, without loss of generality we may assume $j = j_\lambda$ for some $\lambda \in \mathbb{H}$. Then $\phi$ is a biholomorphic map of $(\mathbb{T}^2, j_\lambda)$. Lifting to a translation-invariant automorphism of $\mathbb{C}$ and composing with a suitable translation, we obtain a holomorphic automorphism which fixes 0, 1 and $\lambda$, hence is the identity. □

*Proof of Proposition 4.40.* If $g = 0$ and $m \leq 2$, the uniformisation theorem shows that $(\Sigma, j)$ can be reparametrised as $(\mathbb{S}^2, i)$, hence so can $(\Sigma, j_0)$. Hence, we may take $U = \mathcal{M}(J)$ in this case.

Assume $g \geq 1$ or $m \geq 3$. Then the action of $\mathrm{Diff}_0(\Sigma, \theta_0)$ on $\mathcal{J}(\Sigma)$ is proper [Wen15, Lemma 4.2.8]. As $\mathcal{J}(\Sigma)^A$ is $\mathrm{Diff}_0(\Sigma, \theta_0)$-invariant, the action restricts to a proper action on $\mathcal{J}(\Sigma)^A$. It suffices to consider a sequence of curves: if the statement is false, there exists a sequence of curves $(j'_k, v_k)$ converging to $(j_0, u_0)$ without any such reparametrisation; running the argument of the proof will yield a contradiction.

Suppose $(\Sigma, j_k, \theta, u_k) \in \widetilde{\mathcal{M}}^A(J)$ is a sequence with $j_k \to j_0$ and $u_k \to u_0$. Then, each $j_k$ lies in the stratum $\mathcal{J}(\Sigma)^A$, hence (by Corollary 4.34) each $[j_k]$ lies in the Teichmüller stratum $\mathcal{T}(\Sigma, \theta)^A$ and $[j_k] \to [j]$ in $\mathcal{T}(\Sigma, \theta)^A$. Using Lemma 3.21, choose diffeomorphisms $\phi_k \in \mathrm{Diff}_0(\Sigma, \theta)$ such that $\phi_k^* j_k$ is a sequence in $\mathcal{T}$ approaching $j_0$.

By properness of the $\mathrm{Diff}_0(\Sigma, \theta_0)$-action on $\mathcal{J}(\Sigma)^A$, the sequence $(\phi_k)$ has a subsequence converging to an element $\phi \in \mathrm{Aut}(\Sigma, j_0, \theta_0) \cap \mathrm{Diff}_0(\Sigma, \theta_0)$. Since the action of $\mathrm{Diff}_0(\Sigma, \theta_0)$ on $\mathcal{J}(\Sigma)$ is free (Lemma 4.42), we have $\phi = \mathrm{id}$, hence $\phi_k \to \mathrm{id}$ and $u_k \circ \phi_k \to u_0$. For large $k$, the curve $(\phi^* j_k, u_k \circ \phi_k)$ lies in an arbitrarily small neighbourhood of $(j_0, u_0)$ in $\bar{\partial}_J^{-1}(0)$. Since the projection $\mathcal{T} \to \mathcal{T}(\Sigma, \theta)^A$ is a local diffeomorphism at $j_0$ (by Proposition 4.35), we deduce $j_k \in \mathcal{T}$. Since $\mathcal{T}$ is $A$-invariant (by Lemma 4.41), we conclude $\phi^* j_k \in \mathcal{T}$. □

## 4.3. Local models for $\mathcal{M}^A(J)$ and $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$

Using adapted Teichmüller slices, we can write down local models of the iso-symmetric strata. We begin with a local model for $\mathcal{M}^A(J)$ and refine these to describe the sets $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$. To some extent, these local models will be parametrised.

**Lemma 4.43.** *Let $(\Sigma, j_0, \theta, u_0) \in \widetilde{\mathcal{M}}^A(J)$ be arbitrary; let $\mathcal{T}$ be an A-adapted Teichmüller slice through $j_0$. Then $[u_0]$ has a neighbourhood $U \subset \mathcal{M}(J)$ such that $U \cap \mathcal{M}^A(J)$ is an open neighbourhood $V$ of $u_0$ in*

$$S := \{[(\Sigma, j, \theta, u)] \mid j \in \mathcal{T}, u\colon (\Sigma, j) \to M \text{ is J-holomorphic}, [u] = C\}.$$

*(Recall that $\Sigma$ and $\theta$ are fixed bookkeeping choices, so only $j$ and $u$ are allowed to vary.)*

*Proof.* The inclusion $\supseteq$ is straightforward since $S \subset \mathcal{M}^A(J)$. Conversely, choose $U$ as in Proposition 4.40: then any curve $u \in U \cap \mathcal{M}^A(J)$ has a reparametrisation $(\Sigma, j', \theta, u')$ with $j' \in \mathcal{T}$; that means $[(\Sigma, j', \theta, u')]$ is contained in $S$. □

Since the homology group $H_2(M;\mathbb{Z})$ is a discrete set, perhaps after shrinking $U$ further, every curve in $U$ has homology class $C$. We can further sharpen our local model and present $\mathcal{M}^A(J)$ locally as the zero set of a smooth Banach space bundle section. Fix an integer $p > 2$. Standard arguments (e.g. [Eli67]) show that $\mathcal{B} = W^{1,p}(\Sigma, M)$ is a Banach manifold with tangent space $T_u\mathcal{B} = W^{1,p}(u^*TM)$, i.e. vector fields of $M$ along $u$ of class $W^{1,p}$. (By the Sobolev embedding theorem, each tangent vector admits a continuous representative.) In particular, $\mathcal{T} \times \mathcal{B}$ is also a Banach manifold. Furthermore, the vector bundle $\mathcal{E} \to \mathcal{T} \times \mathcal{B}$ with fibre $\mathcal{E}_{(j,u)} = L^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$ is a Banach space bundle, and the bundle section

$$\overline{\partial}_J \colon \mathcal{T} \times \mathcal{B} \to \mathcal{E}, (j,u) \mapsto J \circ du \circ j + du$$

is smooth. Then, the local model can be refined to the following.

**Corollary 4.44.** *Let $(\Sigma, j_0, \theta, u_0) \in \widetilde{\mathcal{M}}^A(J)$ be arbitrary; let $\mathcal{T}$ be an $A$-adapted Teichmüller slice through $j_0$. Then $[u_0]$ has a neighbourhood $U \subset \mathcal{M}(J)$ such that every $U \cap \mathcal{M}^A(J)$ corresponds to an open neighbourhood $V$ of $(j_0, u_0)$ in the zero set $\overline{\partial}_J^{-1}(0)$: every $[u] \in U \cap \mathcal{M}^A(J)$ has a reparametrisation contained[13] in $V$.*

*Proof.* Choose $U$ as in Lemma 4.43. Shrinking $U$ if necessary, we may assume each curve in $U$ has homology class $C$. The forgetful map $[u] \mapsto (j,u)$ maps $U \cap \mathcal{M}^A(J)$ to a subset of $\overline{\partial}_J^{-1}(0)$. Conversely, for every $(j,u) \in \overline{\partial}_J^{-1}(0)$, elliptic regularity arguments show that $u \in \mathcal{B}$ and $\overline{\partial}_J(j,u) = 0$ imply $u$ is smooth. Since each curve in $U$ has homology class $C$, a suitable open subset of $(j_0, u_0)$ corresponds to $U \cap \mathcal{M}^A(J)$. □

As the next step, let us incorporate the stabilisers $(A \times G)_u$, to obtain a local model for $\mathcal{M}^{A,H}(J)$. At this stage, it is crucial to become aware again of the two group actions in our set-up.

**Observation 4.45.** The group $A \times G$ acts...

- ...on $\mathcal{B}$ via $(\phi, g) \cdot u = \psi_g \circ u \circ \phi^{-1}$,
- ...on the base $\mathcal{T} \times \mathcal{B}$ by $(\phi, g) \cdot (j,u) := (\phi^* j, (\phi, g) \cdot u) = (\phi^* j, \psi_g \circ u \circ \phi^{-1})$
- ...on $\mathcal{E}$ by $(\phi, g) \cdot \eta = d\psi_g \circ \eta \circ d\phi^{-1}$.

This action is equivariant, i.e. the actions on $\mathcal{T} \times \mathcal{B}$ and $\mathcal{E}$ commute with the bundle projection $\mathcal{E} \to \mathcal{T} \times \mathcal{B}$.

*Proof.* Well-definedness follows from standard properties of Sobolev spaces (using $p > 2$) and the fact that $\mathcal{T}$ is $A$-adapted. It is easy to check that these assignments define left actions which are equivariant. □

[13] This is some mild abuse of notation: we identify $[(\Sigma, j, \theta, u)] \in U \cap \mathcal{M}^A(J)$ with the pair $(j,u)$, which has to lie in $\overline{\partial}_J^{-1}(0)$.

Our local model for $\mathcal{M}^A(J)$ can be improved to capture $\mathcal{M}^{A,H}(J)$ by restricting $\overline{\partial}_J$ to a suitable Banach submanifold and sub-bundle. To motivate the details of this construction better, we may consider a finite-dimensional toy model: since this does not affect the results in this thesis, it can be found in Appendix A.1.

**Definition 4.46.** *For a closed subgroup $H \leqslant A \times G$, let $\mathcal{B}^H := \{u \in \mathcal{B} \mid (A \times G)_u \cong H\}$, where $\cong$ denotes conjugate subgroups of $A \times G$.*

The omniscient reader will recognise $\mathcal{B}^H$ as the orbit type[14] of $H$ w.r.t. the $A \times G$-action on $\mathcal{B}$. In particular, it is $A \times G$-invariant, and the $A \times G$-action restricts to $\mathcal{B}^H$. Since $\mathcal{T}$ is $A$-adapted, the orbit types of $H$ on $\mathcal{T} \times \mathcal{B}$ is closely related to $\mathcal{B}^H$.

**Observation 4.47.** For each closed subgroup $H \leqslant A \times G$, we have $\mathcal{T} \times \mathcal{B}^H = \{(j, u) \in \mathcal{T} \times \mathcal{B} \mid (A \times G)_{(j,u)} \cong H\}$.

*Proof.* Since $\mathcal{T}$ is an $A$-adapted Teichmüller slice, $A$ acts trivially on $\mathcal{T}$: for all $\phi \in A$, we have $\phi^* j = j$, hence $(\phi, g) \cdot (j, u) = (j, (\phi, g) \cdot u)$ and $(A \times G)_{(j,u)} = (A \times G)_u$. □

We need $\mathcal{B}^H$ to be a Banach submanifold of $\mathcal{B}$: then $\mathcal{T} \times \mathcal{B}^H$ is a Banach submanifold of the base $\mathcal{T} \times \mathcal{B}$ of $\mathcal{E}$. The next step is to argue that $\overline{\partial}_J|_{\mathcal{T} \times \mathcal{B}^H}$ maps into a suitable sub-bundle of $\mathcal{E}$. The main observation to this end is that $\overline{\partial}_J$ is $G$-equivariant and preserves stabilisers w.r.t. the $(A \times G)$-action.

**Lemma 4.48.** *The section $\overline{\partial}_J \colon \mathcal{T} \times \mathcal{B} \to \mathcal{E}$ is $G$-equivariant: for all $g \in G$ and $(j, u) \in \mathcal{T} \times \mathcal{B}$, we have $g \cdot \overline{\partial}_J(j, u) = \overline{\partial}_J(g \cdot (j, u))$. Moreover, for all $h \in (A \times G)_u$, we have $h \cdot \overline{\partial}_J(j, u) = \overline{\partial}_J(h \cdot (j, u)) = \overline{\partial}_J(j, u)$.*

*Proof.* Let $(j, u) \in \mathcal{T} \times \mathcal{B}$ be arbitrary. For all $g \in G$, we compute

$$\begin{aligned} g \cdot \overline{\partial}_J(j, u, J) &= d\psi_g \circ \overline{\partial}_J(j, u, J) \\ &= d\psi_g \circ du + d\psi_g \circ (J \circ du \circ j) \\ &= d(\psi_g \circ u) + J \circ d(\psi_g \circ u) \circ j \\ &= \overline{\partial}_J(j, g \cdot u, J) \end{aligned} \tag{4.3}$$

In equation $(1)$, we used the $G$-equivariance of $J$. Similarly, we use the equivariance of $J$ to compute, for all $(\phi, g) \in (A \times G)_u$

$$\begin{aligned} (\phi, g) \cdot \overline{\partial}_J(j, u) &= d\psi_g \circ (du + J \circ du \circ j) \circ d\phi^{-1} \\ &= d(\psi_g \circ u \circ \phi^{-1}) + J \circ d\psi_g \circ du \circ d\phi^{-1} \circ (d\phi \circ j \circ d\phi^{-1}) \\ &= d((\phi, g) \cdot u) + J \circ d((\phi, g) \cdot u) \circ \phi^{-1} j = du + J \circ du \circ j = \overline{\partial}_J(j, u) \\ &= \overline{\partial}_J(\phi^* j, (\phi, g) \cdot u) = \overline{\partial}_J((\phi, g) \cdot (j, u)). \end{aligned}$$

□

[14] We caution the reader: we do not assume this action to be differentiable! The definition of orbit types makes perfect sense without any differentiability; only the properties of the orbit type stratification require differentiability. In fact, the $A \times G$-action is *not* differentiable in general, because of the loss of derivatives. See Section 4.7 for details.

**Lemma 4.49.** *For all $(j,u) \in \mathcal{T} \times \mathcal{B}$, we have $(A \times G)_{\overline{\partial}_J(j,u)} = (A \times G)_{(j,u)}$.*

*Proof.* The inclusion "$\subseteq$" always holds: writing $\pi\colon \mathcal{E} \to \mathcal{T} \times \mathcal{B}$ for the bundle projection in $\mathcal{E}$, an equality $h \cdot \overline{\partial}_J(j,u) = \overline{\partial}_J(j,u)$ implies $h \cdot (j,u) = \pi(h \cdot \overline{\partial}_J(j,u)) = \pi(\overline{\partial}_J(j,u)) = (j,u)$. The inclusion "$\supseteq$" follows from the previous lemma: for all $h \in (A \times G)_{(j,u)}$, we have $h \cdot \overline{\partial}_J(j,u) = \overline{\partial}(j,u)$, hence $h \in (A \times G)_{(j,u)}$. $\square$

We denote the induced $G$-action on $\mathcal{T} \times \mathcal{B}^H$ by $\alpha_g\colon \mathcal{T} \times \mathcal{B}^H \to \mathcal{T} \times \mathcal{B}^H$, for each $g \in G$.

**Corollary 4.50.** *$\overline{\partial}_J|_{\mathcal{T}\times\mathcal{B}^H}$ maps into*

$$\mathcal{E}_H := \{\eta \in (\mathcal{E}|_{\mathcal{T}\times\mathcal{B}^H})_v \mid (A \times G)_\eta = (A \times G)_v \cong H\}. \quad \square$$

Finally, a local model for $\mathcal{M}^{A,H}(J)$ is given via the zero set of $\overline{\partial}_J|_{\mathcal{T}\times\mathcal{B}^H}$.

**Corollary 4.51** (Local model for $\mathcal{M}^{A,H}(J)$)**.**
*Let $(\Sigma, j_0, \theta, u_0) \in \widetilde{\mathcal{M}}^{A,H}(J)$ be arbitrary; let $\mathcal{T}$ be an $A$-adapted Teichmüller slice through $j_0$. Then $[u_0]$ has a neighbourhood $U \subset \mathcal{M}(J)$ such that $\mathcal{M}^{A,H}(J) \cap U$ corresponds to an open neighbourhood of $(j_0, u_0)$ in $\overline{\partial}_J|_{\mathcal{T}\times\mathcal{B}^H}^{-1}(0)$. In other words, $\mathcal{M}^{A,H}(J)$ is locally given as the zero set of the restriction $\overline{\partial}_J^H := \overline{\partial}_J|_{\mathcal{T}\times\mathcal{B}^H}\colon \mathcal{T} \times \mathcal{B}^H \to \mathcal{E}_H$.*

*Proof.* Choose a neighbourhood $U$ as in Corollary 4.44. Then, a neighbourhood of $[u_0]$ in $\mathcal{M}^A(J) \cap U$ corresponds to some neighbourhood of $(j_0, u_0)$ in $\overline{\partial}_J^{-1}(0)$. In particular, locally $\mathcal{M}^{A,H}(J) \cap U$ corresponds to $\overline{\partial}_J^{-1}(0) \cap \{(j,u) \mid (A \times G)_u \cong H\}$. Since the second summand is precisely $\mathcal{T} \times \mathcal{B}^H$, the claim follows. $\square$

To make this local model useful, $\mathcal{B}^H$ must be a smooth Banach submanifold of $\mathcal{B}$ and $\mathcal{E}_H$ must be a smooth Banach sub-bundle of $\mathcal{E}$. If $A$ and $G$ are finite, this is the case.

**Lemma 4.52.** *If $A$ and $G$ are finite, $\mathcal{B}^H$ is a smooth Banach submanifold of $\mathcal{B}$.*

*Proof.* We begin by showing that $\mathcal{B}^H$ is a smooth submanifold. Let $u_0 \in \mathcal{B}$ be arbitrary; write $K := (A \times G)_u$. Consider the fixed point set $\mathcal{B}_K := \{u \in \mathcal{B} \mid \forall k \in K, k \cdot u = u \text{ a.e.}\}$ of the $(A \times G)_u$-action on $\mathcal{B}$. By construction, $u \in \mathcal{B}_K$. Then $\mathcal{B}_K$ is a Banach submanifold (e.g. [DK00, p. 108; AB15, Proposition 3.93]) of $\mathcal{B}$. We show that $u$ has a neighbourhood $U \subset \mathcal{B}$ such that $U \cap \mathcal{B}^H = U \cap \mathcal{B}_K$.

Indeed, for each $h \notin K$, we have $h \cdot u_0 \neq u_0$. As $\mathcal{B}^H$ is metrisable, hence Hausdorff and both sides vary continuously with $u$, for each $h \in A \times G \setminus K$ there exists an open neighbourhood around $u_0$ with $h \cdot u \neq u$, hence $n \notin K$. Taking the intersection of these finitely many neighbourhoods yields a neighbourhood $U$ as desired. $\square$

**Lemma 4.53.** *If $A$ and $G$ are finite, $\mathcal{E}_H$ is a smooth Banach sub-bundle of $\mathcal{E}|_{\mathcal{T}\times\mathcal{B}^H}$.*

*Proof.* By Lemma 4.52, $\mathcal{B}^H \subset \mathcal{B}$ is a Banach submanifold, hence $\mathcal{E}|_{\mathcal{T}\times\mathcal{B}^H}$ is a smooth Banach space bundle. Let $v \in \mathcal{E}_H$ be arbitrary. An argument analogous to Lemma 4.52 shows that locally, $\mathcal{E}^H$ agrees with the subset

$$\mathcal{E}^H := \{\eta \in \mathcal{E}|_{\mathcal{T}\times\mathcal{B}} \mid h\cdot\eta = \eta \text{ a.e. for all } h \in H\}.$$

Lemma 4.54 below concludes the proof. □

**Lemma 4.54.** *Suppose $A$ is finite and $G$ acts smoothly and properly on $M$. Then $\mathcal{E}^H$ is a smooth sub-bundle of $\mathcal{E}|_{\mathcal{T}\times\mathcal{B}^H}$.*

*Proof.* We construct local trivialisations for $\mathcal{E}^H$ by hand: more precisely, we claim that the local trivialisations for $\mathcal{E}$, if the auxiliary data in the construction are well-chosen, restrict to local trivialisations of $\mathcal{E}^H$. Let us review how local trivialisations on $\mathcal{E}$ are constructed. For simplicity, let us neglect variation of $j$ at first: fix $j$ and consider the bundle $\mathcal{E}'$ over $\mathcal{B}$ with fibres $\mathcal{E}'_u := \mathcal{E}_{(j,u)}$.

As a first ingredient, we need to relate each fibre $\mathcal{E}'_u = L^p(\overline{\mathrm{Hom}}_{\mathbb{C}}((T\Sigma, j), (u^*TM, J)))$ to fibres $\mathcal{E}'_v$ of $v$ near $u$. This is done using parallel transport on the tangent bundle $TM$. Choose a complex connection $\nabla$ on $TM$. Recall this induces parallel transport maps: for any path $\gamma\colon [0,1] \to M$, there exists an induced isomorphism $P_\gamma\colon T_{\gamma(0)}M \to T_{\gamma(1)}M$ between tangent spaces of $M$, whose inverse is the map $P_{-\gamma}\colon T_{\gamma(1)}M \to T_{\gamma(0)}$ induced by the reverse path $t \mapsto \gamma(1-t)$. The same construction pushed forward to the fibres of $\mathcal{E}$. For any path $\overline{\gamma}\colon [0,1] \to \mathcal{B}$ from $\overline{\gamma}(0) = u$ to $\overline{\gamma}(1) = v$, we have an induced path $\gamma_z\colon [0,1] \to M, \gamma_z(t) := \gamma(t)(z)$. We obtain an induced map

$$\overline{P}_{\overline{\gamma}}\colon L^p(\mathrm{Hom}(T\Sigma, u^*TM)) \to L^p(\mathrm{Hom}(T\Sigma, v^*TM)), \eta \mapsto (z \mapsto P_{\gamma_z} \circ \eta).$$

To make this explicit: for all $\eta \in L^p(\mathrm{Hom}(T\Sigma, u^*TM))$, $z \in \Sigma$ and $X \in T_z\Sigma$, we have $\overline{P}_{\overline{\gamma}}(\eta)(z) = X \mapsto P_{\gamma_z} \circ \eta(X)$.

Since $\nabla$ is a complex connection, each parallel transport map $P_\gamma$ is complex linear, thus $\overline{P}_{\overline{\gamma}}$ maps complex anti-linear sections to complex anti-linear sections, and descends to a well-defined map

$$\overline{P}_{\overline{\gamma}}\colon L^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM)) \to L^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, v^*TM)), \eta \mapsto (z \mapsto P_{\gamma_z} \circ \eta).$$

To describe local trivialisations, we need to choose a canonical path between $u \in \mathcal{B}$ and $v \in \mathcal{B}$ close by. To do so, let us recall how charts for $\mathcal{B}$ are defined, following Eliasson's work.

**Recall.** To define the Banach manifold structure on $\mathcal{B}$, choose a Riemannian metric $g$ on $M$. This defines an exponential map $\exp_p\colon T_pM \to M$ at each point $p \in M$. We can also use this to define an "exponential map" $\exp_{\mathcal{B}}$ on $\mathcal{B}$.[15] For $u \in \mathcal{B}$, we

[15] Let us emphasize that this not related to ideas such as a Riemannian metric (or a spray) on the Banach manifold $\mathcal{B}$. However, it satisfies the same function, without having to think about details such as whether the tangent spaces $T_u\mathcal{B}$ are self-dual: in our setting, they emphatically are *not*.

define the map $(\exp_{\mathcal{B}})_u \colon T_u\mathcal{B} \to \mathcal{B}$ by "following the exponential map at each point", i.e. $(\exp_{\mathcal{B}})_u(X) := (z \mapsto (\exp_g)_{u(z)}X(z) \in M)$. In plain English: a tangent vector $X \in T_u\mathcal{B} = W^{1,p}(u^*TM)$ is a vector field along $u$ (of regularity $W^{1,p}$): at each point $p = u(z)$, we apply the exponential map (in $M$) at $p$ in direction $X(z)$; this defines a new map $v = \exp_u(X) \in W^{1,p}(\Sigma, M) = \mathcal{B}$.
Near $0 \in T_u\mathcal{B}$, this map restricts to a bijection to its image; we define an atlas of $\mathcal{B}$ via these restrictions. In particular, each exponential map $(\exp_{\mathcal{B}})_u$ becomes a local diffeomorphism $U \subset T_u\mathcal{B} \to V \subset \mathcal{B}$ between suitable neighbourhoods $U$ of $0$ and $V$ of $u$.

Having this in mind, a canonical path from $u \in \mathcal{B}$ to $v$ nearby becomes apparent: if $v = \exp_{\mathcal{B}} X$ for some unique $X \in T_uB$, then $\gamma_X \colon [0,1] \to \mathcal{B}, t \mapsto \exp_{\mathcal{B}}(tX)$ is a path in $\mathcal{B}$ from $u$ to $v$. (When there is no risk of confusion, we will write $\exp_{\mathcal{B}}$ for $(\exp_{\mathcal{B}})_u$.) Now we have assembled all ingredients to describe the local trivialisations of $\mathcal{E}'$. A local trivialisation for $\mathcal{E}'$ near $u \in \mathcal{B}$ is given as

$$\phi'_U \colon U \times \mathcal{E}'_u \to \mathcal{E}'|_U, (v = \exp_{\mathcal{B}} X, \eta) \mapsto \overline{P}_{\gamma_X}(\eta) \in \mathcal{E}'_v, \tag{4.4}$$

where $U \subset \mathcal{B}$ is an open neighbourhood of $u$ so the exponential map $(\exp_{\mathcal{B}})_u \colon U \subset T_u\mathcal{B} \to \mathcal{B}$ is a diffeomorphism to its image.

**Lemma 4.55** (folklore)**.** *The maps $\phi'_U$ (for all $u \in \mathcal{B}$) define local trivialisations for $\mathcal{E}'$.* □

Having reviewed the classical case, let us indicate how to adapt this to the $H$-invariant elements of $\mathcal{E}'|_{\mathcal{B}^H}$. As a reminder: we use the same construction, just for a carefully chosen Riemannian metric $g$ on $M$ and a connection $\nabla$ on $TM$. The following lemma is implicitly contained in the computation of the tangent space $T_u\mathcal{B}^H$ in [AB15, Proposition 3.93].

**Lemma 4.56.** *Suppose $g$ is a $G$-invariant Riemannian metric on $M$.*[16] *Then $\exp_{\mathcal{B}}$ is $G$-equivariant, and for $u \in \mathcal{B}^H$, we have $v := (\exp_{\mathcal{B}})_u(X) \in \mathcal{B}^H$ if and only if $X$ is $H$-invariant, i.e. $h \cdot X = X$ a.e. for all $h \in H$. In particular, restricting $\exp_{\mathcal{B}}$ to $H$-invariant vector fields yields slice charts for $\mathcal{B}^H$, and $T_u\mathcal{B}^H$ consists of all $X \in T_u\mathcal{B}$ which are $H$-invariant.* □

Motivated by this lemma, we will choose a $G$-invariant Riemannian metric $g$ on $M$, and define the charts on $\mathcal{B}$ accordingly. Thus, each $v \in \mathcal{B}^H$ near $u \in \mathcal{B}^H$ is described as $v = \exp_{\mathcal{B}} X$ for some $X \in T_u\mathcal{B}^H$, i.e. $X \in W^{1,p}(u^*TM)$ is an $H$-invariant vector field. We would like to choose the complex connection $\nabla$ on $TM$ to be $G$-invariant, so each parallel transport is $G$-equivariant.

**Lemma 4.57.** *Suppose $G$ is compact. There exists a complex connection $\nabla$ on $TM$ which is $G$-invariant, i.e., such that the parallel transport maps $P_\gamma$ satisfy $d\psi_g \circ P_\gamma = P_{\psi_g \cdot \gamma} \circ d\psi_g$ for all $g \in G$ and each path $\gamma \colon [0,1] \to M$.*

[16] Since $G$ acts smoothly and properly on $M$, such an invariant metric always exists.

*Proof.* Observe that $G$ acts linearly on the space $\mathcal{A}(TM)$ of complex connections on $TM$, by $g \cdot \nabla := g \cdot \nabla(g^{-1} \cdot s)$, where $g\cdot$ denotes the $G$-action on $\Gamma(TM)$ and $M$, respectively. Let us check that this in fact defines a complex connection: for $f \in C^\infty(M, \mathbb{C})$ and $s \in \Gamma(TM)$, we compute

$$\begin{aligned}\nabla_g(fs) &= g \cdot \nabla(g^{-1} \cdot (fs)) = g \cdot \nabla(f(g^{-1} \cdot s)) = g \cdot [df(\cdot)(g^{-1} \cdot s) + f \circ \psi_g^{-1} \nabla(g^{-1} \cdot s)] \\ &= df(\cdot)(g \cdot (g^{-1} \cdot s)) + (f \circ \psi_g^{-1} \circ \psi_g) g \cdot \nabla(g^{-1} \cdot s) = df(\cdot)s + f\nabla_g s\end{aligned}$$

using $C^\infty(M, \mathbb{C})$-linearity of the $G$-action. [17] Note that $df(X)$ is a smooth function for each vector field $X$.

Since $\mathcal{A}(TM)$ is an affine space and in particular convex, we can take averages of connections: given any complex connection $\nabla \in \mathcal{A}(TM)$, its average $\nabla^G := \int_G g \cdot \nabla \, \mathrm{d}g$ w.r.t. the Haar measure on $G$ is a well-defined complex connection on $TM$. Observe that $\nabla^G$ is $G$-equivariant: for all $g \in G$ and $s \in \Gamma(TM)$, we compute

$$\begin{aligned}\nabla(g \cdot s) &= \int_G h \cdot \nabla_0(h^{-1} \cdot g \cdot s) = \int_G g \cdot (g^{-1}h)\nabla_0((g^{-1}h)^{-1} \cdot s) \\ &= \int_G g \cdot \tilde{h} \cdot \nabla_0(\tilde{h}^{-1} \cdot s) \, \mathrm{d}\tilde{h} = g \cdot \int_G h \cdot \nabla_0(h^{-1} \cdot s) = g \cdot \nabla(s).\end{aligned}$$

Hence, if $\gamma$ is a horizontal path in $E$ w.r.t. $\nabla^G$, so is $g \cdot \gamma$, and parallel transport w.r.t. $\nabla^G$ is $G$-equivariant. ◻

Choosing $g$ and $\nabla$ as in the two preceding lemmas will yield a local trivialisation as desired. Choose a $G$-invariant Riemannian metric $g$ on $M$ and a $G$-invariant complex connection $\nabla$ on $TM$. Use $g$ and $\nabla$ to define the maps $\exp_{\mathcal{B}}$ and $P_\gamma$ above. Let $u \in \mathcal{B}^H$ be arbitrary. Suppose $U \subset \mathcal{B}$ is an open neighbourhood of $u$ so the exponential map $(\exp_{\mathcal{B}})_u \colon U \subset T_u\mathcal{B} \to \mathcal{B}$ is a diffeomorphism to its image. Consider the subset $\mathcal{E}'^H \subset \mathcal{E}'|_{\mathcal{B}^H}$ of all $H$-invariant $\eta \in \mathcal{E}'|_{\mathcal{B}^H}$. Then local trivialisations of $\mathcal{E}'$ restrict to local trivialisations of $\mathcal{E}'^H$.

**Lemma 4.58.** *Every map $\phi'_U$ from (4.4) restricts to a bijective map*

$$\phi'^H_U \colon (U \cap \mathcal{B}^H) \times \mathcal{E}'^H_u \to \mathcal{E}'^H_U.$$

*Proof.* We need to prove that $\phi'^H_U$ is well-defined and surjective. For well-definedness, let $v = \exp_{\mathcal{B}} X \in U \cap \mathcal{B}^H$ and $\eta \in \mathcal{E}'^H_u$ be arbitrary; in particular, $X \in T_u\mathcal{B}^H$. We need to prove $\phi'_U(v, \eta) = \overline{P}_{\gamma_X}(\eta) \in \mathcal{E}'^H$. Since $\phi'_U(v, \eta) \in \mathcal{E}'_v$, it remains to prove $\overline{P}_{\gamma_X}(\eta)$ is $H$-invariant. To that end, for each $h = (\phi, g) \in H$ we compute

$$\begin{aligned}h \cdot \overline{P}_{\gamma_X}\eta &= d\psi_g \circ \overline{P}_{\gamma_X}\eta \circ d\phi^{-1} = z \mapsto (d\psi_g \circ (P_{(\gamma_X)_z} \circ \eta) \circ d\phi^{-1}) \\ &= z \mapsto (P_{(\gamma_X)_z} \circ (d\psi_g \circ \eta \circ d\phi^{-1})) = z \mapsto (P_{(\gamma_X)_z} \circ h \cdot \eta) \\ &= z \mapsto (P_{(\gamma_X)_z} \circ \eta) = \overline{P}_{\gamma_X}\eta\end{aligned}$$

[17] This uses that $J$ is $G$-equivariant: in particular, each map $d\psi_g$ is a complex bundle isomorphism.

since each map $P_{(\gamma_X)_z}\colon T_{u(z)}M \to T_{v(z)}M$ was $G$-equivariant by hypothesis. For surjectivity, suppose $(v = \exp_{\mathcal{B}} X, \eta) \in (U \cap \mathcal{B}^H) \times \mathcal{E}'_u$ with $\phi'_U(v, X) \in \mathcal{E}'^H_U$, i.e. $\phi'_U(v, X)$ is $H$-invariant: we need to prove that $\eta$ was $H$-invariant to begin with. For all $h = (\phi, g) \in H$ and $z \in \Sigma$, we compute

$$\begin{aligned} P_{(\gamma_X)_z} \circ \eta = \overline{P}_{\gamma_X}\eta(z) &= (h \cdot \overline{P}_{\gamma_X}\eta)(z) = (d\psi_g \circ P_{(\gamma_X)_z} \circ \eta \circ d\phi^{-1})(z) \\ &= (P_{(\gamma_X)_z} \circ d\psi_g \circ \eta \circ d\phi^{-1})(z) = (P_{(\gamma_X)_z} \circ h \cdot \eta)(z); \end{aligned}$$

since each $P_{(\gamma_X)_z}$ is an isomorphism, this implies $h \cdot \eta = \eta$. Since $h \in H$ was arbitrary, $\eta$ is indeed $H$-invariant. ☐

Since the $\phi_U$ form a system of local trivialisations of $\mathcal{E}'$, we deduce that the $\phi'^H_U$ form local trivialisations for $\mathcal{E}'_H$, hence $\mathcal{E}'_H$ is a smooth sub-bundle of $\mathcal{E}'|_{\mathcal{B}^H}$.

Improving this to a proof of Lemma 4.54, about $\mathcal{E}^H \subset \mathcal{E}|_{\mathcal{T}\times\mathcal{B}^H}$ is a standard exercise whose details we omit. This step is fully analogous to improving the statement "$\mathcal{E}' \to \mathcal{B}$ is a smooth Banach space bundle" to "the extension $\mathcal{E} \to \mathcal{T} \times \mathcal{B}^H$ of $\mathcal{E}'$ is a Banach space bundle": the result is not difficult, but I am not aware of any written reference. This thesis is not the place to change this. ☐

If $A$ or $G$ are infinite, the trick in Lemmas 4.52 and 4.53 does not work any more, and a new argument is needed. This is much more involved than one might think; see Section 4.7 for details.

To close this section, let us note that this description also yields a local model for the sets $\mathcal{M}^{A,H}_{\mathcal{U}}(J)$. The idea is that if $[u] \in \mathcal{M}^{A,H}_{\mathcal{U}}(J)$ (i.e., $u \in \mathcal{M}^{A,H}(J)$ has an injective point mapped to $\mathcal{U}$), every holomorphic curve near $u$ must also be simple.

**Proposition 4.59.** *$\{[u] \mid u$ has an injective point mapped to $\mathcal{U}\} \subset \mathcal{M}(J)$ is an open subset.*

Since the topology on $\mathcal{M}^{A,H}(J)$ and $\mathcal{M}^{A,H}_{\mathcal{U}}(J)$ is the subspace topology from $\mathcal{M}(J)$, we deduce that $\mathcal{M}^{A,H}_{\mathcal{U}}(J) \subset \mathcal{M}^{A,H}(J)$ is an open subset. In conclusion, we obtain the following.

**Corollary 4.60** (Local model for $\mathcal{M}^{A,H}_{\mathcal{U}}(J)$)**.** *$\mathcal{M}^{A,H}_{\mathcal{U}}(J)$ is locally given as the zero set of the restriction $\overline{\partial}^H_J := \overline{\partial}_J|_{\mathcal{T}\times\mathcal{B}^H}\colon \mathcal{T} \times \mathcal{B}^H \to \mathcal{E}^H$. That is, each $[(\Sigma, j, \theta, u)] \in \mathcal{M}^{A,H}_{\mathcal{U}}(J)$ has a neighbourhood $U$ in $\mathcal{M}^{A,H}_{\mathcal{U}}(J)$ such that the corresponding subset of $\mathcal{T} \times \mathcal{B}^H$ is a neighbourhood of $(j, u)$ in $(\overline{\partial}^H_J)^{-1}(0)$.*

Proposition 4.59 is classical; let us review its proof since we will use similar ideas later. The main idea is to use the equivalence of simple and somewhere injective curves, and show that being somewhere injective is an open condition. In fact, this result holds in greater generality: being somewhere injective is an open condition in the space $C^1(M, N)$ of $C^1$ maps $f\colon M \to N$. Consequently, the proof uses facts about differential topology, in particular the $C^1_{\mathrm{loc}}$-topology[18].

[18] We omit its definition in this document: since the analogous statement in $C^\infty(M, N)$ also holds, the cautious reader may simply work with smooth maps.

**Lemma 4.61.** *Let $M$ and $N$ be $C^1$ manifold. The set of somewhere injective maps $f\colon M \to N$ is open. Moreover, for any open subset $U \subset N$, the set*

$$S := \{f \in C^1(M,N) \mid f \text{ has an injective point } z \text{ with } f(z) \in U\}$$

*is open in the $C^1_{loc}$-topology.*

*Idea of proof.* Let $f \in S$ be arbitrary; choose an injective point $p \in M$ of $f$ with $f(p) \in U$. This means three conditions are satisfied: $df_p$ is injective, $f^{-1}(f(p)) = \{p\}$ and $f(p) \in U$. Each of these is an open condition in $C^1_{\mathrm{loc}}(M,N)$. □

What is more, injective points are not isolated: the set of injective points is always open (though perhaps empty), as both defining conditions are open within the set of smooth maps. Let us record this fact for later use in this section.

**Lemma 4.62.** *Let $f\colon M \to N$ be a $C^1$ map. Then the set of injective points of $f$ is open (perhaps empty). More generally, for any open subset $U \subset N$, the set of injective points $z$ of $f$ with $f(z) \in U$ is open.* □

Before we close this section, let us mention one last definition, which is very close to a standard one.

**Recall.** A bounded linear operator $L\colon X \to Y$ between normed spaces[19] is called a *Fredholm operator* (or *Fredholm* for short) if and only if its kernel and co-kernel are finite-dimensional and it has closed image.

For $[(\Sigma, j, \theta, u)] \in \mathcal{M}^{A,H}(J)$, let $D\overline{\partial}_J(j,u)\colon T_j\mathcal{T} \times T_u\mathcal{B}^H \to \mathcal{E}^H_{(j,u)}$ be the linearisation of the operator $\overline{\partial}_J(j,u)\colon \mathcal{T} \times \mathcal{B}^H \to \mathcal{E}^H$. One can show that $D\overline{\partial}_J(j,u)$ is a Fredholm operator. (We will prove in Lemma 4.82 that restricting $D\overline{\partial}_J(j,u)$ to the second summand is Fredholm; since $T_j\mathcal{T}$ is finite-dimensional, $D\overline{\partial}_J(j,u)$ also is Fredholm.)

**Definition 4.63** ($H$-Fredholm regular curves)**.** *A curve $u \in \mathcal{M}^{A,H}(J)$ is called* $H$-Fredholm regular[20] *if and only if the linearisation*

$$D\overline{\partial}_J(j,u)\colon T_j\mathcal{T} \times T_u\mathcal{B}^H \to L^p_H(\overline{\mathrm{Hom}}_{\mathbb{C}}((T\Sigma, j), (u^*TM, J))) \tag{4.5}$$

*is surjective.*

[19]Most commonly in symplectic geometry, Fredholm operators are considered between Banach spaces; this is not required for their definition. Over Banach spaces, however, the theory of Fredholm operators is simpler — for instance, the image of a Fredholm operator is always closed and admits a closed complement. In this thesis, we only encounter Fredholm operators between Banach spaces: the linearised Cauchy–Riemann operator $D_u$ or the operator $L$ from the next section are Fredholm operators.

[20]This is a slight adaptation of standard terminology: conventionally, $u$ is called Fredholm regular if the linearisation $D_u$ of $\overline{\partial}_J(j,u)\colon \mathcal{T} \times \mathcal{B} \to \mathcal{E}$ is surjective (where $\mathcal{T}$ is a classical Teichmüller slice). This definition is the natural analogue in our setting. I contemplated re-using the term "Fredholm regular" and decided against it to avoid confusion.

Standard arguments (similar to e.g. [Wen15, Lemma 4.3.2]) show that being $H$-Fredholm regular is independent of the particular choice of the adapted Teichmüller slice $\mathcal{T}$. Like Fredholm regularity, $H$-Fredholm regularity is an open condition. This follows from the argument discussed in Section 5.1 to prove Lemma 5.16. For the overall idea, suppose $u$ is $H$-Fredholm regular, then $D\overline{\partial}_J(j,u)$ is a surjective Fredholm operator, in particular induces an isomorphism $V \to \mathcal{E}^H_{(j,u)}$, where $V$ is a closed complement of $\ker D\overline{\partial}_J(j,u)$ in $T_j\mathcal{T} \oplus T_u\mathcal{B}^H$. Being an isomorphism is an open condition, hence all Fredholm operators sufficiently close to $D\overline{\partial}_J(j,u)$ are also surjective.

Fredholm regular curves have an important property (e.g. [Wen15, Theorem 4.3.6]): if $[u] \in \mathcal{M}(J)$ is Fredholm regular, the set of Fredholm regular curves near $u$ is a smooth orbifold of dimension $\text{vir-dim}(u) := (2-2g)(n-3) + 2c_1(C) + 2m$. When restricting to simple curves, it is also a manifold. Proving this result requires no transversality — but this result could be vacuous, if the set of Fredholm regular curves is empty. The analogous argument applies in our setting and proves the following.

**Lemma 4.64.** *Suppose $A$ is finite. If $[u] \in \mathcal{M}^{A,H}_{\mathcal{U}}(J)$ is $H$-Fredholm regular, then a neighbourhood of $[u]$ in $\mathcal{M}^{A,H}_{\mathcal{U}}(J)$ is a smooth manifold of dimension* $\operatorname{ind}(D\overline{\partial}_J(j,u))$. □

## 4.4. The universal moduli space

Having defined the equivariant $C_\epsilon$-space and adapted Teichmüller-slices, we can now state and prove smoothness of the universal moduli space. In essence, we have to prove two versions of this result, for tame and compatible almost complex structures. We will treat the tame case first; the compatible case is very similar; we will indicate the necessary adjustments later.

For the remainder of this section, fix $J_{\text{ref}} \in \mathcal{J}^G_\tau(M,\omega;\mathcal{U},J_{\text{fix}})$ and consider the corresponding spaces $C_\epsilon(\overline{\text{End}}_{\mathbb{C}}(TM, J_{\text{ref}});\mathcal{U})$ and $\mathcal{J}_\epsilon$.

**Definition 4.65** (Universal moduli space)**.** *The* universal moduli space *associated to the sequence $\epsilon \in \boldsymbol{\mathcal{E}}$ and $J_{ref}$ is*

$$\mathcal{U}^*(\mathcal{J}_\epsilon) := \{(u,J) \mid J \in \mathcal{J}_\epsilon, u \in \widetilde{\mathcal{M}}^{A,H}_{\mathcal{U}}(J)\}.$$

In this section, we aim to prove the following.

**Proposition 4.66.** *For all $\epsilon \in \boldsymbol{\mathcal{E}}$ decaying sufficiently fast*[21]*, the universal moduli space $\mathcal{U}^*(\mathcal{J}_\epsilon)$ is a smooth Banach manifold, separable and metrisable, and the canonical projection $\pi\colon \mathcal{U}^*(\mathcal{J}_\epsilon) \to \mathcal{J}_\epsilon, (u,J) \mapsto J$ is smooth.*

We show this locally, using the implicit function theorem: hence, as in the previous section, the first task is to find a suitable local model for $\mathcal{U}^*(\mathcal{J}_\epsilon)$.

[21] by this, we mean: there exists a sequence $\epsilon_0 \in \boldsymbol{\mathcal{E}}$ such that for all $\epsilon \prec \epsilon_0$, ..."

### 4.4.1. Local models for the universal moduli space

Fortunately, it is straight-forward to modify the local model for $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$ from the previous section to obtain a local model for $\mathcal{U}^*(\mathcal{J}_\epsilon)$: we just need to take the additional factor $J$ into account.

Let $(\Sigma, j, \theta) \in \widetilde{\mathcal{M}}_{g,m}^A$ be arbitrary. Choose an $A$-adapted Teichmüller slice $\mathcal{T}$ through $j$. Consider the Banach manifold $\mathcal{B} = W_\theta^{1,p}(\Sigma, M)$. We consider the Banach space bundle $\mathcal{E} \to \mathcal{T} \times \mathcal{B} \times \mathcal{J}_\epsilon$ with fibres $\mathcal{E}_{(j,u,J)} := L^p(\overline{\mathrm{Hom}}_{\mathbb{C}}((T\Sigma, j), (u^*TM, J)))$, and the smooth section

$$\overline{\partial}_J \colon \mathcal{T} \times \mathcal{B} \times \mathcal{J}_\epsilon \to \mathcal{E}, \quad (j, u, J) \mapsto du + J \circ du \circ j.$$

The $A$- and $G$-action on the moduli space $\mathcal{M}(J)$ carry over to the bundle $\mathcal{E} \to \mathcal{T} \times \mathcal{B} \times \mathcal{J}_\epsilon$. The proof proceeds exactly as for Observation 4.45; we omit the details.

**Observation 4.67.** The group $A \times G$ acts...

- ...on the bundle $\mathcal{E}$ via $(\phi, g) \circ \eta = d\psi_g \circ \eta \circ d\phi^{-1}$;
- ...on the base $\mathcal{T} \times \mathcal{B} \times \mathcal{J}_\epsilon$ by $(\phi, g) \cdot (j, u, J) := (\phi^* j, \psi_g \circ u \circ \phi^{-1}, (\psi_g)_* J = J)$;
- ...on $\mathcal{B}$ by $(\phi, g) \cdot u = \psi_g \circ u \circ \phi^{-1}$.

The actions on $\mathcal{E}$ and the base are compatible with the projection $\mathcal{E} \to \mathcal{T} \times \mathcal{B} \times \mathcal{J}_\epsilon$. □

We consider the restriction of $\overline{\partial}_J$ to the sub-manifold $\mathcal{T} \times \mathcal{B}^H \times \mathcal{J}_\epsilon$ of the base: it will take values in a suitable sub-bundle.

**Lemma 4.68.** $\mathcal{E}_H := \{\eta \in (\mathcal{E}|_{\mathcal{T} \times \mathcal{B}^H \times \mathcal{J}_\epsilon})_v \mid (A \times G)_\eta = (A \times G)_v \cong H\}$ *is a smooth Banach space sub-bundle of the restriction* $\mathcal{E}|_{\mathcal{T} \times \mathcal{B}^H \times \mathcal{J}_\epsilon}$.

*Proof.* Repeat the proof of Lemma 4.53 mutatis mutandis. □

**Observation 4.69.** $\overline{\partial}_J|_{\mathcal{T} \times \mathcal{B}^H \times \mathcal{J}_\epsilon}$ maps into $\mathcal{E}_H$.

*Proof.* Repeat the proof of Corollary 4.50 mutatis mutandis. □

The proof of the previous lemma shows that $\mathcal{E}_H$ locally agrees with the smooth sub-bundle

$$\mathcal{E}^H := \{\eta \in \mathcal{E}|_{\mathcal{T} \times \mathcal{B} \times \mathcal{J}_\epsilon} \mid h \cdot \eta = \eta \text{ a.e. for all } h \in H\},$$

and $\overline{\partial}_J$ locally maps into $\mathcal{E}^H$. The zero set of $\overline{\partial}_J^H$ yields a local model for the universal moduli space: the proof is the same as for Corollary 4.60.

**Corollary 4.70** (Local model for $\mathcal{U}^*(\mathcal{J}_\epsilon)$)**.** *The universal moduli space* $\mathcal{U}^*(\mathcal{J}_\epsilon)$ *is locally given as the zero set of the restriction* $\overline{\partial}_J^H := \overline{\partial}_J|_{\mathcal{T} \times \mathcal{B}^H \times \mathcal{J}_\epsilon} \colon \mathcal{T} \times \mathcal{B}^H \times \mathcal{J}_\epsilon \to \mathcal{E}^H$. □

### 4.4.2. Smoothness of the universal moduli space

Now, let us prove that $(\overline{\partial}_J^H)^{-1}(0)$ is a smooth (Banach) manifold near each $(j,u,J)$ for $(\Sigma, j, u, \theta) \in \widetilde{\mathcal{M}}_{\mathcal{U}}^{A,H}(J)$.[22] We do this using the implicit function theorem: we need to prove that the linearisation $D\overline{\partial}_J^H(j,u,J)$ of $\overline{\partial}_J^H$ at $(j,u,J)$ is surjective with a bounded right inverse. Since $\overline{\partial}_J^H$ maps to the sub-bundle $\mathcal{E}^H$, so does $D\overline{\partial}_J^H(j,u,J)$. Let us prove two related lemmas.

**Lemma 4.71.** *For all $v \in \mathcal{M}(J)$ and $h \in H$, we have $\overline{\partial}_J(h \cdot (j,v,J)) = h \cdot \overline{\partial}_J(j,v,J)$.*

*Proof.* Write $h = (\phi, g) \in A \times G$. Since $j$ is $A$-invariant and $J$ is $G$-equivariant, we have

$$h \cdot (j,v,J) = (\phi^* j, h \cdot v, (\psi_g)_* J) = (j, h \cdot v, J);$$

we deduce

$$\begin{aligned}\overline{\partial}_J(h \cdot (j,v,J_{\text{ref}})) &= \overline{\partial}_J(j, \psi_g \circ v \circ \phi^{-1}, J) = d\psi_g \circ dv \circ d\phi^{-1} + J \circ d\psi_g \circ dv \circ d\phi^{-1} \circ j \\ &= d\psi_g \circ dv \circ d\phi^{-1} + d\psi_g \circ J_{\text{ref}} \circ dv \circ (d\phi^{-1} \circ j \circ d\phi) \circ d\phi^{-1} \\ &= d\psi_g \circ (dv + J_{\text{ref}} \circ dv \circ j) \circ d\phi^{-1} = h \cdot \overline{\partial}_J(j,v,J_{\text{ref}}). \qquad \square\end{aligned}$$

**Corollary 4.72.** *For all $h \in H$ we have $D_u(h \cdot \eta) = h \cdot D_u\eta$, where $D_u\eta := D\overline{\partial}_{J_{ref}}(u)$ is the linearisation of the operator $\overline{\partial}_{J_{ref}} \colon T_u\mathcal{B} \to \mathcal{E}_{(j,u,J_{ref})}, u \mapsto \overline{\partial}_{J_{ref}}(j,u,J_{ref})$ at $u$.*

*Proof.* Let $h = (\phi, g) \in H$ and $\eta = \partial_\tau(u_\tau)|_{\tau=0} \in T_u\mathcal{B}$ be arbitrary. Using the previous lemma, we compute

$$\begin{aligned}D\overline{\partial}_{J_{\text{ref}}}(u)(h \cdot \eta) &= \nabla_\tau \overline{\partial}_{J_{\text{ref}}}(h \cdot u_\tau)|_{\tau=0} = \nabla_\tau h \cdot \overline{\partial}_{J_{\text{ref}}}(u_\tau)|_{\tau=0} \\ &= \nabla_\tau d\psi_g \circ \overline{\partial}_{J_{\text{ref}}}(u_\tau) \circ d\phi^{-1}|_{\tau=0} = d\psi_g \circ \nabla_\tau \overline{\partial}_{J_{\text{ref}}}(u_\tau)|_{\tau=0} \circ d\phi^{-1} \\ &= h \cdot D\overline{\partial}_J(u)\eta. \qquad \square\end{aligned}$$

For the operator $\overline{\partial}_{J_{\text{ref}}}^H$, a simple computation shows

$$\begin{aligned}D\overline{\partial}_{J_{\text{ref}}}^H(j,u,J) \colon T_j\mathcal{T} \oplus T_u\mathcal{B}^H \oplus T_{J_{\text{ref}}}\mathcal{J}_\epsilon &\to \mathcal{E}_{(j,u,J_{\text{ref}})}^H, \\ (y, \eta, Y) &\mapsto J_{\text{ref}} \circ du \circ y + D_u^H \eta + Y \circ du \circ j,\end{aligned}$$

where $D_u^H \colon T_u\mathcal{B}^H \to L_H^p(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$ is the restriction of $D_u$. Recall that locally, $\mathcal{B}^H$ is the fixed point set of $H$ and has tangent space (by e.g. [AB15, Proposition 3.93])

$$T_u\mathcal{B}^H = \{\eta \in W_\theta^{1,p}(u^*TM) \mid h \cdot \eta = \eta \text{ a.e. for all } h \in H\}.$$

Moreover, $T_{J_{\text{ref}}}\mathcal{J}_\epsilon = C_\epsilon^G(\overline{\text{End}}_{\mathbb{C}}(TM, J_{\text{ref}}); \mathcal{U})$ by construction. Thus, we have a well-defined bounded linear operator

$$L \colon T_u\mathcal{B}^H \oplus C_\epsilon^G(\overline{\text{End}}_{\mathbb{C}}(TM, J_{\text{ref}}); \mathcal{U}) \to \mathcal{E}_{(j,u,J_{\text{ref}})}^H, (\eta, Y) \mapsto D_u^H \eta + Y \circ du \circ j. \quad (4.6)$$

We better make sure that $L$ also maps into the sub-bundle $\mathcal{E}^H$: indeed, it does.

[22] A posteriori, it will be finite-dimensional.

**Lemma 4.73.** *$L$ is $H$-invariant: for all $h \in H$ and $(\eta, Y) \in T_u\mathcal{B}^H \oplus C^G_\epsilon(\overline{\mathrm{End}}_\mathbb{C}(TM, J_{ref}); \mathcal{U})$, we have $h \cdot L(\eta, Y) = L(\eta, Y)$. In particular,* $\mathrm{im}(L) \subset \mathcal{E}^H_{(j,u,J_{ref})}$.

*Proof.* Let $(\eta, Y) \in T_u\mathcal{B}^H \oplus C^G_\epsilon(\overline{\mathrm{End}}_\mathbb{C}(TM, J_{\mathrm{ref}}); \mathcal{U})$ and $h = (\phi, g) \in H$ be arbitrary. It suffices to check that the first summand $T\colon (y, \eta, Y) \mapsto J_{\mathrm{ref}} \circ du \circ y$ of $D\overline{\partial}_{J_{\mathrm{ref}}}(j, u, J)$ is $H$-invariant. Indeed, we compute

$$\begin{aligned} h \cdot T(y, \eta, Y) &= (\phi, g) \cdot J_{\mathrm{ref}} \circ du \circ y = d\psi_g \circ J_{\mathrm{ref}} \circ du \circ y \circ d\phi^{-1} \\ &= J_{\mathrm{ref}} \circ d\psi_g \circ du \circ d\phi \circ (d\phi^{-1} \circ y \circ d\phi) \\ &= J_{\mathrm{ref}} \circ d(\psi_g \circ u \circ \phi^{-1}) \circ y = J_{\mathrm{ref}} \circ du \circ y = T(y, \eta, Y), \end{aligned}$$

using the $G$-equivariance of $J_{\mathrm{ref}}$ and the $A$-invariance of $y$. □

The key lemma in the proof of smoothness is the following.

**Lemma 4.74** (Workhorse Lemma)**.** *For $\epsilon \in \boldsymbol{\mathcal{E}}$ decaying sufficiently fast, if $u$ has an injective point which is mapped to $\mathcal{U}$, then $L$ is surjective and has a bounded right inverse.*

At a key point in the proof of the Workhorse Lemma, we need to use a well-chosen injective point of $u$. The following lemma summarises the main requirements posed on this choice. Denote by $\pi_1\colon A \times G \to A$ the projection to the first component, similarly for $\pi_2\colon A \times G \to G$.

**Lemma 4.75** ($u$ has many good injective points)**.** *If $u$ has an injective point mapped into $\mathcal{U}$, the set of injective points $z_0$ of $u$ with $u(z_0) \in \mathcal{U}$, $G_{u(z_0)} = G_u$ and*

$$\mathrm{im}(u) \cap G \cdot u(z_0) = \{u \circ \phi(z_0) \mid \phi \in \pi_1(H)\} = \{g \cdot u(z_0) \mid g \in \pi_2(H)\}$$

*is open and non-empty.*

The main step in Lemma 4.75 is the following. It uses the following version of the identity theorem for holomorphic curves.

**Fact.** Two closed holomorphic curves with non-identical images have only finitely many intersection points. □

**Lemma 4.76.** *If $G$ is finite, there exists an open dense subset $U \subset \Sigma$ such that $G_{u(z)} = G_u$ for all $p \in U$.*

*Proof.* For each $g \in G \setminus G_u$, by the identity theorem either $g \cdot u$ and $u$ has distinct images (thus, only intersect in finitely many points), or $g \cdot u$ is a reparametrisation of $u$. Each reparametrisation of $u$ has only finitely many fixed points (by the identity theorem again, since we consider closed holomorphic curves). Therefore, for all but finitely many $z \in \Sigma$, we have $G_u = G_{u(z)}$. □

*Proof of Lemma 4.75.* Since $u$ is simple, its set of injective points is open. Since $\mathcal{U}$ is open, the set of injective points of $u$ mapping to $\mathcal{U}$ is open. By hypothesis, there exists such an injective point $z_0 \in \Sigma$. By Lemma 4.76, we can assume $G_u = G_{u(z_0)}$. Now an easy computation proves $\mathrm{im}(u) \cap G \cdot u(z_0) = \{u \circ \phi(z_0) \mid \phi \in \pi_1(H)\}$. □

For later use in Chapter 5, let us also note the following phrasing of Lemma 4.75: the conclusion is exactly the same, just phrased in terms of $\pi_2(H)$ instead of $\pi_1(H)$.

**Lemma 4.77.** *Let $u \in \mathcal{M}_{\mathcal{U}}^{A,H}(J)$ be a simple curve. Then $u$ has a non-empty open set $S \subset u^{-1}(\mathcal{U})$ of injective points such that* $\operatorname{im}(u) \cap G \cdot u(s) = \pi_2(H) \cdot u(s)$ *for all $s \in S$.* □

The core of the proof of Lemma 4.74 is showing that $L$ has dense image: since $D_u$ is a Fredholm operator (by Lemma 4.82), by Lemma 4.88 below, $L$ always has closed image. We will show $\operatorname{im} L$ is dense using the Hahn–Banach theorem: otherwise, there exists a non-zero linear functional on it which annihilates $\operatorname{im} L$. To make this argument work, we need to understand such functionals better: we argue a functional on $(\mathcal{E}^H)_{(j,u,J_{\text{ref}})}$ can be extended to $\mathcal{E}_{(j,u,J_{\text{ref}})} = L^p(\overline{\operatorname{Hom}}_{\mathbb{C}}((T\Sigma, j), (u^*TM, J_{\text{ref}})))$, which even corresponds to an $H$-invariant $L^q$-section of $\overline{\operatorname{Hom}}_{\mathbb{C}}((T\Sigma, j), (u^*TM, J_{\text{ref}}))$. Henceforth, let us write $F := \overline{\operatorname{Hom}}_{\mathbb{C}}((T\Sigma, j), (u^*TM, J_{\text{ref}}))$ for brevity.

**Observation 4.78.** $H$ acts continously and linearly on $L^p(F)$ by $(\phi, g) \cdot \eta := d\psi_g \circ \eta \circ d\phi^{-1}$. □

For the ease of discussion, a section $\eta \in L^p(F)$ will be called *$H$-invariant* if and only if $h \cdot \eta = \eta$ for all $h \in H$. In other words, $(\mathcal{E}^H)_{(j,u,J_{\text{ref}})}$ consists of all $H$-invariant sections in $L^p(F)$.

The main tool for extending a linear map on $(\mathcal{E}^H)_{(j,u,J_{\text{ref}})}$ to $L^p(F)$ is an averaging operation: each section of $L^p(F)$ can be continously "averaged" to an $H$-invariant section.

**Proposition 4.79.** *There exists a continuous idempotent linear map* $\operatorname{av}\colon L^p(F) \to L^p(F)$ *with* $\operatorname{im} \operatorname{av} \subset (\mathcal{E}^H)_{(j,u,J_{\text{ref}})}$.

To convert a section of $F$ to one in $(\mathcal{E}^H)_{(j,u,J_{\text{ref}})}$, we need to make it $H$-invariant.

**Lemma 4.80.** *For all $1 \leq p' \leq \infty$, the map* $\operatorname{av}\colon L^{p'}(F) \to L^{p'}(F), \eta \mapsto \frac{1}{|H|} \sum_{h \in H} h \cdot \eta$ *is continuous and linear. Each section $\operatorname{av} \eta$ is $H$-invariant and $\operatorname{av}^H$ is idempotent.*

*Proof.* Since $H$ acts continuously linearly on $L^{p'}(F)$, each map $\eta \mapsto h \cdot \eta$ and thus av is continuous and linear. For each $\eta \in L^{p'}(F)$, the section $\operatorname{av} \eta$ is $H$-invariant: for all $k \in H$, we compute

$$k \cdot \operatorname{av} \eta = k \cdot \frac{1}{|H|} \sum_{h \in H} h \cdot \eta = \frac{1}{|H|} \sum_{h \in H} kh \cdot \eta = \frac{1}{|H|} \sum_{l \in H} l \cdot \eta = \operatorname{av} \eta.$$

Finally, $\operatorname{av}^H$ preserves $H$-invariant sections: if $\eta$ is $H$-equivariant, we compute

$$\operatorname{av} \eta = \frac{1}{|H|} \sum_{h \in H} h \cdot \eta = \frac{1}{|H|} \sum_{h \in H} \eta = \eta.$$

In particular, we deduce $\operatorname{im} \operatorname{av} \subset (\mathcal{E}^H)_{(j,u,J_{\text{ref}})}$ and $\operatorname{av} \circ \operatorname{av} = \operatorname{av}$. □

*Proof of Proposition 4.79.* Use the map av from the previous lemma. □

At a later point in the proof, we also need to know that av is $H$-invariant.

**Lemma 4.81** (Averaging is $H$-invariant)**.** *For all $h \in H$, we have* $\mathrm{av}(h \cdot \eta) = \mathrm{av}(\eta)$.

*Proof.* Indeed, for any $h \in H$ we compute

$$\mathrm{av}(h \cdot \eta) = \frac{1}{|H|} \sum_{k \in H} k \cdot (h \cdot \eta) = \frac{1}{|H|} \sum_{k \in H} (kh) \cdot \eta = \frac{1}{|H|} \sum_{l \in H} l \cdot \eta = \mathrm{av}^H(\eta). \qquad \square$$

As a final preparatory lemma, let us prove that the first summand of $L$ is a Fredholm operator. For later use, let us also record the Fredholm index of the restricted operator. Recall that if $\rho$ is a finite-dimensional $H$-representation, the multiplicity of the trivial representation in $\rho$ by $m_1^H(\rho)$.

**Lemma 4.82.** *The restricted operator $D_u^H \colon T_u\mathcal{B}^H \to L_H^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$ is Fredholm. Its Fredholm index is $m_1^H(\ker D_u) - m_1^H(\mathrm{coker}\, D_u)$. In particular, if $D_u^H$ is surjective, this simplifies to $\mathrm{ind}(D_u^H) = m_1^H(\ker D_u)$.*

*Proof.* By construction, $D_u^H$ is the restriction of the linearised Cauchy–Riemann operator $D_u \colon T_u\mathcal{B} = W_\theta^{1,p}(u^*TM) \to L^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$, which is Fredholm (e.g. [Wen15, Theorem 3.3.1]). We prove that

$$\ker D_u^H = \ker D_u \cap T_u\mathcal{B}^H \quad \text{and} \quad \mathrm{coker}\, D_u^H = \mathrm{coker}\, D_u \cap L_H^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM)).$$

Since the kernel and co-kernel of $D_u$ are both finite-dimensional, this will imply $D_u^H$ is also Fredholm.

The first relation is obvious since $D_u^H$ is the restriction of $D_u$. The second equation follows from a small trick. Since $D_u$ is $H$-equivariant, its kernel is $H$-invariant. Choose an $H$-invariant closed complement $V \subset W_\theta^{1,p}(u^*TM)$ of $\ker D_u$. Then $D_u$ restricts to an isomorphism from $V$ to $\mathrm{im}\, D_u$, and $\mathrm{im}\, D_u = D_u(V)$ in particular. The claim now follows by proving that

$$\mathrm{im}\, D_u^H = \mathrm{im}\, D_u \cap L_H^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM)).$$

Direction $\subseteq$ is obvious. For the converse implication $\supseteq$, consider any $w \in \mathrm{im}\, D_u \cap L_H^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$. Write $w = D_u v$, for $v \in V$. For each $h \in H$, we observe

$$D_u(h \cdot v) = h \cdot D_u v = h \cdot w = w$$

by $H$-equivariance of $D_u$, hence $h \cdot v - v \in \ker D_u$. Since $V$ is $H$-invariant, we have $h \cdot v - v \in V$ and $h \cdot v = v$ follows. We deduce $w = D_u v = D_u^H v \in \mathrm{im}\, D_u^H$.

This completes the proof that $D_u^H$ is Fredholm. For its Fredholm index, note that $\ker D_u^H = \ker D_u \cap T_u\mathcal{B}^H$ is precisely the fixed subspace of $\ker D_u$ under the $H$-action: thus, its dimension is precisely the multiplicity of the trivial representation in $\ker D_u$. The formula $\dim \mathrm{coker}\, D_u^H = m_1^H(\mathrm{coker}\, D_u)$ follows similarly. □

Now, we are prepared to prove Lemma 4.74. One inessential detail of the proof is the occurrence of the $C_\epsilon$-space, giving rise to the question "is this section of class $C_\epsilon$?" (which will be answered affirmatively by choosing $\epsilon$ to make this true). Let us ignore this detail for now and consider the extension of $L$ to *all* smooth $G$-equivariant perturbations of $J_{\text{ref}}$ first.

**Lemma 4.83.** *Suppose $u$ has an injective point which is mapped to $\mathcal{U}$, then the natural extension of $L$ to an operator $\tilde{L}\colon T_u\mathcal{B}^H \oplus \Gamma^G(\overline{\text{End}}_{\mathbb{C}}(TM, J_{ref}); \mathcal{U}) \to \mathcal{E}^H_{(j,u,J_{ref})}$ is surjective.*

*Proof.* By Lemma 4.82, the operator $D^H_u$ is Fredholm. Therefore, Lemma 4.88 below shows that $\text{im}(L)$ is closed and $\ker L$ has a closed complement. If $L$ is surjective, the existence of a bounded right inverse follows. Hence, it suffices to show that $L$ has dense image. Suppose otherwise, then by the Hahn–Banach theorem, there exists a functional $\alpha \in (\mathcal{E}^H_{(j,u,J_{\text{ref}})})^*$ with $\alpha \neq 0$ such that $\alpha|_{\text{im}\, L} = 0$.

**Claim 1.** *$\alpha$ extends to a linear functional $\hat{\alpha}$ on $L^p(F) = L^p(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$.*

*Proof.* We define by $\hat{\alpha}$ by combining $\alpha$ with averaging w.r.t. the $H$-action: for each section $\eta \in L^p(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$, we define $\hat{\alpha}(\eta) := \alpha(\text{av}\,\eta)$. By Proposition 4.79, this is well-defined and $\hat{\alpha}|_{\mathcal{E}^H_{(j,u,J_{\text{ref}})}} = \alpha$. △

Choosing a suitable $L^2$-pairing of $L^p$ and $L^q$, the functional $\hat{\alpha}$ corresponds to a section $\check{\alpha} \in L^q(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$. The construction of $\hat{\alpha}$ via averaged sections can be used to ensure $\check{\alpha}$ is $H$-invariant— provided we choose a pairing respecting the $H$-action. To make this precise, we begin by reviewing the setup for the pairing.

**Recall.** Let $(E, J) \to (\Sigma, j)$ be a complex vector bundle over a Riemann surface $(\Sigma, j)$, let $g$ be a Riemannian metric on $\Sigma$ compatible with $j$ and $\mu_g$ be the induced volume form on $\Sigma$. Any bundle metric $(,)_g$ on $\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, E)$ defines an $L^2$ inner product on $\Gamma(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, E))$ by $\langle \alpha, \beta\rangle_{L^2} := \int_\Sigma (\alpha, \beta)_g \,\mathrm{d}\mu_g$. The functional $\hat{\alpha} \in (L_p(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, E)))^*$ corresponds to a unique section $\check{\alpha} \in L^q(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, E))$ which satisfies $\hat{\alpha}(\eta) = \langle \eta, \hat{\alpha}\rangle_{L^2}$ for all $\eta$.

In our case, we consider the complex vector bundle $E = (u^*TM, J_{\text{ref}})$. Choose an $A$-invariant bundle metric on $\Sigma$ and a $G$-invariant bundle metric on $TM \to M$. These induce a Hermitian metric on $u^*TM$, and we readily verify that the induced $L^2$-pairing on $\Gamma(u^*TM)$ is $H$-invariant: for all $\eta, \xi \in \Gamma(E)$ and $h = (\phi, g) \in H$, we compute

$$
\begin{aligned}
\langle h\cdot\eta, h\cdot\eta\rangle_{L^2} &= \int_\Sigma ((h\cdot\eta)(z), (h\cdot\xi)(z))\,\mathrm{d}z \\
&= \int_\Sigma ((d\psi_g)_{u(z)}\eta(\phi^{-1}(z)), (d\psi_g)_{u(z)}\xi(\phi^{-1}(z)))\,\mathrm{d}z \\
&= \int_\Sigma (\eta(\phi^{-1}(z)), \xi(\phi^{-1}(z)))\,\mathrm{d}z = \int_\Sigma (\eta(\tilde{z}), \xi(\tilde{(z)}))\,\mathrm{d}\tilde{z} = \langle \eta, \xi\rangle_{L^2}
\end{aligned}
$$

using the $G$- and $A$-invariance of the metrics on $TM$ resp. $\Sigma$. Similarly, the induced $L^2$-pairing on $\Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$ is $H$-invariant. An easy computation shows that the formal adjoint $(D_u^H)^*$ of $D_u^H$ is then $H$-equivariant.

**Lemma 4.84.** *Suppose a group $H$ acts on normed space $X$ and $Y$. Suppose $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$ are non-degenerate $H$-invariant bilinear forms on $X$ and $Y$, respectively. Let $D\colon X \to Y$ be a bounded $H$-equivariant linear operator. Then the formal adjoint $D^*\colon Y \to X$ of $D$ with respect to $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$ is $H$-equivariant as well.*

*Proof.* Let $h \in H$ and $y \in Y$ be arbitrary; we need to show $D^*(h\cdot y) = h\cdot D^*y$. For all $x \in X$, writing $x = h\cdot\tilde{x}$, we compute

$$\begin{aligned}\langle x, h\cdot D^*y\rangle_X &= \langle \tilde{x}, D^*y\rangle_X = \langle D\tilde{x}, y\rangle_Y \\ &= \langle h\cdot D\tilde{x}, h\cdot y\rangle_Y = \langle D(h\cdot\tilde{x}), h\cdot y\rangle_Y = \langle x, D^*(h\cdot y)\rangle_X\end{aligned}$$

using $H$-invariance of the pairing, $H$-equivariance of $D$ and the adjoint property. Since $x$ was arbitrary and $\langle\cdot,\cdot\rangle_X$ is non-degenerate, we deduce $h\cdot D^*y = D^*hy$ as desired. □

Additionally, the section $\check{\alpha}$ is $H$-invariant.

**Lemma 4.85.** *The section $\check{\alpha}$ is $H$-invariant: for all $h \in H$ and $\eta \in L^p(F)$, we have $\check{\alpha}(h\cdot\eta) = \check{\alpha}\eta$. In particular, $\mathrm{av}\,\check{\alpha} = \check{\alpha}$.*

*Proof.* Let $h \in H$ be arbitrary. Since $(,)_g$ is $H$-invariant, for each $\eta \in L^p(F)$, we have $\langle \eta, h\cdot\check{\alpha}\rangle_{L^2} = \langle h^{-1}\cdot\eta, \check{\alpha}\rangle_{L^2}$. Using Lemma 4.81, we compute

$$\langle \eta, h\cdot\check{\alpha}\rangle_{L^2} = \langle h^{-1}\cdot\eta, \check{\alpha}\rangle_{L^2} = \hat{\alpha}(h^{-1}\cdot\eta) = \alpha(\mathrm{av}(h^{-1}\cdot\eta)) = \alpha(\mathrm{av}\,\eta) = \hat{\alpha}(\eta) = \langle \eta, \check{\alpha}\rangle_{L^2}$$

for all $\eta \in L^p(F)$, hence $\check{\alpha} = h\cdot\check{\alpha}$ follows. □

Recall that $\alpha|_{\mathrm{im}(L)} = 0$, i.e. $\langle L(\eta, Y), \check{\alpha}\rangle_{L^2} = 0$ for all sections $\eta \in T_u\mathcal{B}^H$ and $Y \in C^G_\epsilon(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U})$. Equivalently, that means

$$\begin{aligned}\langle D_u^H\eta, \check{\alpha}\rangle_{L^2} &= 0 \qquad \text{for all } \eta \in T_u\mathcal{B}^H\text{, and} \\ \langle Y \circ du \circ j, \check{\alpha}\rangle_{L^2} &= 0 \qquad \text{for all } Y \in C^G_\epsilon(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U}).\end{aligned}$$

In particular, the first relation is valid for all $H$-invariant vector fields $\eta$ of $u^*TM$ with $\eta|_\theta = 0$. This "almost" implies $\check{\alpha}$ is a weak solution of the formal adjoint equation $(D_u^H)^*\check{\alpha} = 0$ on $\Sigma \setminus \theta$: except that the first equation only holds for $H$-invariant sections. However, using the $H$-invariance of $D_u^H\eta$ (by Corollary 4.72) and $\check{\alpha}$, we conclude this for all sections. The $H$-invariance of $D_u^H$ implies, for any $\eta \in L^p(F)$,

$$\begin{aligned}D_u^H(\mathrm{av}\,\eta) &= D_u^H\Big(\frac{1}{|H|}\sum_{h\in H} h\cdot\eta\Big) = \frac{1}{|H|}\sum_{h\in H} D_u^H(h\cdot\eta) \\ &= \frac{1}{|H|}\sum_{h\in H} h\cdot(D_u^H\eta) = \mathrm{av}(D_u^H\eta).\end{aligned}$$

Next, we observe that the $H$-invarance of our $L^2$-pairing implies that averaging is compatible with formal adjoints. Because the map $\eta \mapsto h \cdot \eta$ is an isometry, for any two sections $\alpha \in L^p(F)$ and $\beta \in L^q(F)$ we have

$$\begin{aligned}\langle \operatorname{av}(\alpha), \beta \rangle_{L^2} &= \langle \frac{1}{|H|} \sum_{h \in H} h \cdot \alpha, \beta \rangle_{L^2} = \frac{1}{|H|} \sum_{h \in H} \langle h \cdot \alpha, \beta \rangle_{L^2} \\ &= \frac{1}{|H|} \sum_{h \in H} \langle \alpha, h^{-1} \cdot \beta \rangle_{L^2} = \langle \alpha, \frac{1}{|H|} \sum_{h \in H} h^{-1} \cdot \beta \rangle_{L^2} = \langle \alpha, \operatorname{av}(\beta) \rangle_{L^2}.\end{aligned}$$

Putting these together, we obtain the desired result. For any section $\eta \in W^{1,p}_\theta(u^*TM)$, its average $\operatorname{av}(\eta)$ is $H$-invariant, hence lies in $T_u\mathcal{B}^H$, so hence $0 = \langle D^H_u \operatorname{av}(\eta), \check{\alpha} \rangle_{L^2}$. Using the $H$-invariance of $D^H_u$, the $L^2$-pairings and $\check{\alpha}$, we deduce

$$0 = \langle D^H_u \operatorname{av} \eta, \check{\alpha} \rangle_{L^2} = \langle \operatorname{av}(D^H_u \eta), \check{\alpha} \rangle_{L^2} = \langle D^H_u \eta, \operatorname{av} \check{\alpha} \rangle_{L^2} = \langle D^H_u \eta, \check{\alpha} \rangle_{L^2},$$

thus $\langle \eta, (D^H_u)^* \check{\alpha} \rangle_{L^2} = 0$ for all $\eta \in W^{1,p}_\theta(u^*TM)$, in particular for any smooth compactly supported section on $\Sigma \setminus \theta$. Therefore, $\check{\alpha}$ is a solution of the formal adjoint equation $(D^H_u)^* \check{\alpha} = 0$ on $\Sigma \setminus \theta$. By standard elliptic regularity results, this means that $\check{\alpha}$ is smooth on $\Sigma \setminus \theta$; by the similarity principle, $\check{\alpha} \neq 0$ implies that $\check{\alpha}$ has only isolated zeroes.

**Claim 2.** *There exists a $G$-equivariant section $Y \in C^G_\epsilon(\overline{\operatorname{End}}_{\mathbb{C}}(TM, J_{ref}); \mathcal{U})$ such that $\langle Y(u) \circ du \circ j, \alpha \rangle > 0$.*

*Proof of claim.* Choose an injective point $z_0 \in \Sigma$ of $u$ as in Lemma 4.75, such that $z_0 \notin \theta$ and $\alpha(z_0) \neq 0$. (The last conditions both hold everywhere except on a finite set of points, hence can be ensured as well.) Using Lemma 4.89 below, we choose a smooth section $Y \in \Gamma(\overline{\operatorname{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U})$ whose value at $p := u(z_0)$ is chosen such that $Y(u) \circ du \circ j = \check{\alpha}$ at $z_0$ and $Y|_{G \cdot p}$ is $G$-equivariant. Note that this step uses the condition $G_{u(z_0)} = G_u$: the inclusion $\supset$ always holds (by Observation 3.12); if $G_{u(z_0)}$ were a superset of $G_u$, the $G$-equivariance of $Y$ would place a further constraint on the choice of $Y(u(z_0))$.
Averaging over the $G$-action, we may assume $Y$ is $G$-equivariant.

Consider the discrete set

$$S := \bigcup_{g \in G \setminus \pi_2(H)} \operatorname{im}(u) \cap \operatorname{im}(g \cdot u) \subset M.$$

By construction, $u(z_0) \notin S$. Using a $G$-invariant cut-off function, we may assume $Y$ is compactly supported, and in fact $\operatorname{supp} Y \subset G \cdot U_0$ for some open neighbourhood $U_0$ of $z_0$ such that $G \cdot U_0$ is disjoint from $S$. All in all, we obtain a $G$-equivariant smooth compactly supported section $Y$ with $(Y(u) \circ du \circ j(z), \check{\alpha}(z))_g > 0$ on a neighbourhood $U$ of $z_0$ such that $U \cap S = \emptyset$.

We claim that $\langle Y(u) \circ du \circ j, \check{\alpha} \rangle_{L^2} > 0$. Consider the function $f \colon \Sigma \to \mathbb{R}$, $f(z) := (Y \circ du \circ j(z), \check{\alpha}(z))_g$; we have $\langle Y \circ du \circ j, \check{\alpha} \rangle_{L^2} = \int_\Sigma f(z) \, d\mu_g$ by definition. Note that $f$ is smooth on $\Sigma \setminus \theta$ since each component is: $\check{\alpha}|_{\Sigma\setminus\theta}$ and $u$ are smooth by elliptic regularity; $Y$, $j$ and the pairing $(,)_g$ are smooth by construction.

The idea underlying the remaining computation is simple: away from the nowhere dense set $S$, the orbit of each point $p \in \mathrm{im}(u)$ is described by the group $\pi_1(H) \leqslant A$. By construction of $Y$, the integrand $f$ vanishes outside of $u^{-1}(G \cdot U_0)$, and we know $f|_{U_0} > 0$ by construction. The final piece in the puzzle is the invariance of $f$ under the action of $\pi_1(H)$.

**Lemma 4.86.** *We have $f \circ \phi = f$ for all $\phi \in \pi_1(H)$.*

*Proof.* Let $z \in \Sigma$ be arbitrary. For $(\phi, g) \in H$, we compute

$$\begin{aligned} Y \circ du \circ j(\phi(z)) \circ d\phi_z &= Y \circ du \circ d\phi_z \circ \underbrace{d\phi_z^{-1} \circ j(\phi(z)) \circ d\phi_z}_{(\phi^* j)_z} \\ &= Y \circ d\psi_g \circ d\psi_g^{-1} \circ du \circ d\phi_z \circ \phi^* j(z) \\ &= d\psi_g \circ Y \circ d(\psi_g^{-1} \circ u \circ \phi) \circ \phi^* j(z)(z) \\ &= d\psi_g \circ Y \circ du \circ j(z) \end{aligned}$$

using $G$-equivariance of $Y$, $A$-invariance of $j$ and $H$-invariance of $u$. Thus, we have $Y \circ du \circ j(\phi(z)) = d\psi_g \circ Y \circ du \circ j \circ d\phi^{-1}(z)$.

Similarly, since $\check{\alpha}$ is $H$-invariant, we compute

$$\check{\alpha}(\phi(z)) \circ d\phi_z = d\psi_g \circ d\psi_g^{-1} \circ \check{\alpha}(\phi(z)) \circ d\phi_z = d\psi_g \circ \check{\alpha}(z),$$

thus $\check{\alpha}(\phi(z)) = d\psi_g \circ \check{\alpha}(z) \circ d\phi_z^{-1}$. Combining these and using the $H$-invariance of $(,)_g$, we obtain

$$\begin{aligned} f(\phi(z)) &= (Y \circ du \circ j(\phi(z)), \check{\alpha}(\phi(z)))_g \\ &= (d\psi_g \circ (Y \circ du \circ j)(z) \circ d\phi_z^{-1}, d\psi_g \circ \check{\alpha}(z) \circ d\phi_z^{-1})_g \\ &= (Y \circ du \circ j(z), \check{\alpha}(z))_g = f(z). \end{aligned}$$

△

Since $f = 0$ on $S$ and $\mathrm{supp}(Y) \subset G \cdot U_0$ by construction of $Y$, we compute

$$\langle Y \circ du \circ j, \check{\alpha} \rangle_{L^2} = \int_\Sigma f(z) \, \mathrm{d}\mu_g = \int_{\Sigma\setminus S} f(z) \, \mathrm{d}\mu_g = \int_{u^{-1}(G\cdot U_0)} f \, \mathrm{d}\mu_g.$$

Since $S$ and $U_0$ are disjoint, every point in $z \in u^{-1}(G \cdot U_0)$ satisfies

$$G \cdot u(z) \cap \mathrm{im}(u) = \{u \circ \phi(z) \mid \phi \in \pi_1(H)\}$$

by Lemma 4.75. Therefore, $\int_{u^{-1}(G\cdot U_0)} f \, \mathrm{d}\mu_g = \int_{U_0} \sum_{\phi\in\pi_1(H)} f(\phi(z)) \, \mathrm{d}\mu_g$. By Lemma 4.86, we have $f(\phi(z)) = f(z)$, hence

$$\langle Y \circ du \circ j, \check{\alpha} \rangle_{L^2} = |\pi_2(H)| \int_{U_0} f(z) \, \mathrm{d}\mu_g > 0.$$

△

The last claim contradicts the second equation, hence $L$ is indeed surjective. This completes the proof of Lemma 4.83. □

**Remark 4.87.** This proof is where our chosen definition of the iso-symmetric strata was really important: had we considered merely the stabiliser w.r.t. the $G$-action, the pairing $\langle Y \circ du \circ j, \breve{\alpha} \rangle_{L^2}$ could have picked up further contributions along the orbit $G \cdot u(z_0)$, spoiling the argument.
See also [Bar24, Proposition 2.2.10] for an alternative proof of this lemma (in the non-equivariant case).

In the proof of Lemma 4.83, we used the two standard results. The first is a standard exercise in functional analysis.

**Lemma 4.88** ([Wen20, Exercise 8.11])**.** *Given Banach spaces $X$, $Y$ and $Z$, a Fredholm operator $T\colon X \to Y$ and a bounded linear operator $A\colon Z \to Y$, consider the operator $L\colon X \oplus Z \to Y, (x, z) \mapsto Tx + Az$. Then the kernel of $L$ has a closed complement in $X \oplus Z$, and the image of $L$ is closed in $Y$.* □

Secondly, we also used an elementary, but slightly non-trivial lemma from linear algebra. Recall that for a symplectic vector space $(V, \omega)$ with complex structure $J$, one can choose a basis to identify $J$ with $i$ and $\omega$ with the standard symplectic structure $\omega_{\text{std}}$ on $\mathbb{R}^{2n}$. Under this identification, the linear maps $Y$ which anti-commute with $i$ and satisfy $\omega_{\text{std}}(\cdot, Y\cdot) + \omega_{\text{std}}(Y\cdot, \cdot) = 0$ are precisely the symmetric complex anti-linear matrices.

**Lemma 4.89** ([MS12, Lemma 3.2.2])**.** *For any non-zero vectors $v, w \in \mathbb{R}^{2n}$, there exists a symmetric matrix $Y$ which anti-commutes with $i$ and satisfies $Yv = w$.* □

Let us upgrade this to a theorem about the actual operator $L$, hence about the universal moduli space. The following definition, taken from Wendl's blog [Wen21], will be useful.

**Definition 4.90** ($\epsilon$-regular curve)**.** *An element $(u, J) \in \mathcal{U}^*(\mathcal{J}_\epsilon)$ is called* $\epsilon$-regular *if and only if the operator $L$ defined by (4.6) is surjective. Given any almost complex structure $J$, a curve $u \in \mathcal{M}(J)$ is called* $\epsilon$-regular *if $J \in \mathcal{J}_\epsilon$ and the pair $(u, J)$ is $\epsilon$-regular.*[23]

Clearly, $\epsilon$-regularity is an open condition, so

$$\mathcal{U}^*_{\text{reg}}(\mathcal{J}_\epsilon) := \{(u, J) \in \mathcal{U}^*(\mathcal{J}_\epsilon) \mid (u, J) \text{ is } \epsilon\text{-regular}\}$$

is an open subset of $\mathcal{U}^*(\mathcal{J}_\epsilon)$. So far, it could be empty: taking a close look at the previous proof shows it *never* is.

**Lemma 4.91** ([Wen21])**.** *For any given curve $u \in \mathcal{M}(J_{ref})$, for all $\epsilon \in \boldsymbol{\mathcal{E}}$ with sufficiently rapid decay, $u$ is $\epsilon$-regular.*

[23] For the experts, we note that an $\epsilon$-regular curve $u \in \mathcal{M}(J)$ need not be Fredholm regular, as the latter is a smoothness condition concerning a neighbourhood of $u$ in the moduli space $\mathcal{M}(J)$, while $\epsilon$-regularity is about $(u, J)$ in the universal moduli space.

*Proof.* Lemma 4.83 shows that $\tilde{L}\colon T_u\mathcal{B} \times \Gamma^G(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U}) \to \mathcal{E}$ has dense image. Since $\mathcal{E}^H_{(j,u,J_{\mathrm{ref}})} = L^p(F)$ is a separable Banach space, we may choose a dense sequence $(\xi_k)$ in $\mathcal{E}^H_{(j,u,J_{\mathrm{ref}})}$ together with a sequence $(\eta_k, Y_k) \in T_j\mathcal{B}^H \oplus \Gamma^G(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}))$ such that $\tilde{L}(\eta_k, Y_k) = \xi_k$ for all $k$. By Lemma 4.11, for some $\epsilon \in \boldsymbol{\mathcal{E}}$ of sufficiently rapid decay, all the sections $Y_k$ are of class $C_\epsilon$, hence are in $C^G_\epsilon(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U})$. Hence, for such epsilon, the image of $L$ is dense. Since it is closed, the result follows. □

Finally, a change of quantifiers upgrades this to the statement we need. Let us recall Lemma 4.74 for convenience.

**Lemma 4.92.** *For $\epsilon \in \boldsymbol{\mathcal{E}}$ decaying sufficiently fast, if $u$ has an injective point which is mapped to $\mathcal{U}$, then $L$ is surjective and has a bounded right inverse.*

*Proof.* Since $T_j\mathcal{T} \times T_u\mathcal{B}^H \oplus C^G_\epsilon(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U})$ is a separable metrizable space, so is the zero set $(\overline{\partial}^H_{J_{\mathrm{ref}}})^{-1}(0) \subset C^G_\epsilon(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U})$. In particular, it is second-countable and Lindelöf: every open cover has a countable sub-cover. Since $\epsilon$-regularity is an open condition, we apply the preceding Lemma 4.91: to each $(j, u, J) \in (\overline{\partial}^H_{J_{\mathrm{ref}}})^{-1}(0)$, we associate some $\epsilon^u$ and an open neighbourhood $U_u \subset (\overline{\partial}^H_{J_{\mathrm{ref}}})^{-1}(0)$ such that each $(v, J) \in U_u$ is $\epsilon^u$-regular. Choose a sequence $(u_k)$ in $(\overline{\partial}^H_{J_{\mathrm{ref}}})^{-1}(0)$ such that the open sets $U_{u_k}$ still cover $(\overline{\partial}^H_{J_{\mathrm{ref}}})^{-1}(0)$. Then, the lemma holds for any $\epsilon$ which is a lower bound for the countable set $\{\epsilon^{u_k}\}$. □

We conclude this section by proving Proposition 4.66.

*Proof of Proposition 4.66.* A neighbourhood of $(j, u, J)$ in the universal moduli space $\mathcal{U}^*(\mathcal{J}_\epsilon)$ is described by the zero set of $\overline{\partial}^H_J$. By Lemma 4.74, for all $\epsilon$ decaying sufficiently rapidly, the operator $L$ is surjective at each $(u, J) \in \mathcal{U}^*(\mathcal{J}_\epsilon)$, hence in particular $D\overline{\partial}^H_J$ is surjective. Since $\mathcal{T}$ is finite-dimensional, $T_u\mathcal{B}^H \oplus C^G_\epsilon(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U})$ is a closed subspace in $T_j\mathcal{T} \oplus T_u\mathcal{B}^H \oplus C^G_\epsilon(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J_{\mathrm{ref}}); \mathcal{U})$ and $D\overline{\partial}^H_J$ has a bounded right inverse. Thus, the implicit function theorem proves that a neighbourhood of $(j, u, J)$ in $(\overline{\partial}^H_J)^{-1}(0)$ is a smooth Banach manifold of $\mathcal{T} \times \mathcal{B}^H \times \mathcal{J}_\epsilon$.

Now, standard arguments (similar to e.g. [Wen15, Theorem 4.3.6]) imply that $\mathcal{U}^*(\mathcal{J}_\epsilon)$ is a smooth Banach manifold: this exploits that each curve in the universal moduli space is simple. The zero set $(\overline{\partial}^H_J)^{-1}(0) \subset \mathcal{T} \times \mathcal{B}^H \times \mathcal{J}_\epsilon$ is a closed subset of a separable metrisable space, hence also separable and metrisable: therefore, so is $\mathcal{U}^*(\mathcal{J}_\epsilon)$. Finally, under the above identification, the projection $\mathcal{U}^*(\mathcal{J}_\epsilon) \to \mathcal{J}_\epsilon, (u, J) \mapsto J$ is the restriction of the smooth map $\mathcal{T} \times \mathcal{B} \times \mathcal{J}_\epsilon \to \mathcal{J}_\epsilon, (j, u, J) \mapsto J$, hence smooth as well. □

## 4.5. Smoothness of $\mathcal{M}^{A,H}_{\mathcal{U}}(J)$

In this section, we use smoothness of the universal moduli space to prove smoothness of each $\mathcal{M}^{A,H}_{\mathcal{U}}(J)$, for generic $J \in \mathcal{J}_\epsilon$. As mentioned before, the starting point is Smale's generalisation of the Morse–Sard theorem to smooth Banach manifolds.

**Theorem 4.93** (Sard–Smale theorem, [Sma65])**.** *Suppose $X$ and $Y$ are smooth separable paracompact Banach manifolds. Suppose $f\colon X \to Y$ is a smooth map whose differential $df_x\colon T_xX \to T_{f(x)}Y$ is a Fredholm operator, for every $x \in X$. Then the regular values of $f$ form a co-meagre subset of $Y$.* $\square$

The following observation shows how the Sard–Smale theorem is useful for our purposes.

**Lemma 4.94.** *Each linearisation $d\pi(u, J)$ of the canonical projection $\mathcal{U}^*(\mathcal{J}_\epsilon) \ni (u, J) \mapsto J \in \mathcal{J}_\epsilon$ is a Fredholm operator. If $J$ is a regular value of $\pi$, the pair $(u, J) \in \mathcal{U}^*(\mathcal{J}_\epsilon)$ is $\epsilon$-regular.*

*Proof.* Let $(u, J) \in \mathcal{U}^*(\mathcal{J}_\epsilon)$. We prove that the linearisation $d\pi(u, J)$ is a Fredholm operator with the same index as the linearisation (4.5) that defines $H$-Fredholm regularity: therefore, every regular value of $\pi$ belongs to $\mathcal{J}_{\text{reg}}$. In the local identification of $\mathcal{U}^*(\mathcal{J}_\epsilon)$ with $(\overline{\partial}_J^H)^{-1}(0)$, the smooth projection $\pi$ has derivative at $(u, J)$ equivalent to the linear projection

$$\ker D\overline{\partial}_J(j, u, J) \to T_J\mathcal{J}_\epsilon, (y, \eta, Y) \mapsto Y.$$

This yields a natural identification of $\ker d\pi(u, J)$ with the kernel of the operator

$$D\overline{\partial}(j, u)\colon T_j\mathcal{T} \oplus T_u\mathcal{B}^H \to L^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$$

which is used to define $H$-Fredholm regularity. The following general fact from linear functional analysis shows that the co-kernels of $d\pi(u, J)$ and $D\overline{\partial}_J(j, u)$ are also isomorphic: thus, both are Fredholm operators, with the same index. $\square$

**Lemma 4.95** ([Wen15, Lemma 4.4.13])**.** *Suppose $X$, $Y$ and $Z$ are Banach spaces, $D\colon X \to Z$ is a Fredholm operator, $A\colon Y \to Z$ is another bounded linear operator and $L\colon X \oplus Y \to Z, (x, y) \mapsto Dx + Ay$ is surjective. Then the projection $\Pi\colon \ker L \to Y, (x, y) \mapsto y$ is a Fredholm operator, and there are natural isomorphisms $\ker \Pi \cong \ker D$ and $\operatorname{coker} \Pi \cong \operatorname{coker} D$.* $\square$

Therefore, the Sard–Smale theorem shows the following.

**Corollary 4.96.** *For all $\epsilon$ decaying sufficiently rapidly, there exists a co-meagre set $\mathcal{J}_{reg} \subset \mathcal{J}_\epsilon$ such that $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$ is a smooth manifold for all $J \in \mathcal{J}_{reg}$.* $\square$

Each $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$ is finite-dimensional with explicitly given dimension.

**Lemma 4.97.** *For $J \in \mathcal{J}_{reg}$, the dimension of $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$ near $[(j, \theta, u)]$ is $m_1^A(\operatorname{coker} D_{(j,\theta)}) + m_1^H(\ker D_u)$.*

*Proof.* Let $J \in \mathcal{J}_{\text{reg}}$ be arbitrary. Then each $u \in \mathcal{M}_{\mathcal{U}}^{A,H}(J)$ is $H$-Fredholm regular. Thus, Lemma 4.64 shows that $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$ is a finite-dimensional manifold, whose dimension near $[u] \in \mathcal{M}_{\mathcal{U}}^{A,H}(J)$ is $\operatorname{ind}(D\overline{\partial}_J(j,u))$. Let $[u] \in \mathcal{M}_{\mathcal{U}}^{A,H}(J)$ be arbitrary. Recall that $D\overline{\partial}_J(j,u)(0,\eta) = D_u^H\eta$: since $T_j\mathcal{T}$ is finite-dimensional, we have

$$\dim \mathcal{M}_{\mathcal{U}}^{A,H}(J) = \operatorname{ind} D\overline{\partial}_J(j,u) = \dim T_j\mathcal{T} + \operatorname{ind}(D_u^H).$$

For the first summand, Lemma 4.30 shows $\dim T_j\mathcal{T} = m_1^A(\operatorname{coker} D_{(j,\theta)})$. Since $u$ is $H$-Fredholm regular, the operator $D_u^H$ is surjective, and Lemma 4.82 implies $\operatorname{ind}(D_u^H) = m_1^H(\ker D_u)$. Combining these, we obtain

$$\dim \mathcal{M}_{\mathcal{U}}^{A,H}(J) = \dim T_j\mathcal{T} + \operatorname{ind}(D_u^H) = m_1^A(\operatorname{coker} D_{(j,\theta)}) + m_1^H(\ker D_u). \qquad \square$$

**Remark 4.98.** If $A$ and $H$ are the trivial group, this reduces to the virtual dimension $(2-2g)(n-3) + 2c_1(C) + 2m$ of the non-equivariant setting. Since $A$ is finite, the operator $D_{(j,\theta)}$ is injective [Wen15, Proposition 4.2.12], thus we have

$$m_1^A(\operatorname{coker} D_{(j,\theta)}) = \dim\operatorname{coker} D_{(j,\theta)} = -\operatorname{ind} D_{(j,\theta)} = -(3\chi(\Sigma) - 2m) = 6 - 6g + 2m.$$

Since $H$ is trivial, $D_u^H = D_u$; since $D_u^H$ is surjective by $H$-Fredholm regularity of $u$,

$$m_1^H(\ker D_u) = \dim\ker D_u = -\operatorname{ind} D_u = n\chi(\Sigma) + 2c_1(C);$$

altogether, we obtain

$$\begin{aligned}\dim \mathcal{M}_{\mathcal{U}}^{A,H}(J) &= m_1^A(\operatorname{coker} D_{(j,\theta)}) + m_1^H(\ker D_u) = 6 - 6g + 2m + n(2-2g) + 2c_1(C) \\ &= (2-2g)(n-3) + 2c_1(C) + 2m = \text{vir-dim}(u).\end{aligned}$$

Lemma 4.97 is still a statement about the $C_\epsilon$-space. We can easily deduce that a *dense* subset of *all* equivariant tame (resp. compatible) almost complex structures is regular.

**Corollary 4.99.** *There exists a dense subset $\mathcal{J}_{reg} \subset \mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{fix})$ (resp. $\mathcal{J}_{reg} \subset \mathcal{J}^G(M,\omega;\mathcal{U},J_{fix})$) such that $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$ is a smooth finite-dimensional manifold for all $J \in \mathcal{J}_{reg}$.*

*Proof.* Let us prove the result for $\mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{\text{fix}})$; the compatible case is analogous. Let $J_0 \in \mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{\text{fix}})$ be arbitrary; we consider the space $\mathcal{J}_\epsilon$, defined w.r.t. $J_{\text{ref}} = J_0$, for some $\epsilon$ as in Corollary 4.96. Since co-meagre subsets are also dense, the set $\mathcal{J}_{\text{reg}}$ of regular values contains arbitrarily close approximations to $J_0$ in the $C_\epsilon$-topology, and therefore also in the $C^\infty_{\text{loc}}$-topology. Since $J_0$ was chosen arbitrarily, this implies that $\mathcal{J}_{\text{reg}}$ is dense in $\mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{\text{fix}})$. $\square$

The final detail to take care of is upgrading this to a co-meagre subset, i.e. proving that $\mathcal{J}_{\text{reg}}$ is the intersection of countably many open dense subsets. We use an argument originally due to Taubes [Tau96, Section 5]. This depends on the fact that the moduli space of somewhere injective curves, and the sets $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$, can be exhausted — in a way which depends continuously on $J$ — by a countable collection of compact sets. Note that the notion of convergence in $\mathcal{M}(J)$ does not depend on $J$ in an essential way: hence, we can speak of a sequence of curves $u_k \in \mathcal{M}(J_k)$ converging, where $J_k \in \mathcal{J}(M)$ are allowed to be different almost complex structures. The key step we need is the following.

**Lemma 4.100.** *For every $J \in \mathcal{J}(M)$ and every positive real number $c > 0$, there exists a subset $\mathcal{M}_{\mathcal{U}}^{A,H}(J, c) \subset \mathcal{M}_{\mathcal{U}}^{A,H}(J)$ with the properties*

(1) *Every curve in $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$ belongs to $\mathcal{M}_{\mathcal{U}}^{A,H}(J, c)$ for some $c > 0$.*

(2) *For each $c > 0$ and any sequence $J_k \to J \in \mathcal{J}(M)$, every sequence $u_k \in \mathcal{M}_{\mathcal{U}}^{A,H}(J_k, c)$ has a subsequence converging to an element of $\mathcal{M}_{\mathcal{U}}^{A,H}(J, c)$.*

Let us postpone the proof of this lemma for a moment, and prove how to find a co-meagre set $\mathcal{J}_{\text{reg}}$ first. Again, the argument is exactly the same for tame and compatible almost complex structures; let us just write down the tame case. For each $c > 0$, we define an open and dense subset $\mathcal{J}_{\text{reg}}^c \subset \mathcal{J}_\tau^G(M, \omega; \mathcal{U}, J_{\text{fix}})$, by saying $J \in \mathcal{J}_\tau^G(M, \omega; \mathcal{U}, J_{\text{fix}})$ belongs to $\mathcal{J}_{\text{reg}}^c$ if and only if every curve $u \in \mathcal{M}_{\mathcal{U}}^{A,H}(J, c)$ is $H$-Fredholm regular. Then, $\mathcal{J}_{\text{reg}} := \bigcap_{n\in\mathbb{N}} \mathcal{J}_{\text{reg}}^{1/n} \subset \mathcal{J}_\tau^G(M, \omega; \mathcal{U}, J_{\text{fix}})$ is a co-meagre subset such that for every $J \in \mathcal{J}_{\text{reg}}$, every curve $u \in \mathcal{M}_{\mathcal{U}}^{A,H}(J)$ is $H$-Fredholm regular, and $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$ is a smooth finite-dimensional manifold.

Each set $\mathcal{J}_{\text{reg}}^c$ clearly contains $\mathcal{J}_{\text{reg}}$, and is therefore dense. To prove that $\mathcal{J}_{\text{reg}}^c$ is open, we argue by contradiction. If $\mathcal{J}_{\text{reg}}^c$ is not open, there exists a sequence $J_k \in \mathcal{J}_\tau^G(M, \omega; \mathcal{U}, J_{\text{fix}}) \setminus \mathcal{J}_{\text{reg}}^c$ which converges to some $J \in \mathcal{J}_{\text{reg}}^c$. Then, there also exists a sequence $u_k \in \mathcal{M}_{\mathcal{U}}^{A,H}(J_k, c)$ of curves which are not $H$-Fredholm regular. By property (2) of the sets $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$, a subsequence of $(u_k)$ converges to some $u \in \mathcal{M}_{\mathcal{U}}^{A,H}(J, c)$. Since $J \in \mathcal{J}_{\text{reg}}^c$, the curve $u$ is $H$-Fredholm regular. But $H$-Fredholm regularity is an open condition, so some $u_k$ must also be $H$-Fredholm regular, contradiction!

Let us now prove Lemma 4.100. To begin, we recall the corresponding fact in the standard setting (without a symplectic group action on $(M, \omega)$).

**Lemma 4.101** (e.g. [Wen15, Lemma 4.4.14])**.** *For every $J \in \mathcal{J}(M)$, there exists a collection of subsets $\{\mathcal{M}(J, c)\}_{c\in\mathbb{R}^+}$ of $\mathcal{M}(J)$ with the properties*

- *For all $J \in \mathcal{J}(M)$ and $c < c'$, we have $\mathcal{M}(J, c') \subset \mathcal{M}(J, c)$.*[24]
- *Every curve in $\mathcal{M}(J)$ with an injective point mapped into $\mathcal{U}$ belongs to $\mathcal{M}(J, c)$ for some $c > 0$.*
- *For each $c > 0$ and any sequence $J_k \to J \in \mathcal{J}(M)$, every sequence $u_k \in \mathcal{M}(J_k, c)$ has a subsequence converging to an element of $\mathcal{M}(J, c)$.* □

To prove Lemma 4.100, we improve the construction in Lemma 4.101 for our purposes. For motivation, let us first consider the possible sources of non-compactness: suppose $(J_k)$ is a sequence in $\mathcal{J}(M)$ converging to $J$ and a sequence $[(\Sigma, j_k, \theta, u_k)] \in \mathcal{M}_{\mathcal{U}}^{A,H}(J_k)$ has no subsequence converging to an element of $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$. Then, one of the following must happen:

[24] This property is usually not mentioned explicitly, but is implicitly proven in e.g. the proof referenced above.

(1) there is bubbling or the domains $(\Sigma, j_k, \theta)$ converge to a non-smooth element of Deligne-Mumford space

(2) some subsequence converges to some curve $[u] \in \mathcal{M}(J)$, but no subsequence converges to a curve $u$ with a somewhere injective point mapped into $\mathcal{U}$

(3) the limit $[(\Sigma, j, \theta, u)]$ satisfies $A \subsetneq \mathrm{Aut}(\Sigma, j, \theta)$.
(The inclusion $\subset$ always holds, by continuity.)

(4) the limit $[(\Sigma, j, \theta, u)]$ lies in $\mathcal{M}^A(J)$, but the stabiliser subgroup $(A \times G)_u$ is *larger* than $H$

If we choose the subsets $\mathcal{M}_{\mathcal{U}}^{A,H}(J, c) := \mathcal{M}_{\mathcal{U}}^{A,H}(J) \cap \mathcal{M}(J, c)$ for all $J \in \mathcal{J}(M)$ and $c > 0$, possibility (1) is already excluded. The standard construction of $\mathcal{M}(J, c)$ also includes a condition about having an injective point "at least distance $c$ away from the boundary of $\mathcal{U}$", preventing item (2) from happening: for clarity, we explicitly add this to our definition of $\mathcal{M}_{\mathcal{U}}^{A,H}(J, c)$ below. It remains to add conditions excluding the last two scenarios.

Let us explain the condition concerning the group $A$ separately. Choose a metric $d$ metric on $\mathcal{J}(\Sigma)$. For brevity, we will write Aut$(j)$ (instead of Aut$(\Sigma, j, \theta)$), leaving $\Sigma$ and $\theta$ understood. For any closed subgroup $A \leqslant \mathrm{Diff}_+(\Sigma, \theta)$, let us abbreviate

$$\mathcal{J}(\Sigma)_A := \{j \in \mathcal{J}(\Sigma) \mid \mathrm{Aut}(\Sigma, j_0, \theta) = A\}.$$

For each real number $c > 0$, consider the set $\mathcal{J}(\Sigma)^{A,c} \subset \mathcal{J}(\Sigma)_A$ given by

$$\mathcal{J}(\Sigma)^{A,c} := \{j_0 \in \mathcal{J}(\Sigma)_A \mid \forall j \in B_c(j_0), A \subset \mathrm{Aut}(\Sigma, j, \theta) \text{ implies } \mathrm{Aut}(\Sigma, j, \theta) = A\}.$$

Intuitively speaking, $\mathcal{J}(\Sigma)^{A,c}$ consists of all $A$-invariant complex structures $j \in \mathcal{J}(\Sigma)$ which are "not close to" having automorphism group larger than $A$. The arguments from the previous chapter imply that the sets $\mathcal{J}(\Sigma)^{A,c}$ indeed cover $\mathcal{J}(\Sigma)_A$.

**Lemma 4.102.** *For every closed subgroup $A \leqslant \mathrm{Diff}(\Sigma, \theta)$, we have $\mathcal{J}(\Sigma)_A = \bigcup_{c>0} \mathcal{J}(\Sigma)^{A,c}$.*

*Proof.* The inclusion $\supset$ holds by construction. For $\supset$, let $j \in \mathcal{J}(\Sigma)_A$ be arbitrary. By Lemma 4.28, $j$ has a neighbourhood $U \subset \mathcal{J}(\Sigma)$ such that for all $j' \in U$, the automorphism group Aut$(j')$ is conjugate to a subgroup of Aut$(j)$. Choose $c > 0$ so $B_c(j) \subset U$; we claim that $j \in \mathcal{J}(\Sigma)^{A,c}$. Indeed, let $j' \in B_c(j)$ be any $A$-invariant complex structure. Since $j' \in U$, Aut$(j')$ is conjugate to a subgroup of Aut$(j) = A$. As $A \subset \mathrm{Aut}(j')$, in fact equality must hold. This proves that $j \in \mathcal{J}(\Sigma)^{A,c}$. □

The following result is the core of why this definition is a useful criterion to apply Taubes' trick.

**Lemma 4.103.** *If $(j_k)$ is a sequence in $\mathcal{J}(\Sigma)^{A,c}$ converging to some $j \in \mathcal{J}(\Sigma)$, then $j \in \mathcal{J}(\Sigma)^{A,c}$.*

*Proof.* Suppose $(j_k)$ is a sequence in $\mathcal{J}(\Sigma)^{A,c}$ converging to some $j_0 \in \mathcal{J}(\Sigma)$; we need to show $j_0 \in \mathcal{J}(\Sigma)^{A,c}$. Observe that the set of $A$-invariant complex structures in $\mathcal{J}(\Sigma)$ is closed: hence, $A \subset \mathrm{Aut}(\Sigma, j_0, \theta)$. In fact, this inclusion is an equality: we have $j_0 \in B_c(j_k)$ for some $k$ (in fact, any sufficiently large $k$). Since $j_k \in \mathcal{J}(\Sigma)^{A,c}$, by definition of $\mathcal{J}(\Sigma)^{A,c}$ we conclude $\mathrm{Aut}(\Sigma, j_0, \theta) = A$.

If $j_0 \notin \mathcal{J}(\Sigma)^{A,c}$, some $j \in \mathcal{J}(\Sigma)$ would satisfy $c' := d(j_0, j) < c$ and $A \subsetneq \mathrm{Aut}(\Sigma, j, \theta)$. Choose $k$ sufficiently large so $d(j_k, j_0) < c - c'$, then the triangle inequality implies $d(j_k, j) < c$. But then $j_k \in \mathcal{J}(\Sigma)^{A,c}$ implies $\mathrm{Aut}(j) = A$, contradiction! □

*Proof of Lemma 4.100.* Choose subsets $\mathcal{M}(J, c) \subset \mathcal{M}(J)$ for each $J \in \mathcal{J}(M)$ and $c > 0$ as in Lemma 4.101. To define the sets $\mathcal{M}_{\mathcal{U}}^{A,H}(J, c) \subset \mathcal{M}_{\mathcal{U}}^{A,H}(J)$, let us introduce some notation. Fix some metric on $M$ and consider the induced Hausdorff distance $d_H$ of two subsets $S, T \subset M$. For all $A$, $H$ and $c > 0$, consider the set $S(A, H, c)$ of all parametrised curves $(\Sigma, j, \theta, u) \in \widetilde{\mathcal{M}}^{A,H}(J)$ such that[25]

(1) $j \in \mathcal{J}(\Sigma)^{A,c}$

(2) $d_H(\mathrm{im}\, u, \mathrm{im}\, h \cdot u) \geq c$ for all $h \in A \times G \setminus (A \times G)_u$, and

(3) "$u$ is not close to losing an injective point mapped to $\mathcal{U}$": there exists a point $z_0 \in \Sigma$ such that $d_H(u(z_0), M \setminus \mathcal{U}) \geq c$, $\|du(z_0)\| \geq c$ and $\inf_{z \in \Sigma \setminus \{z_0\}} \frac{d_H(u(z_0), u(z))}{d_H(z_0, z)} \geq c$.

In particular, the last condition ensures that $u \in S(A, H, c)$ has an injective point which is mapped into $\mathcal{U}$. The set $S(A, H, c)$ contains the essential conditions to add to the sets $\mathcal{M}(J, c)$ to make them apply in our setting: for each $J \in \mathcal{J}(M)$ and $c > 0$, we define

$$\mathcal{M}_{\mathcal{U}}^{A,H}(J, c) := \mathcal{M}(J, c) \cap \{[u] \mid u \in S(A, H, c)\}. \tag{4.7}$$

We claim these sets $\mathcal{M}_{\mathcal{U}}^{A,H}(J, c)$ enjoy the properties we want. For the first condition, let $(\Sigma, j, \theta, u) \in \widetilde{\mathcal{M}}_{\mathcal{U}}^{A,H}(J)$ be arbitrary; we show $[u] \in \mathcal{M}_{\mathcal{U}}^{A,H}(J, c)$ for some $c > 0$. Since $\mathrm{Aut}(\Sigma, j, \theta) = A$ by definition, we have $j \in \mathcal{J}(\Sigma)_A$, thus Lemma 4.102 implies $j \in \mathcal{J}(\Sigma)^{A,c_1}$ for some $c_1 > 0$. Moreover, for each $h \in A \times G \setminus (A \times G)_u$, the images of $u$ and $h \cdot u$ are distinct closed subsets of $M$, hence have positive Hausdorff distance; thus $c_2 := \min\{d_H(\mathrm{im}(u), \mathrm{im}(h \cdot u)) | h \in A \times G \setminus (A \times G)_u\}$ is positive.

By definition, $u$ has an injective point $z_0$ mapped into $\mathcal{U}$. Let us show that $u$ satisfies item (3) of $S(A, H, c')$ for some $c' > 0$. This argument is exactly the same as in standard proofs of Lemma 4.101. Denote $c_3 := d_H(u(z_0), M \setminus \mathcal{U})$; this distance is positive since $\mathcal{U}$ is open. Since $du(z_0)$ is injective and $T_{z_0}\Sigma$ is finite-dimensional, the operator norm $c_4 := \|du_{z_0}\|$ is positive. Finally, $c_5 := \inf_{z \in \Sigma \setminus \{z_0\}} \frac{d_H(u(z_0), u(z))}{d_H(z_0, z)} > 0$: for $z$ sufficiently near $z_0$, say $z_0 \in U$ for some open neighbourhood $U \subset \Sigma$ of $z_0$, we have $\frac{d_H(u(z_0), u(z))}{d_H(z_0, z)} \geq \frac{c_2}{2}$ since $u$ is $C^1$. The function $z \mapsto \frac{d_H(u(z_0), u(z))}{d_H(z_0, z)}$ is continuous

[25] Recall that in this chapter, $\Sigma$ and $\theta$ are our fixed bookkeeping choices.

and never zero on the compact set $\Sigma \setminus U$ (since $u^{-1}(u(z_0)) = \{z_0\}$ by hypothesis of $z_0$ being an injective point), hence has a positive lower bound. Thus, $u$ satisfies item (3) in the definition of $S(A, H, c')$ for $c' = \min(c_3, c_4, c_5)$. Altogether, we deduce $u \in S(A, H, \min(c_1, \ldots, c_5))$. By property (3) of the sets $\mathcal{M}(J, c)$, we have $[u] \in \mathcal{M}(J, c_6)$ for some $c_6 > 0$. Choosing $c := \min(c_1, c_2, c_3, c_4, c_5, c_6)$, we obtain $u \in S(A, H, c)$ and $[u] \in \mathcal{M}(J, c)$, hence $[u] \in \mathcal{M}_{\mathcal{U}}^{A,H}(J, c)$.

For the second condition, let $c > 0$ and a sequence $J_k \to J$ in $\mathcal{J}(M)$ be arbitrary; suppose $([\Sigma, j_k, \theta, u_k])$ is a sequence in $\mathcal{M}_{\mathcal{U}}^{A,H}(J_k, c)$. Tweaking the chosen representatives if needed, we assume $(\Sigma, j_k, \theta, u_k) \in S(A, H, c)$. We need to exhibit a subsequence converging to some $u \in \mathcal{M}_{\mathcal{U}}^{A,H}(J, c)$. By property (3) of the sets $\mathcal{M}(J, c)$, by passing to a suitable sub-sequence, we may assume $(u_k)$ converges to some $u \in \mathcal{M}(J, c)$. By property (1) of $S(A, H, c)$, we have $j_k \in \mathcal{J}(\Sigma)^{A,c}$ for all $k$. By Lemma 4.103, after passing to a further subsequence, we may assume $(j_k)$ converges to some $j \in \mathcal{J}(\Sigma)^{A,c}$.
By hypothesis, each $u_k$ satisfies the conditions $(A \times G)_{u_k} \cong H$ and $d_H(\operatorname{im} u_k, \operatorname{im} h \cdot u_k) \geq c$ for all $h \in A \times G \setminus (A \times G)_{u_k}$. Since $A \times G$ is finite, there are only finitely many subgroups of $A \times G$. Thus, passing to a further subsequence if necessary, we may assume $(A \times G)_{u_k} = H_0$ for all $k$, where $H_0 \leqslant A \times G$ is some fixed subgroup conjugate to $H$. By hypothesis, each curve $u_k$ has an injective point $z_k \in \Sigma$ such that

$$d(u_k(z_k), M \setminus \mathcal{U}) \geq c, \|du_k(z_k)\| \geq c \text{ and } \inf_{z \in \Sigma \setminus \{z_k\}} \frac{d(u_k(z_k), u_k(z))}{d(z_k, z)} \geq c. \tag{4.8}$$

Restricting to a further subsequence, we may assume the sequence $(z_k)$ converges to some $z_0 \in \Sigma$.

Now, $[(\Sigma, j, \theta, u)] \in \mathcal{M}_{\mathcal{U}}^{A,H}(J, c)$ follows: we already proved $[u] \in \mathcal{M}(J, c)$ and $j \in \mathcal{J}(\Sigma)^{A,c}$. By continuity of the $A \times G$-action, we have $H_0 \subset (A \times G)_u$. For $h \in A \times G \setminus H_0$, for each $k$ we have $d_H(\operatorname{im} u_k, \operatorname{im} h \cdot u_k) \geq c$ by hypothesis. Since the Hausdorff distance is continuous, we deduce $d_H(\operatorname{im} u, \operatorname{im} h \cdot u) \geq c$ for all $h \in A \times G \setminus H_0$. This proves item (2) in the definition of $S(A, H, c)$, and also implies $(A \times G)_u = H_0$: in particular, $u \in \widetilde{\mathcal{M}}^{A,H}(J)$ follows. Item (3) defining $S(A, H, c)$ follows by continuity of $d_H$: using (4.8) and $u_k \to u$, we deduce

$$d(u(z_0), M \setminus \mathcal{U}) \geq c, \|du(z_0)\| \geq c \text{ and } \inf_{z \in \Sigma \setminus \{z_0\}} \frac{d(u(z_0), u(z))}{d(z_0, z)} \geq c. \qquad \square$$

## 4.6. Completing the proof: deducing smoothness of iso-symmetric strata

In this section, we complete the proof of Theorem 4.3 by deducing it from Theorem 4.4. We only present the details for compatible $G$-equivariant almost complex structures; the argument in the tame case is exactly the same.

The main new feature is including the orders $\mathbf{l}$ of the critical points: this can be handled using standard methods (see e.g. [Wen23d, Appendix A].) We omit the details.

**Theorem 4.104.** *Suppose $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$ is a smooth manifold, then for all $k$-tuples $\boldsymbol{l}$ of positive integers, the iso-symmetric stratum $\mathcal{M}_{\mathcal{U},\boldsymbol{l}}^{A,H}(J)$ is a smooth submanifold of $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$ of co-dimension $2n\sum_{i=1}^{k} l_i$.* □

*Proof of Theorem 4.3.* Since $2g+m \geq 3$, for all $j$ the automorphism group $\mathrm{Aut}(\Sigma, j, \theta)$ is finite (by Lemma 3.44). Thus, for all non-empty pre-strata $\widetilde{\mathcal{M}}^A(J)$, the group $A$ is finite.

By Proposition 3.63, the overall number of distinct non-empty sets $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$ is countable. Denote this collection by $\{S_n(J)\}_{n\in I}$ for some subset $I \subset \mathbb{N}$. For each stratum $S_n(j)$, using Theorem 4.4 we find a co-meagre set $J_{\mathrm{reg},n} \subset \mathcal{J}^G(M,\omega;\mathcal{U},J_{\mathrm{fix}})$ such that $S_n(J)$ is a smooth manifold for all $J \in \mathcal{J}_{\mathrm{reg},n}$. Then $J_{\mathrm{reg}} := \cap_{n\in I} J_{\mathrm{reg},n}$ is a countable intersection of co-meagre sets (hence co-meagre), and for each $J \in J_{\mathrm{reg}}$, all sets $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$ are smooth manifolds. For each such $J$, each iso-symmetric stratum $\mathcal{M}_{\mathcal{U},\mathbf{l}}^{A,H}(J)$ is a smooth submanifold of $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$, by Theorem 4.104 above; its dimension is

$$\begin{aligned}\dim \mathcal{M}_{\mathcal{U},\mathbf{l}}^{A,H}(J) &= \dim \mathcal{M}_{\mathcal{U}}^{A,H}(J) - 2n{\textstyle\sum_{i=1}^{k}} l_i \\ &= m_1^A(\operatorname{coker} D_{(j,\theta)}) + m_1^H(\ker D_u) - 2n\sum_{i=1}^{k} l_i.\end{aligned}$$ □

## 4.7. Generalising to infinite $A$ or compact $G$

The proofs so far require $A$ and $G$ to be finite. There are good reasons to expect this to be overly restrictive — but more general proofs require further ideas and substantial effort. Let us comment on this.

The main reason for this restriction is mostly technical and not conceptual: to write down a local model for the sets $\mathcal{M}_{\mathcal{U}}^{A,H}(J)$, we need the base $\mathcal{T} \times \mathcal{B}^H$ and in particular the orbit type

$$\mathcal{B}^H := \{u \in \mathcal{B} \mid (A\times G)_u \cong H\}$$

of $H$ in $\mathcal{B} = W^{1,p}(\Sigma, M)$ to be a smooth submanifold of $\mathcal{B}$. If $A \times G$ is finite, this was not difficult to show, as $\mathcal{B}^H$ then locally coincides with the fixed point set $\mathcal{B}_H$ of $H$. Put differently, the stabilisers of $u \in \mathcal{B}^H$ are, in fact, locally *equal* to $H$. If $A \times G$ is a Lie group of positive dimension, this is generally no longer true, and stabilisers may only be equal up to conjugation.

In the finite-dimensional setting of the orbit type stratification, this is addressed via the *slice theorem*: this allows reducing the description of $\mathcal{B}^H$ to considering the fixed point set $\mathcal{B}_H$. The slice theorem is a classical fact for finite-dimensional smooth manifolds with a proper $C^1$ action. It also holds holds in infinite dimensions, for instance for a smooth and proper action of a finite-dimensional Lie group on a smooth

Banach manifold.[26] In our setting, however, this is not useful at all, since the $A \times G$-action on $\mathcal{B}$ is not differentiable, let alone smooth. While $G$ acts smoothly on $\mathcal{B}$, the $A$-action is only continuous and cannot be differentiable, because of the loss of derivatives.

Therefore, proving smoothness of the iso-symmetric strata for $A \times G$ an infinite group requires new ideas, beyond what is presented in this thesis. One promising approach is using the *global deformation operator* of $u$ [Wen23a; Bar24] instead: we refer the reader to Wendl's blog ([Wen23a] and the not yet published part 3 of that series) for an overview of this idea.

One necessary side effect of this approach — which we consider a positive feature — is removing the need to construct adapted Teichmüller slices. This required technical effort to construct (in particular, for the unmarked torus, where the obvious adaptation of the standard slice is not obviously a valid construction). It is also a non-canonical choice, and it requires understanding the moduli spaces $\mathcal{M}_{g,m}$ first: one could argue (e.g. [Wen23a]) this should not be necessary.

The global deformation operator dispenses with the adapted Teichmüller slices in local models and the pre-strata in the definition; let us briefly indicate how this is possible. In their stead, we consider just the $\mathrm{Diff}_+(\Sigma,\theta) \times G$-action on the space $\widetilde{\mathcal{N}}(J)$ from Observation 3.58. While $\mathrm{Diff}_+(\Sigma,\theta)$ is not a finite-dimensional Lie group, this action is still well-defined and continuous. The iso-symmetric stratum $\mathcal{M}^H(J)$ corresponding to a closed subgroup $H \leqslant \mathrm{Diff}_+(\Sigma,\theta)\times G$ is then defined as $\mathcal{M}^H(J) := \{[u] \mid (j,u) \in X\}$, where

$$X := \{(j,u) \in \widetilde{\mathcal{N}}(J) \mid (\mathrm{Diff}_+(\Sigma,\theta) \times G)_u \cong H\}.$$

As the $\mathrm{Diff}_+(\Sigma,\theta) \times G$-action is proper on stable curves (by Remark 3.62), $H$ is still a compact Lie group. Countability of the strata needs additional work: it turns out that the definition is related to the sets $\mathcal{M}^{A,H}(J)$.

**Lemma 4.105.** *For all closed subgroups $H \leqslant \mathrm{Diff}_+(\Sigma,\theta) \times G$, we have*

$$\mathcal{M}^H(J) = \bigcup_{A \leqslant \mathrm{Diff}_+(\Sigma,\theta)} \mathcal{M}^{A,H}(J).$$

*Proof.* Observe that $(j,u) \in \mathcal{J}(\Sigma) \times \widetilde{\mathcal{M}}(J)$ lies in $X$ if and only if $(\Sigma, j, \theta)$ is stable and

$$(\mathrm{Diff}_+(\Sigma,\theta) \times G)_u \overset{3.58}{=} (\mathrm{Aut}(\Sigma,j,\theta) \times G)_u \cong H.$$

We prove both inclusions separately.

"$\Rightarrow$": Suppose $[u] \in \mathcal{M}^H(J)$, i.e. $(j,u) \in X$. Write $A := \mathrm{Aut}(\Sigma,j,\theta)$; this is a closed subgroup of $\mathrm{Diff}_+(\Sigma,\theta)$. By the above, $(j,u) \in X$ implies $(\mathrm{Aut}(\Sigma,j,\theta) \times G)_u \cong H$; thus $(A \times G)_u \cong H$ and $u \in \widetilde{\mathcal{M}}^{A,H}(J)$ follows.

[26] The author is not aware of any explicit written reference for this setting, but the classical proof (found in e.g. [AB15, Theorem 3.49] or [DK00, Theorem 2.3.3]) generalises almost verbatim.

"$\Leftarrow$": Suppose $[u] \in \mathcal{M}^{A,H}(J)$ for some closed subgroup $A \leqslant \text{Diff}_+(\Sigma, \theta)$. Then $(A \times G)_u \cong H$ by hypothesis. By definition, $u \in \widetilde{\mathcal{M}}^A(J)$ implies $\text{Aut}(\Sigma, j, \theta) = A$, so $(\text{Diff}_+(\Sigma, \theta) \times G)_u \overset{3.58}{=} (\text{Aut}(\Sigma, j, \theta) \times G)_u = (A \times G)_u \cong H$ follows, proving $(j, u) \in X$ and $[u] \in \mathcal{M}^H(J)$. □

# 5. Definition and smoothness of walls

In the previous two chapters, we decomposed the moduli space $\mathcal{M}(J)$ into countably many iso-symmetric strata $\mathcal{M}^{A,H}_{\mathcal{U},1}(J)$ and proved that, for generic $G$-equivariant $J$, every stratum is a smooth finite-dimensional manifold. As the next step, we decompose each iso-symmetric stratum further into countable many *walls* and prove that these are (generically) smooth manifolds.

Recall that to each curve in $\mathcal{M}^{A,H}_{\mathcal{U},1}(J)$, we associate an equivariant Fredholm operator. A fruitful operator to consider was already used in Wendl's solution of the super-rigidity conjecture [Wen23d]: in fact, there are two closely related operators, the *linearised Cauchy–Riemann operator* $D_u$ of each holomorphic curve $u$, and its *normal Cauchy–Riemann operator* $D^N_u$. These are Fredholm operators, hence have finite-dimensional kernels and co-kernels. In Wendl's setting (of multiply covered curves, but no symplectic group action on $M$), there is a symmetry through the (generalised) automorphism group $G$ of the curve, and the operators are $G$-equivariant. Their kernels and co-kernels then define $G$-representations, which can be used to define the walls. In our particular situation, we want to obtain $G_u$-equivariant Fredholm operators, where $G_u$ is the stabiliser of each curve $u \in \mathcal{M}^{A,H}_{\mathcal{U},1}(J)$ under the $G_u$-action.[1] The normal Cauchy–Riemann operator is useful, but in general, the correct operator to consider is the *restricted normal Cauchy–Riemann operator*.

In Section 5.1, we recall the definition of the normal Cauchy–Riemann operator and show it is $H$-equivariant in our setting. We explain how this induces the restricted normal Cauchy–Riemann operator (which is $G_u$-equivariant), and define walls using this restricted operator. We state the main result of this chapter and prove that there are countably many non-empty distinct walls.
The remainder of this chapter is devoted to proving that walls are generically smooth. Like for the iso-symmetric strata, smoothness of the walls is proved using the implicit function theorem. In Section 5.2, we explain the basic set-up for the proof. The main argument has two parts: the first one (called "flexibility" in Doan–Walpuski's terminology) is carried out in Section 5.3. The second part is the crux of the argument: we prove that for generic $J$, the restricted normal Cauchy–Riemann operators satisfy a condition known as *Petri's condition* (see Definition 5.23). In general, proving that an equivariant transversality problem satisfies Petri's condition is the hardest part: for $H$ finite, this work reduces to the non-equivariant case. We complete the proof of smoothness in Section 5.5.

[1] If $G_u$ is trivial, we actually about $H$-equivariance: this is a more exceptional case.

## 5.1. Definition of walls

In this section, we explain how to split each iso-symmetric stratum $\mathcal{M}^{A,H}_{\mathcal{U},\mathrm{I}}(J)$ into walls. The basic idea is not difficult: to each curve $u \in \mathcal{M}^{A,H}_{\mathcal{U},\mathrm{I}}(J)$, we associate an equivariant Fredholm operator $D^{N,\mathrm{restr}}_u$ which varies smoothly with $u$. Then, $\mathcal{M}^{A,H}_{\mathcal{U},\mathrm{I}}(J)$ decomposes into walls according to the dimension of the kernel and cokernel of $D^{N,\mathrm{restr}}_u$.

**Definition 5.1.** *For all $J \in \mathcal{J}^G_\tau(M,\omega)$ and integers $k, c \geq 0$, the corresponding* wall $\mathcal{M}(J;k,c) \subset \mathcal{M}^{A,H}_{\mathcal{U},I}(J)$ *is defined as*

$$\mathcal{M}(J;k,c) := \{[u] \in \mathcal{M}^{A,H}_{\mathcal{U},I}(J) \mid \dim\ker D^{N,restr}_u = k, \dim\operatorname{coker} D^{N,restr}_u = c\}.$$

As hinted already, the Fredholm operator $D^{N,\mathrm{restr}}_u$ is the restricted normal Cauchy–Riemann operator of $u$ (for suitably chosen auxiliary data): let us introduce its definition. We begin by reviewing the definition of the “unrestricted” normal Cauchy–Riemann operator.

**Recall.** If $u\colon \Sigma \to M$ is a smooth embedding, the subset $T_u := du(T\Sigma) \subset u^*TM$ is a smooth sub-bundle of $u^*TM$, called the *tangent bundle* to $u$. Choosing a bundle metric on $u^*TM$, the complement $N_u \subset u^*TM$ of $T_u$ is called the *normal bundle* to $u$. If $u$ is $J$-holomorphic, $T_u$ is a complex sub-bundle of $u^*TM$, and so is $N_u$ if we chose a complex bundle metric.

If $u$ is a simple holomorphic curve, it is embedded away from a finite set of self-intersections and critical points. In fact, self-intersections are not an issue for defining the tangent and normal bundles to $u$; the above definition works verbatim for immersed curves. If $u$ has a critical point $z_0 \in \Sigma$, however, the space $du_{z_0}(T_{z_0}\Sigma) \subset T_{u(z_0)}M$ is zero-dimensional, hence we need to tweak the definition. The *generalised tangent bundle* and *generalised normal bundle* to $u$ extend the above to curves with isolated critical points. This idea goes back to Ivashkovich and Shevchishin [IS99]. We follow Wendl's presentation [Wen10], which uses less algebraic language (avoiding, for instance, any mention of sheaves and exact sequences).

**Lemma 5.2** ([Wen10, Section 3.3])**.** *There exists a holomorphic rank one sub-bundle $T_u \subset u^*TM$, called the* generalised tangent bundle *to $u$, such that $(T_u)_z = \operatorname{im} du_z$ whenever $z$ is not a critical point of $u$.* □

Since we need to use this information later, let us briefly indicate how to define $(T_u)_z$ for a critical point $z \in \Sigma$ of $u$. The linearised Cauchy–Riemann operator $D_u$ defines a holomorphic structure on the bundle $u^*TM$ (uniquely characterised by the condition that local holomorphic sections vanish under $D_u$). This induces a holomorphic structure on $\mathrm{Hom}_{\mathbb{C}}(T\Sigma, u^*TM)$, and one can show that $du \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(T\Sigma, u^*TM))$ is a holomorphic section. Thus, choose a holomorphic trivialisation of $\mathrm{Hom}_{\mathbb{C}}(T\Sigma, u^*TM)$ near $z_0$ and express $u$ locally as a $\mathbb{C}^n$-valued function (where $\dim M = 2n$) of the form $(z - z_0)^k F(z)$ for some $k \geq 1$ and a $\mathbb{C}^n$-valued meromorphic function $F$ with $F(z_0) \neq 0$. (In fact, $k$ is the order of the critical point $z_0$.) We define $(T_u)z_0$ as the complex subspace spanned, in the trivialisation, by $F(z_0) \in \mathbb{C}^n \setminus \{0\}$.

**Definition 5.3.** *The* generalised normal bundle $N_u \subset u^*TM$ *to* $u$ *is the complement of* $T_u$ *w.r.t. to some complex bundle metric. In particular,* $N_u$ *is a complex vector bundle.*

Thus, we have a splitting $u^*TM = T_u \oplus N_u$. Decomposing the linearised Cauchy–Riemann operator $D_u = D\overline{\partial}_J(u)\colon \Gamma(u^*TM) \to \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$ according to this splitting yields a block decomposition

$$D_u = \begin{pmatrix} D_u^T & D_u^{TN} \\ D_u^{NT} & D_u^N \end{pmatrix},$$

so e.g. $D_u^{NT}\colon \Gamma(T_u) \to \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u))$.

**Definition 5.4.** *The* normal Cauchy–Riemann operator *of* $u$ *is the composition*

$$D_u^N := \pi_N \circ D_u|_{\Gamma(N_u)}\colon \Gamma(N_u) \to \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u)),$$

*where* $\pi_N\colon \Gamma(u^*TM) \to \Gamma(N_u)$ *is the quotient projection along* $N_u$*.*

Recall that $D_u$ and $D_u^N$ are Fredholm operators and vary smoothly with $u$. An important observation, leading to the definition of the restricted normal Cauchy–Riemann operator, is that for curves in a given iso-symmetric stratum, these are also equivariant: if $u \in \widetilde{\mathcal{M}}_{\mathcal{U}}^{A,H}(J)$, then $D_u$ and $D_u^N$ are $H$-equivariant Fredholm operators. Let us begin by proving the equivariance of $D_u$.

**Observation 5.5.** For every $u \in \widetilde{\mathcal{M}}_{\mathcal{U}}^{A,H}(J)$, the stabiliser $H$ acts linearly on $\Gamma(u^*TM)$ by $(\phi, g) \cdot \eta := d\psi_g \circ \eta \circ \phi^{-1}$.[2] Similarly, $H$ acts linearly on $\Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$ by $(\phi, g) \cdot \eta := d\psi_g \circ \eta \circ d\phi^{-1}$.[3]

*Proof.* We only prove the first part; the second statement is analogous. It is easy to check by hand that this is well-defined and defines a left action, using $h \cdot u = u$ for all $h \in H$. A more conceptual proof observes that this action is the linearisation of the $A \times G$-action on the bundle $\mathcal{E} \to \mathcal{B}$ from the previous chapter: for $\eta = \partial_t u_t|_{t=0} \in \Gamma(u^*TM)$, we compute

$$(\phi, g) \cdot \eta = d\psi_g(\eta(\phi^{-1})) = d\psi_g(\partial_t u_t)|_{t=0} \circ \phi^{-1} = \partial_t(d\psi_g \circ u_t \circ \phi^{-1})|_{t_0} = \partial_t(\phi, g) \cdot u_t|_{t=0},$$

thus $(\phi, g) \cdot \eta = \partial_t(\phi, g) \cdot u_t|_{t=0} = \partial_t u_t|_{t=0} = \eta$. $\square$

Note that $H$ does *not* act on $u^*TM$ (nor on $N_u$), as $G$ does not act over the identity map on the base. However, we get a linear action on the space of sections of $u^*TM$ (and of $N_u$), which is all we need.

[2] More explicitly, for each $z \in \Sigma$ we have $((\phi, g) \cdot \eta)(z) = d\psi_g(\eta(\phi^{-1}(z))) \in T_{g \cdot u(\phi^{-1}(z))}M = T_{u(z)}M$.
[3] More explicitly, we have $((\phi, g)\eta)(z) := (X \mapsto d\psi_g \circ \eta_{\phi^{-1}(z)}(d\phi^{-1}(X)))$ for all $z \in \Sigma$ and $X \in T_z\Sigma$.

**Observation 5.6.** For all $u \in \widetilde{\mathcal{M}}_{\mathcal{U}}^{A,H}(J)$, the linearised Cauchy–Riemann operator $D_u$ is $H$-equivariant.

*Proof.* Let $h \in H$ be arbitrary. Recall that $h \cdot \overline{\partial}_J(u) = \overline{\partial}_J(h \cdot u)$. Differentiating and using the linearity of the $H$-action, for each $\eta = \partial_t u_t|_{t=0} \in \Gamma(u^*TM)$ we obtain

$$\begin{aligned} D_u(h \cdot \eta) &= D_u(h \cdot \partial_t u_t|_{t=0}) = D_u(\partial_t(h \cdot u_t)|_{t=0}) = \nabla_t \overline{\partial}_J(h \cdot u_t)|_{t=0} \\ &= \nabla_t h \cdot \overline{\partial}_J(u_t)|_{t=0} = h \cdot \nabla_t \overline{\partial}_J(u_t)|_{t=0} = h \cdot D_u(\partial_t u_t|_{t=0}) = h \cdot D_u(\eta). \square \end{aligned}$$

**Lemma 5.7.** *The space $\Gamma(T_u)$ is $H$-invariant.*

*Proof.* Let $\eta \in \Gamma(T_u)$ and $h = (\phi, g) \in H$ be arbitrary; we must show $h \cdot \eta \in \Gamma(T_u)$. This follows from two observations: they imply $h \cdot \eta$ is tangent to $d\psi_g \circ u(\phi^{-1}) = u$ as desired.

- For any $\eta \in \Gamma(T_u)$ and $g \in G$, the section $d\psi_g \circ \eta$ lies in $\Gamma(T_{g \cdot u})$.
- For any $\eta \in \Gamma(T_u)$ and $\phi \in A$, the section $\eta(\phi^{-1})$ lies in $\Gamma(T_{u \circ \phi^{-1}})$.

At immersed points of $u$, both observations are obvious. They is also not hard to prove at critical points; one just needs to unfold the definition of the generalised tangent bundle $T_u$. We omit the details. $\square$

Next, we turn to $H$-equivariance of the normal Cauchy–Riemann operator $D_u^N$. To boot, this requires the space of sections $\Gamma(N_u)$ and $\Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u))$ to be $H$-invariant, wherefore we need to choose our bundle metrics with some care. As in the previous chapter, choose, once and for all, an $A$-invariant Riemannian metric on $\Sigma$ and a $G$-invariant bundle metric on $TM$. This induces a bundle metric on $u^*TM$ for any $u \in \widetilde{\mathcal{M}}_{\mathcal{U}}^{A,H}(J)$, and also an $H$-invariant $L^2$-pairings on $\Gamma(u^*TM)$ and $\Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$. We define $N_u$ using this bundle metric. We claim that $D_u^N$ is an $H$-equivariant Fredholm operator $\Gamma(N_u) \to \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u))$. The first step is proving that $\Gamma(N_u)$ is $H$-invariant.

**Lemma 5.8.** *The space $\Gamma(N_u)$, defined using the above bundle metric, is $H$-invariant.*

*Proof.* This is a general fact about complements of vector sub-bundles and follows from the $H$-invariance of $\Gamma(T_u)$ and the bundle metric. Observe that $\Gamma(N_u)$ is the orthogonal complement of $\Gamma(T_u)$ with respect to the $L^2$-pairing on $\Gamma(u^*TM)$. Suppose $\xi \in \Gamma(N_u)$ and $h \in H$. For all $\eta \in \Gamma(T_u)$, we compute

$$\langle \eta, h \cdot \xi \rangle = \langle h^{-1}\eta, \xi \rangle = 0$$

using the $H$-invariance of the pairing, therefore $h \cdot \xi \in \Gamma(N_u)$ follows. $\square$

Secondly, $\Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u))$ is $H$-invariant.

**Observation 5.9.** $\Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u))$ is $H$-invariant.

*Proof.* Completely analogously to $\Gamma(T_u)$, we prove that $\Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, T_u))$ is $H$-invariant. Observe that $\Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u))$ is the complement of $\Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, T_u))$ w.r.t. our chosen $H$-invariant pairing: therefore, it must also be $H$-invariant. □

**Corollary 5.10.** *$D_u^N$ is an $H$-equivariant Fredholm operator which varies smoothly with $u$.*

*Proof.* It just remains to show $H$-equivariance. By definition, $D_u^N = \pi_N \circ D_u|_{\Gamma(N_u)}$. The operator $D_u$ is $H$-equivariant (as above); $\pi_N$ is $H$-equivariant since $H$ acts linearly and both $\Gamma(T_u)$ and $\Gamma(N_u)$ are $H$-invariant. □

The normal Cauchy–Riemann operator is *almost* the right operator to consider: we have to make one last change, to account for another, better hidden, symmetry inherent in our setting. So far, we accounted for the stabiliser $H = (A \times G)_u$, which corresponds to group elements $g \in G$ whose action yields a reparametrisation of $u$. However, it is also necessary to note which $g \in G$ leave $u$ fixed *point-wise*: the reason is that Petri's condition is a local condition. If a point $p \in M$ is fixed under the $G$-action, $G$-equivariance places another constraint on local perturbations near $M$: we need to remember this information. We encode this additional constraint by considering a suitable sub-bundle of the normal bundle $N_u$. If $u$ has non-trivial (point-wise) stabiliser $G_u$ under the $G$-action, then $u$ maps into a symplectic submanifold of $M$, and the normal Cauchy-Riemann operator induces an operator on a corresponding sub-bundle of $N_u$; we call this operator the *restricted normal Cauchy–Riemann operator*.

Let us make this precise. Fix a curve $u \in \mathcal{M}_{\mathcal{U},1}^{A,H}(J)$ and consider the stabiliser $G_u \leqslant G$. Then $u$ takes values in the fixed-point set $M^K := \mathrm{Fix}(G_u) \subset M$. The starting observation is that $M^K$ is a symplectic submanifold, so we may study $u$ as a holomorphic curve into $M^K$.

**Observation 5.11.** For every finite subgroup $G_0 \leqslant G$, the fixed point set $M^{G_0} := \mathrm{Fix}(G_0)$ is a symplectic and almost complex submanifold of $M$.

*Proof.* It is a classical result that $M^{G_0}$ is a smooth submanifold; see Lemma 4.52 resp. [DK00, p. 108; AB15, Proposition 3.93]. Since $J$ is $G$-equivariant, $M^{G_0}$ must also be an almost complex submanifold: we prove this using Lemma 4.56. For all $X \in T_pM^{G_0}$, we have $g \cdot (JX) = J(g \cdot X) = JX$ for all $g \in G_0$, hence $JX \in T_pM^{G_0}$ as well. Since $J$ is $\omega$-tame, $M^{G_0}$ is also symplectic: for any $X \in T_pM^{G_0}$, tameness implies $\omega(X, JX) > 0$, thus $\omega|_{TM^{G_0}}$ is non-degenerate. □

Let $u^K : \Sigma \to M^K$ denote the co-restriction of $u$. Let $E^K := N_{u^K} \subset u^{K*}TM^K$ be the generalised normal bundle of $u^K$, denote $F^K := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_{u^K})$. Observe that there are natural injective bundle maps $E^K \to E$ and $F^K \to F$, induced by the inclusion $\iota : M^K \hookrightarrow M$. More precisely, $\iota$ induces an injective linear bundle map

$$u_k^* TM^K \hookrightarrow u^* TM, \eta \to d\iota(\eta),$$

which maps $T_{u^K}$ to $T_u$. (This is easy to verify at the non-critical points, for instance.) The (image of the) sub-bundle $TM^K$ is $G_u$-invariant. Since $N_u$ was defined using an $H$-invariant bundle metric, the complement $N_{u^K}$ of $T_{u^K}$ is a sub-bundle of $E^K$. For $F^K$, we argue similarly, using that $(N_{u^K})_z = (N_u)_z \cap d\iota(T_{u(z)}M^K)$. In Section 5.4, we prove (in Lemma 5.32) that any $H$-equivariant Cauchy–Riemann type operator $\Gamma(E) \to \Gamma(F)$ maps $\Gamma(E^K)$ to $\Gamma(F^K)$. In particular, this applies to $D_u^N$.

**Definition 5.12** (Restricted normal Cauchy–Riemann operator)**.** *The* restricted normal Cauchy–Riemann type operator *of $u \in \mathcal{M}^{A,H}_{\mathcal{U},I}(J)$ is the Cauchy–Riemann type operator $D_u^{N,\text{restr}} \colon \Gamma(E^K) \to \Gamma(F^K)$ induced from $D_u^N$, i.e. it is characterised by $D_u^N(\iota(\eta)) = \iota(D_u^{N,\text{restr}}(\eta))$ for all $\eta \in \Gamma(E^K)$. Equivalently, $D_u^{N,\text{restr}}$ is the normal Cauchy–Riemann operator of $u^K$.*

In particular, $D_u^{N,\text{restr}}$ is still a Fredholm operator which depends smoothly on $u$. It has equivariance properties analogous to $D_u^N$.

**Lemma 5.13.** *The restricted normal Cauchy–Riemann operator $D_u^{N,\text{restr}} \colon \Gamma(E^K) \to \Gamma(F^K)$ is a $G_u$-equivariant Fredholm operator.*

*Proof.* This follows from exactly the same reasoning as for the normal Cauchy–Riemann operator $D_u^N$; we use that $D_u^{N,\text{restr}}$ is the normal Cauchy–Riemann operator of the $J$-holomorphic curve $u^K \colon \Sigma \to M^K$. We merely need to compute that $G_u$ is the stabiliser analogous to $H$.

Consider $\widetilde{C} := {u^K}_*[\Sigma] \in H_2(M^K)$, so $\iota_*\widetilde{C} = C$ is the homology class defining the moduli space $\mathcal{M}(J)$ which $u$ lives in. Consider the analogous parametrised moduli space

$$\widetilde{\mathcal{M}}(\widetilde{C}, J; M^K) := \{(\Sigma, j, \theta, v^K) \mid (\Sigma, j, \theta) \in \widetilde{\mathcal{M}}_{g,m}, v^K \colon \Sigma \to M^K \text{ is } J\text{-holomorphic}, \\ v^K(\theta) = \theta, {v^K}_*[\Sigma] = \widetilde{C}\}$$

of curves in $M^K$. This space splits into iso-symmetric strata analogous to $\mathcal{M}(J)$: for each closed subgroup $A \leqslant \text{Diff}_+(\Sigma, \theta)$, there is a pre-stratum

$$\widetilde{\mathcal{M}}^A(\widetilde{C}, J; M^K) = \{(\Sigma, j, \theta, v) \in \widetilde{\mathcal{M}}(\widetilde{C}, J; M^K) \mid (\Sigma, j, \theta) \in \widetilde{\mathcal{M}}^A_{g,m}\},$$

which admits an $A$-action by reparametrisation. The group $G_u$ acts trivially on $M^K$ (since $M^K = \text{Fix}(G_u)$ by definition), hence also on $\widetilde{\mathcal{M}}(\widetilde{C}, J; M^K)$. In particular, each pre-stratum has an $A \times G_u$-action by $(\phi, g) \cdot v := \psi_g \circ (v \circ \phi^{-1}) = v \circ \phi^{-1}$. This implies that for simple curves, each pre-stratum only consists of a single iso-symmetric stratum: for all simple $v^K \in \widetilde{\mathcal{M}}(\widetilde{C}, J; M^K)$, we have $(A \times G_u)_{v^K} = \{\text{id}\} \times G_u$ since

$$(\phi, g) \in (A \times G_u)_{v^K} \Leftrightarrow v = (\phi, g) \circ v = v \circ \phi^{-1}$$

implies $\phi = \text{id}$ for $v$ simple.

The curve $u^K$ has still genus $g$ and $m$ marked points, its domain is the domain of $u$, thus has automorphism group $A$. Therefore, $u^K \in \widetilde{\mathcal{M}}^A(\widetilde{C}, J; M^K)$, and we deduce

$(A \times G_u)_{u^K} = G_u$. Applying Lemma 5.8, Observation 5.9 and Corollary 5.10 to this setting, we deduce that $\Gamma(E^K)$ and $\Gamma(F^K)$ are $G_u$-invariant, and that $D_u^{N,\text{restr}} \colon \Gamma(E^K) \to \Gamma(F^K)$ is $G_u$-equivariant. □

One last observation is important. If $G_u$ is trivial, the above lemma is correct, but vacuous. This does not mean $D_u^{N,\text{restr}}$ has no symmetry then — to the contrary, it is even $H$-equivariant (not just $G_u$-equivariant). The reason is that in this case, $E^K$ resp. $F^K$ agree with $E$ and $F$, respectively and the restricted normal Cauchy–Riemann operator $D_u^{N,\text{restr}}$ is precisely the normal Cauchy–Riemann operator. Thus, the correct symmetry to consider depends on whether $G_u$ is trivial or not.

In the remainder of this chapter, we prove the following.

**Theorem 5.14.** *There exists a co-meagre subset $\mathcal{J}_{reg} \subset \mathcal{J}^G(M, \omega; \mathcal{U}, J_{fix})$ (resp. $\mathcal{J}_{reg} \subset \mathcal{J}_\tau^G(M, \omega; \mathcal{U}, J_{fix})$) such that for every $J \in \mathcal{J}_{reg}$, each wall $\mathcal{M}(J; k, c) \subset \mathcal{M}_{\mathcal{U},\mathbf{l}}^{A,H}(J)$ containing a given curve $u$ is a smooth finite-dimensional manifold. Its co-dimension is* $\operatorname{Hom}_H(\ker D_u^N, \operatorname{coker} D_u^N)$ *if $G_u$ is trivial, and* $\operatorname{Hom}_{G_u}(\ker D_u^{N,\text{restr}}, \operatorname{coker} D_u^{N,\text{restr}})$ *otherwise.*

Since the integers are countable, Proposition 3.63 readily implies

**Theorem 5.15.** *For each $J$, the number of non-empty distinct walls $\mathcal{M}(J; k, c)$ is countable.* □

## 5.2. Setting up the proof of smoothness

The proof of Theorem 5.14 is again an application of the implicit function theorem: we exhibit each wall as the zero set of a suitable smooth section in a Banach space bundle. Let us explain this set-up in detail. During this argument, we need to consider how the Fredholm operators $D_u^{N,\text{restr}}$ vary with $u$.

In the following, consider a *parameter space* $P$, a smooth Banach manifold. (In this thesis, $P$ is simply an iso-symmetric stratum. Other related settings require a different choice of parameter space: for bifurcation analyses similar to Bai and Swaminathan, one studies *parametric* moduli spaces, with e.g. $P$ being one-dimensional.) Take a smooth family of curves $\{[u_\sigma]\}_{\sigma \in P}$ in $\mathcal{M}_{\mathcal{U},\mathbf{l}}^{A,H}(J)$. Choose the representatives $u_\sigma \colon (\Sigma, j_\sigma) \to M$ so that $(A \times G)_{u_\sigma} = H$ and both $u_\sigma$ and $j_\sigma$ vary smoothly with $\sigma$ and $J$. (This is possible by definition of the topology on the iso-symmetric stratum.) As the first step towards the proof, let us locally describe the parameters in $P$ intersecting a given wall. Intuitively, because the Fredholm index of $D_u^{N,\text{restr}}$ is locally constant, it suffices to prescribe the dimension of $\ker D_u^{N,\text{restr}}$. To make this precise, consider the subset

$$P(k, c) := \{\tau \in P \mid \dim \ker D_{u_\tau}^{N,\text{restr}} = k, \dim \operatorname{coker} D_{u_\tau}^{N,\text{restr}} = c\}.$$

**Lemma 5.16.** *Each $\sigma \in P(k, c)$ has a neighbourhood $U_\sigma \subset P$ such that $U_\sigma \cap P(k, c)$ is the set of all $\tau \in U_\sigma$ such that* $\dim \ker D_{u_\tau}^{N,\text{restr}} = k$.

*Proof.* Since $\sigma \in P(k,c)$, we have $\operatorname{ind} D^{N,\text{restr}}_{u_\sigma} = k - c$. Since the family $\{D^{N,\text{restr}}_{u_\tau}\}_{\tau \in P}$ depends continuously on $\tau$ and the Fredholm index is locally constant, for all $\tau$ sufficiently close to $\sigma$ we have $\operatorname{ind} D^{N,\text{restr}}_{u_\tau} = \operatorname{ind} D^{N,\text{restr}}_{u_\sigma}$. Then, $\dim\ker D^{N,\text{restr}}_{u_\tau} = k$ implies

$$\dim\operatorname{coker} D^{N,\text{restr}}_{u_\tau} = \dim\ker D^{N,\text{restr}}_{u_\tau} - \operatorname{ind} D^{N,\text{restr}}_{u_\tau} = \dim\ker D^{N,\text{restr}}_{u_\tau} - \operatorname{ind} D^{N,\text{restr}}_{u_\sigma} = c,$$

as desired. □

For the remainder of this section, let $K$ denote the group $H$ if $G_u$ is trivial, and $G_u$ otherwise. In the previous section, we have shown that $\Gamma(E^K)$ and $\Gamma(F^K)$ are $K$-invariant, and that $D^{N,\text{restr}}_u$ is $K$-equivariant. The domain of the operators $D^{N,\text{restr}}_u$ depends on $u$: this is a bit annoying for later analysis, so let us construct local smooth families of operators on the same domain. For convenience, we abbreviate $D_\tau := D^{N,\text{restr}}_{u_\tau}$ for $\tau \in P$. Shrinking $U_\sigma$ if necessary, we can assume that $U_\sigma$ is convex, hence the maps $u_\sigma$ and $u_\tau$ are homotopic (for each $\tau \in U_\sigma$), hence $N_{u_\sigma}$ and $N_{u_\tau}$ are isomorphic. Fix two parameters $k$ and $p$ with $kp > 2$. Given a smooth family of bundle isomorphisms $\Psi_\tau\colon N_{u_\sigma} \to N_{u_\tau}$ for each $\tau \in U_\sigma$ with $\Psi_\sigma = \text{id}$, there are induced smooth isomorphisms $W^{k,p}(N_{u_\sigma}) \to W^{k,p}(N_{u_\tau}), \eta \mapsto \Psi_\tau \circ \eta$ and

$$W^{k,p}(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, N_{u_\sigma})) \to W^{k,p}(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, N_{u_\sigma})), \eta \mapsto \Psi_\tau \circ \eta,$$

which we will both denote by $\Psi_\tau$ for convenience. Thus, we obtain a smooth family of Fredholm operators

$$\widehat{D}_\tau = \Psi_\tau^{-1} \circ D_\tau \circ \Psi_\tau \colon W^{k,p}(N_{u_\sigma}) \to W^{k-1,p}(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, N_{u_\sigma}))$$

on a fixed domain. If $\Psi_\tau$ was just an arbitrary bundle isomorphism, however, there is no reason why $\widehat{D}_\tau$ would be $K$-equivariant: we have to work a bit harder to ensure this.

The final construction proceeds in two steps. First, we find *any* smooth family of bundle isomorphisms $\Psi_\tau\colon N_{u_\sigma} \to N_{u_\tau}$; then, we upgrade these to $K$-equivariant isomorphisms. For $\tau \in P$, let us write $E_\tau := N_{u_\tau}$ for brevity.

The first step is is straightforward using parallel transport on $M$. Consider the smooth finite rank vector bundle $\widehat{E} \to U_\sigma \times \Sigma$ with fibres $\widehat{E}_{(\sigma,z)} = (N_{u_\sigma})_z \subset T_{u_\sigma(z)}M$. Choose a connection on $\widehat{E}$. For each $\tau$, consider the path $\gamma_\tau\colon [0,1] \to U_\sigma \times \Sigma, t \to ((1-t)\sigma + t\tau, z)$ connecting $(\sigma, z)$ to $(\tau, z)$. Parallel transport along $\gamma_\tau$ yields an isomorphism $(N_{u_\sigma})_z = \widehat{E}_{(\sigma,z)} = \widehat{E}_{\gamma(0)} \to \widehat{E}_{\gamma(1)} = \widehat{E}_{(\tau,z)} = (N_{u_\tau})_z$. These isomorphism combine to a diffeomorphism $\Psi_\tau\colon N_{u_\sigma} \to N_{u_\tau}$; these diffeomorphisms depend smoothly on $\tau$.[4] Each $\Psi_\tau$ induces a linear isomorphism

$$A_\tau\colon W^{k,p}(E_\sigma) \to W^{k,p}(E_\tau), \eta \mapsto \Psi_\tau \circ \eta$$

[4] For the pedantic reader, let us elaborate on why. Basically, parallel transport is an ODE, and unique solutions to ODEs depend smoothly on their initial conditions. The map $\Psi_\tau$ depends smoothly on $\gamma_0 = (\sigma, z)$, hence each $\Psi_\tau$ is smooth. The maps $\Psi_\tau$ depend smoothly on $\tau$ the coefficients of the ODE describing parallel transport depend smoothly on $\tau$, hence so do the solutions $\Psi_\tau$.

with $A_\sigma = \mathrm{id}$.

Next, we "average" the $A_\tau$ using the $K$-action to obtain an $K$-equivariant map $\widetilde{A}_\tau$. For each $k \in K$, consider the map

$$A_\tau^k := k^{-1} \cdot A_\tau \cdot k \colon W^{k,p}(E_\sigma) \to W^{k,p}(E_\tau),$$

combining the $K$-actions on $W^{k,p}(E_\sigma)$ and $W^{k,p}(E_\tau)$. It is easy to verify $A_\tau^k(z) \in (E_\tau)_z$ for all $z \in \Sigma$. We also check that $A_\tau^k$ is linear — since the differential $d\psi_g$ and the $K$-action on $W^{k,p}(E_\tau)$ are linear. Consider now the map

$$\widetilde{A}_\tau \colon W^{k,p}(E_\sigma) \to W^{k,p}(E_\tau), \widetilde{A}_\tau = \int_K A_\tau^k \,\mathrm{d}k,$$

where we integrate w.r.t. the Haar measure of the compact group $K$. We will prove that $\widetilde{A}_\tau$ is a $K$-equivariant linear isomorphism for all $\tau$ sufficiently close to $\sigma$. Clearly, $\widetilde{A}_\tau$ is linear; the proof of $K$-equivariance is standard at this point, so we omit the details.

We observe that $\widetilde{A}_\sigma = \mathrm{id}$ (since $A_\sigma^k = \mathrm{id}$ for all $k \in K$) and that the family $\{\widetilde{A}_\tau\}$ is continuous in $\tau$.[5] It remains to prove that $\widetilde{A}_\tau$ is an isomorphism for $\tau$ near $\sigma$. This is the crux of this construction, and boils down to the fact that diffeomorphisms are an open subset.

**Lemma 5.17.** *If the open neighbourhood $U_\sigma \subset P$ of $\sigma$ is sufficiently small, every map $\widetilde{A}_\tau$ for $\tau \in U_\sigma$ is a continuous linear isomorphism.*

*Proof.* Consider the space $\mathcal{L} := \mathcal{L}(W^{k,p}(E_\sigma), W^{k,p}(E_\sigma))$ of continuous linear maps $W^{k,p}(E_\sigma) \to W^{k,p}(E_\sigma)$. Since $W^{k,p}(E_\sigma)$ is a Banach space, the subset

$$V := \{A \in \mathcal{L} \colon A \text{ is an isomorphism}\} \subset \mathcal{L}$$

is open and non-empty (since $\mathrm{id} \in V$).

This basically proves the claim: we just need to apply a small trick to convert the family $\{\widetilde{A}_\tau\}$ to one with constant target. Consider the family $\{B_\tau := \widetilde{A}_\tau \circ A_\tau^{-1}\}_{\tau \in U_\sigma}$ in $\mathcal{L}$. Since $A_\tau$ and $\widetilde{A}_\tau$ depend continuously on $\tau$, so does $B_\tau = \widetilde{A}_\tau \circ A_\tau^{-1}$. Since $\widetilde{A}_\sigma = \mathrm{id}$, also $B_\sigma = \mathrm{id}$. Shrinking $U_\sigma$, we can ensure all $B_\tau$ lie in $V$, i.e. each $\widetilde{A}_\tau \circ A_\tau^{-1}$ is an isomorphism. But since $A_\tau$ is an isomorphism itself, then so is $\widetilde{A}_\tau$. □

Pre- and post-composing $D_\tau$ with $\widetilde{A}_\tau$ and the $K$-equivariant isomorphism

$$B_\tau \colon W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(\Sigma, N_{u_\sigma})) \to W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(\Sigma, N_{u_\tau}))$$

induced from $\widetilde{A_\tau}$ yields a family of $K$-equivariant Fredholm operators

$$\widehat{D}_\tau := B_\tau^{-1} \circ D_\tau \circ \widetilde{A}_\tau \colon W^{k,p}(N_{u_\sigma}) \to W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(\Sigma, N_{u_\sigma})).$$

To summarise, on the last two pages we have proven the following result.

[5] For the latter, note that the family $\{A_\tau^k\}$ is continuous in both $\tau$ and $k$: each $A_\tau^k$ is just a composition of linear bundle maps; composition is continuous in the weak topology. The claim follows since $A_\tau$ is continuous in $\tau$ and $K$ acts continuously by continuous maps. Integration over $K$ is also continuous.

**Lemma 5.18.** *Each $\sigma \in P$ has a neighbourhood $U_\sigma \subset P$ such that there exists a smooth family $\{\widehat{D}_\tau\}$ of $K$-equivariant Fredholm operators $\widehat{D}_\tau \colon W^{k,p}(N_{u_\sigma}) \to W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(\Sigma, N_{u_\sigma}))$ with fixed domain and co-domain, such that* $\dim\ker \widehat{D}_\tau = \dim\ker D_\tau$ *and* $\dim\operatorname{coker} \widehat{D}_\tau = \dim\operatorname{coker} D_\tau$ *for all $\tau \in U_\sigma$, in particular $U_\sigma \cap P(k,c) = \{t \in U_\sigma \mid$ $\dim\ker \widehat{D}_\tau =$ $k\}$.* □

**Remark 5.19.** A similar argument allows upgrading a family of diffeomorphisms to equivariant diffeomorphisms: we will not have use for this in this thesis.

Let us present the right hand side as the zero set of a suitable smooth map. Abbreviate $X_\sigma := W^{k,p}(N_{u_\sigma})$ and $Y_\sigma := W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(\Sigma, N_{u_\sigma}))$ for brevity, so we have a smooth map

$$U_\sigma \to \mathcal{L}_K(X_\sigma, Y_\sigma), \tau \mapsto \widehat{D}_\tau$$

into the Banach space $\mathcal{L}_K(X_\sigma, Y_\sigma)$ of $K$-equivariant bounded real-linear maps $X_\sigma \to Y_\sigma$. Since $\widehat{D}_\sigma$ is Fredholm, we may choose a splitting $X_\sigma = V_\sigma \oplus \ker \widehat{D}_\sigma$ such that $V_\sigma \subset X_\sigma$ is a closed subspace and $\widehat{D}_\sigma$ maps $V_\sigma$ isomorphically to its image. Standard arguments (similar to [Wen15, Proposition 3.3.4] or [Wen23d, Proposition 3.13]) show that we can split $Y_\sigma = \operatorname{im} \widehat{D}_\sigma \oplus \ker \widehat{D}^*_\sigma$, where $\ker \widehat{D}^*_\sigma$ is equivalent to the space of all sections in $Y_\sigma$ which are $L^2$-orthogonal to $\operatorname{im} \widehat{D}_\sigma$. In terms of these splittings, we write $\widehat{D}_\tau$ in block form as

$$\widehat{D}_\tau = \begin{pmatrix} D^{11}_\tau & D^{12}_\tau \\ D^{21}_\tau & D^{22}_\tau \end{pmatrix};$$

after shrinking $U_\sigma$ if necessary, we may assume that $D^{11}_\tau \colon V_\sigma \to \operatorname{im} \widehat{D}_\sigma$ is invertible for all $\tau \in U_\sigma$. Therefore, there is a well-defined map

$$F_\sigma \colon U_\sigma \to \mathrm{Hom}_K(\ker \widehat{D}_\sigma, \ker \widehat{D}^*_\sigma), \tau \mapsto D^{22}_\tau - D^{21}_\tau \circ (D^{11}_\tau)^{-1} \circ D^{12}_\tau.$$

**Lemma 5.20.** *A parameter $\tau \in U_\sigma$ belongs to $P(k,c)$ if and only if $F_\sigma(\tau) = 0$.*

*Proof.* This is fully analogous to [Wen23d, Lemma 3.25]. □

The implicit function theorem shows that a neighbourhood of $\sigma$ in $P(k,c)$ is a smooth submanifold of co-dimension $\dim \mathrm{Hom}_K(\ker \widehat{D}_\sigma, \ker \widehat{D}^*_\sigma)$ whenever the linearization

$$dF_\sigma(\sigma) \colon T_\sigma P \to \mathrm{Hom}_K(\ker \widehat{D}_\sigma, \ker \widehat{D}^*_\sigma) \tag{5.1}$$

is surjective. The remainder of this chapter is devoted to proving this surjectivity.

## 5.3. The flexibility condition

In this section (and the next one), we establish the mathematical key ideas for proving surjectivity of the linearisation $dF_\sigma$ from (5.1). The first idea was dubbed "flexibility" by Doan and Walpuski: in broad terms, the space of normal Cauchy–Riemann operators is large enough to exhaust all $H$-equivariant Cauchy–Riemann type operators.[6] This is often much easier than the next step (dealing with Petri's condition). The precise statement to prove is modelled after [Wen23d, Lemma 5.27]. As before, we need to prove two versions of this, for tame and compatible equivariant almost complex structures.

**Lemma 5.21.** *Assume $J \in \mathcal{J}^G(M,\omega;\mathcal{U},J_{\mathit{fix}})$. Let $v \in \widetilde{\mathcal{M}}^{A,H}_{\mathcal{U}}(J)$ be a simple $J$-holomorphic curve with generalised normal bundle $N_v \subset v^*TM$ and $(A \times G)_v = H$, such that $N_v$ is the $\omega$-complement of the generalised tangent bundle $T_v \subset v^*TM$. Let $S \subset v^{-1}(\mathcal{U})$ be an open set of injective points such that $\mathrm{im}(v) \cap G \cdot v(s) = \pi_2(H) \cdot v(s)$ for all $s \in S$.*

*Given any $H$-equivariant section $A \in \Gamma^H(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, \mathrm{End}_{\mathbb{R}}(N_v)))$ with support contained in $S$, there exists a smooth family*

$$\{J_\tau \in \mathcal{J}^G(M,\omega;\mathcal{U},J_{\mathit{fix}})\}_{\tau\in(-\epsilon,\epsilon)}$$

*such that $J_0 = J$, $J_\tau(v(z)) = J(v(z))$ for all $\tau$ and $z$, and the resulting family $D^N_{v,\tau}$ of normal Cauchy–Riemann operators for $v$ w.r.t. $J_\tau$ satisfy*

$$\partial_\tau D^N_{v,\tau}\eta|_{\tau=0} = \pi_N \circ \nabla_\eta Y \circ Tv \circ j = A\eta$$

*for $\eta \in \Gamma(N_v)$, where $\nabla$ is any connection on $M$, $Y := \partial_\tau J_\tau|_{\tau=0} \in \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(TM,J))$, and $\pi_N \colon v^*TM \to N_v$ denotes the projection along $T_v$.*
*An analogous result holds for a smooth family $(J_t)$ in $\mathcal{J}^G_\tau(M,\omega;\mathcal{U},J_{\mathit{fix}})$.*

Let us begin with a motivational lemma, which is almost obvious: the variation of $G$-equivariant compatible (resp. tame) almost complex structures is again $G$-equivariant. Intuitively, this follows from the statement $T_J\mathcal{J}^G_\tau(M,\omega) = \Gamma^G(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma,J))$ — but making sense of this statement requires considering $\mathcal{J}^G(M,\omega)$ as a Fréchet manifold, which we would like to avoid.

**Lemma 5.22.** *Suppose $(J_\tau)_{\tau\in(-\epsilon,\epsilon)}$ is a smooth family in $\mathcal{J}^G_\tau(M,\omega)$ with $J_0 := J$. Then $Y := \partial_\tau J_\tau|_{\tau=0} \in \Gamma^G(\overline{\mathrm{End}}_{\mathbb{C}}(TM,J))$.*

*Proof.* Let $g \in G$ and $Y = \partial_t J_t|_{t=0} \in T_J\mathcal{J}^G_\tau(M,\omega)$ be arbitrary; we show $d\psi_g \circ Y = Y \circ d\psi_g$. By definition, for all $t$ we have $d\psi_g \circ J_t = J_t \circ d\psi_g$. Hence, for any $p \in M$ and $X \in T_pM$, we compute

$$\begin{aligned}(d\psi_g \circ Y)(p)(X) &= d\psi_g(Y(p)X) = d\psi_g(\partial_t J_t(p)X)|_{t=0} = \partial_t(d\psi_g(J_tX))|_{t=0} \\ &= \partial_t(J_t \circ d\psi_g(X))|_{t=0} = (\partial_t J_t)|_{t=0}(d\psi_g(X)) = (Y \circ d\psi_g)(p)(X)\end{aligned}$$

using linearity of $d\psi_g$ and the chain rule. □

[6] Perhaps the reader is wondering why we study the normal Cauchy–Riemann operators: analyzing $D^N_u$ is a useful intermediate step for proving Petri's condition. We analyse the passage from $D^N_u$ to its restriction $D^{N,\mathrm{restr}}_u$ in the next section.

To prepare the proof of Lemma 5.21, recall that we chose $N_v$ as the complement of $T_v \subset v^*TM$ w.r.t. some $H$-invariant bundle metric; in particular, $\Gamma(N_v)$ is $H$-invariant. In our setting, choose the bundle metric induced by $\omega(\cdot, J\cdot)$. This metric is $H$-invariant, since $H$ acts symplectically. Note that $T_v$ is $J$-invariant by construction. Recall that $D_v^N\colon \Gamma(N_v) \to \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma, N_v))$ is $H$-equivariant. Observe that each operator $D_{v,\tau}^N$ defined using $v$ and $J_\tau$ is also $H$-equivariant: $v$ is $J_\tau$-holomorphic w.r.t. each $\tau$ by hypothesis, since $J_\tau(v(z)) = J(v(z))$ for all $\tau$ implies $J_\tau \circ dv = J \circ dv = dv \circ j$.

Let $A := \partial_\tau D_{v,\tau}^N \eta|_{\tau=0} = \pi_N \circ \nabla_\eta Y \circ Tv \circ j$, where $\nabla$ is any connection on $M$.[7] Since all $D_{v,\tau}^N$ are $H$-equivariant, so is $A$, hence $A \in \Gamma^H(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma, N_v))$.

*Proof of Lemma 5.21.* We only prove the compatible case: the tame case is exactly the same, except with one fewer condition to check. For $\eta \in \Gamma(N_v)$, write $\nabla_\eta Y$ in block form as

$$\nabla_\eta Y = \begin{pmatrix} \nabla_\eta^T Y & \nabla_\eta^{TN} Y \\ \nabla_\eta^{NT} Y & \nabla_\eta^N Y \end{pmatrix} \in \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(v^*TM, J)),$$

with respect to the splitting $v^*TM = T_v \oplus N_v$. Then, using the definition of the normal Cauchy–Riemann operator, we compute

$$\partial_\tau D_{v,\tau}^N \eta|_{\tau=0} = \pi_N \circ \nabla_\eta Y \circ Tv \circ j = \nabla_\eta^{NT} Y \circ Tv \circ j$$

for any $\eta \in \Gamma(N_v)$.

Let $A \in \Gamma^H(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma, N_v))$ with $\mathrm{supp}(A) \subset S$ be arbitrary. We construct a smooth family $\{J_\tau\}$ so the normal derivatives $\nabla_\eta^{NT} Y$ match $A$ on $S$: i.e., for all $s \in S$, we have

$$A(s) = \nabla_\eta^{NT} Y \circ v \circ j(s). \tag{5.2}$$

Since $v$ has no critical nor double points on $S$, this condition determines $\nabla_\eta^{NT} Y$ on $S$.

Since $N_v$ is the $\omega$-symplectic orthogonal complement of $T_v$, the compatibility of the $J_\tau$ translates into two separate conditions on $\nabla_N^T Y$ and $\nabla_\eta^N Y$, and another condition linking $\nabla_\eta^{TN} Y$ to $\nabla_\eta^{NT} Y$, namely $\omega((\nabla_\eta^{NT} Y)v, w) + \omega(v, (\nabla_\eta^{TN})w) = 0$ for all $(v, w) \in T_v \oplus N_v$. Hence, compatibility of the $\{J_\tau\}$ does not prevent us from freely choosing $\nabla_\eta^{NT} Y$, as long as we choose $\nabla_\eta^{TN} Y$ in accordance and only do this in regions where $v$ has no double points, so the splitting $v^*TM = T_v \oplus N_v$ is unambiguous. By hypothesis, the set $S$ satisfies this constraint.

This choice should be compatible with the $G$-invariance of the $J_\tau$. The overall reason is the following: by $G$-equivariance, choosing $\{J_\tau(v(z))\}$ and $Y(v(z))$ also determines each $\{J_\tau(g \cdot v(z))\}$ and $Y(g \cdot v(z))$ by $J_\tau(g \cdot v(z)) = d\psi_g J_\tau(v(z))$ and $Y(g \cdot v(z)) = d\psi_g Y(v(z))$. For $g \in H$, this just matches $A(g \cdot z)$ since $A$ is $H$-invariant. If $g \notin H$, the condition $z \in S$ guarantees $g \cdot v(z) \notin \mathrm{im}(v)$, hence the condition $\nabla_\eta^{NT} Y \circ v \circ j(s) = A(s)$ is not affected.

[7] Since $J_\tau(v) \equiv J(v)$ for all $\tau$, we have $Y(v) \equiv 0$ and $\nabla Y$ is well-defined along $v$, independently of any connection.

Let us now make this construction of $Y$ precise, to convince also the very sceptic reader. We begin by choosing $Y(v) \equiv 0$. Then, we choose $\nabla_\eta^{NT} Y$ along $y(S)$ as determined by $A$: through Equation (5.2), $A(s)$ determines the value $\nabla_\eta^{NT} Y(y(s))$ uniquely. Since $S$ consists of injective points of $v$, choosing $\nabla_\eta^{NT} Y$ this way is possible. Choose $\nabla_\eta^{TN} Y$ along $y(S)$ as determined by the compatibility condition. By construction, $\nabla_\eta^{NT} Y$ is $H$-invariant, hence so is $\nabla_\eta^{TN} Y$. Finally, choose $\nabla_\eta^{N} Y$ and $\nabla_\eta^{T} Y$ along $y(S)$ to be $H$-invariant, and satisfy the constraint from the compatibility condition. Altogether, we have determined $Y$ along $y(S)$, and the result is $H$-invariant. As a final step, we extend $Y$ to all of $\Sigma$, using a $G$-invariant partition of unity. Averaging this extension ensures that the resulting section $Y$ is $G$-equivariant. This averaging leaves the value of $Y$ on $y(S)$ unchanged, as argued in the preceding paragraph: $Y$ was already $H$-equivariant there, and $g \cdot v(s) \notin S$ for $s \in S$ and $g \notin H$.
The choice of $Y$ determines a smooth family $J_\tau$ whose derivative is $Y$, using the exponential map (2.7) from Chapter 2: we set $J_t := J_{tY}$ for all $t$. By construction of $Y$, all $J_\tau$ are compatible. As $Y$ is $G$-equivariant, so are the resulting almost complex structures $J_\tau$ (as proven in Proposition 2.39); since $Y(v) \equiv 0$, all $J_\tau$ agree with $J = J_0$ along $v$. □

## 5.4. Petri's condition is generic

Let us present the main mathematical idea of this section: in Doan–Walpuski's terminology, this is about *Petri's condition* being generically satisfied (for the restricted normal Cauchy–Riemann operators). Since we only deal with walls for $H$ being a finite group, we can reduce our theorem to Wendl's results: while not trivial, it means we don't have to re-prove *everything*.

Let us begin by explaining what Petri's condition *means*. While we only consider Cauchy–Riemann type operators (see Definition 5.26), Petri's condition is defined more generally for linear partial differential operators. Let $E$ and $F$ be smooth real vector bundles over the same manifold $M$. Petri's condition is related to the so-called *Petri map*, which is defined by

$$\Pi\colon \Gamma(E) \otimes \Gamma(F) \to \Gamma(E \otimes F), \Pi(\eta \otimes \xi)(p) := \eta(p) \otimes \xi(p).$$

We will discuss purely local conditions, hence let us consider a local version of this map: fix a point $p \in M$ and consider the space $\Gamma_p(E) := \Gamma(E)/_\sim$ of *germs* of smooth sections at $p$, where $\eta, \eta' \in \Gamma(E)$ represent the same germ if and only if they coincide on some neighbourhood of $p$. The Petri map descends to a *local Petri map* at $p$,

$$\Pi\colon \Gamma_p(E) \otimes \Gamma_p(F) \to \Gamma_p(E \otimes F).$$

The local Petri map is never injective, for uninteresting reasons: for every smooth function $f\colon M \to \mathbb{R}$, the section $f\eta \otimes \xi - \eta \otimes (f\xi)$ lies in the kernel of $\Pi$. However, it *can* become injective when restricted to the space of solutions of suitable linear PDEs; this is what Petri's condition is about. Petri's condition is some kind of unique continuation condition. Unlike e.g. the similarity principle, it also contains an additional "quadratic" local condition. Proving that Petri's condition is satisfied (after generic perturbations) is the missing ingredient for completing the proof of the transversality theorem.

To make this precise, assume $D\colon \Gamma(E) \to \Gamma(F)$ is a linear partial differential operator with smooth coefficients. Let $D^*\colon \Gamma(F) \to \Gamma(E)$ denote its formal adjoint, with respect to some choice of $L^2$-pairings on $\Gamma(E)$ and $\Gamma(F)$. These pairings are induced from a choice of bundle metrics on $E$ and $F$ and a volume form on $M$. In the next section, we will consider more particular choices of pairings, which are invariant under a group action. For any point $p \in M$, both $D$ and $D^*$ descend to linear maps on spaces of germs of smooth sections at $p$, which we will denote by

$$D_p\colon \Gamma_p(E) \to \Gamma_p(F), \qquad D_p^*\colon \Gamma_p(F) \to \Gamma_p(E).$$

We also assume that $D$ and $D^*$ uniquely determine (say, via extension or restriction) linear maps

$$D\colon X(E) \to Y(F) \qquad \text{and} \qquad D^*\colon X^*(F) \to Y^*(E),$$

where $X(E)$, $Y^*(E)$, $X(F)$ and $Y^*(F)$ are vector spaces of sections (or equivalence classes of sections defined almost everywhere) of the respective bundles. In our applications later, these will be suitable Sobolev spaces. Let us add two further local conditions — both of which are satisfied for a large class of elliptic operators, including those of Cauchy–Riemann type.

- **(Regularity)** Every section in $\ker(D) \subset X(E)$ or $\ker(D^*) \subset X^*(F)$ is smooth.
- **(Unique continuation)** The maps $\ker D \to \ker D_p$ and $\ker D^* \to \ker D_p^*$ that send each section to its germ at $p$ are injective.

**Definition 5.23** (Petri's condition, [Wen23d, Definition 5.1])**.** *Suppose $D\colon X(E) \to Y(F)$ is a differential operator with formal adjoint $D^*\colon X^*(F) \to Y^*(E)$ satisfying the conditions specified above, and $p \in \mathcal{U} \subset M$. We say that $D$ satisfies*

(1) Petri's condition *if the restricted Petri map* $\ker D \otimes \ker D^* \to \Gamma(E \otimes F)$ *is injective;*

(2) Petri's condition over $\mathcal{U}$ *if there is no non-trivial element $t \in \ker D \otimes \ker D^*$ such that $\Pi(t) \in \Gamma(E \otimes F)$ vanishes identically on $\mathcal{U}$;*

(3) *the* local Petri condition *at $p$ if the restricted local Petri map* $\ker D_p \otimes \ker D_p^* \xrightarrow{\Pi} \Gamma_p(E \otimes F)$ *is injective;*

(4) Petri's condition to infinite order *at $p$ if there is no non-trivial element $t \in \ker D_p \otimes \ker D_p^*$ such that $\Pi(t)$ has vanishing derivatives of all orders at $p$.*

*Let us emphasize that the symbol "$\otimes$" always denotes* real *tensor products, even when the operator $D$ happens to be complex linear.*

**Remark 5.24** ([Wen23d, Remark 5.3])**.** It is clear from the definition that the set of points $p \in M$ at which the local Petri condition is not satisfied is open. In particular, proving that Petri's condition is satisfied to infinite order at a dense set of points in some $U \subset M$ implies the local Petri condition holds at *every* point in $U$.

Regarding the differences between these four conditions, let us quote Wendl directly [Wen23d, p. 163].

> Every condition on the list in [Definition 5.23] implies the previous one; note that the implication (3) $\Rightarrow$ (2) in particular follows from our regularity and unique continuation assumptions. The first two conditions are global in nature, as ker $D$ and ker $D^*$ depend on the global properties of $D$, including the choice of domains $X(E)$ and $X^*(F)$. These kernels will always be finite dimensional in the cases we consider, so that it seems unsurprising (if non-obvious) that Petri's condition might hold. In contrast, the third and fourth conditions are much stronger and more surprising because ker $D_p$ and ker $D_p^*$ are in general infinite dimensional, but the local conditions are also more powerful, e.g., it will be extremely useful to observe that they are preserved under pullbacks via branched covers of the base.

We will not yet have use for the invariance under pullback by branched covers; it will be extremely helpful when including multiply covered curves.

**Remark 5.25** ([Wen23d, Remark 5.2])**.** In general, the global Petri conditions may depend on only on the operator $D$, but also on the chosen of $L^2$-pairings on $\Gamma(E)$ and $\Gamma(F)$, via the choice of the induced formal adjoint $D^*$. However, let us emphasize that the local conditions are independent of this choice: indeed, whenever $D_1^*$ and $D_2^*$ are two operators arising as formal adjoints of $D$ via different choices of the $L^2$-pairings, there is a smooth bundle automorphism $\Phi\colon F \to F$ that maps local solutions of $D_1^*\xi = 0$ to local solutions of $D_2^*\xi = 0$, so that $\mathrm{id} \otimes \Phi\colon E \otimes F \to E \otimes F$ identifies the two different versions of $\ker \Pi \subset \ker D_p \otimes \ker D_p^*$.

Having encountered Petri's condition, let us now turn to the setting and main results of this section. Throughout this section, let us fix a Riemann surface $\Sigma$ and a Hermitian bundle metric $\langle \cdot, \cdot \rangle_\Sigma$ on $T\Sigma$. We do not require $\Sigma$ to be compact, as all discussion will be purely local. Fix a smooth complex vector bundle $E \to \Sigma$ with a Hermitian bundle metric, and denote $F := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E)$. We only consider Cauchy–Riemann type differential operators.

**Definition 5.26** (e.g. [Wen15, Definition 2.3.3, Definition 2.3.10])**.** *A* complex-linear Cauchy–Riemann type operator $D\colon \Gamma(E) \to \Gamma(F)$ *is a complex linear map* $D\colon \Gamma(E) \to \Gamma(F)$ *which satisfies the Leibniz rule*

$$D(fv) = (\overline{\partial} f)s + f(Ds) \tag{5.3}$$

*for all* $f \in C^\infty(\Sigma, \mathbb{C})$ *and* $s \in \Gamma(E)$*. A* real-linear Cauchy–Riemann type operator $D\colon \Gamma(E) \to \Gamma(F)$ *is a real linear map* $D\colon \Gamma(E) \to \Gamma(F)$ *such that* (5.3) *is satisfied for all* $f \in C^\infty(\Sigma, \mathbb{R})$ *and* $s \in \Gamma(E)$*.*

Let $\mathcal{CR}(E)$ denote the space of real Cauchy–Riemann type operators $D\colon \Gamma(E) \to \Gamma(F)$. The main result of this section is the following. It uses the results of Sections 5.3.

**Theorem 5.27.** *There exists a co-meagre subset $\mathcal{J}_{reg} \subset \mathcal{J}_\tau^G(M, \omega; \mathcal{U}, J_{fix})$ resp. $\mathcal{J}_{reg} \subset \mathcal{J}_\tau^G(M, \omega; \mathcal{U}, J_{fix})$ such that for all $J \in \mathcal{J}_{reg}$ and every curve $[u] \in \mathcal{M}_{\mathcal{U},1}^{A,H}(J)$, the restricted normal Cauchy-Riemann operator $D_u^{N,restr} \in \mathcal{CR}(N_{u^K})$ satisfies Petri's condition to infinite order on an open and dense set of points in $u^{-1}(\mathcal{U})$. In particular, $D_u^{N,restr}$ satisfies the local Petri condition at every point in $u^{-1}(\mathcal{U})$.*

The overall idea of the proof is to use the finiteness of $H$ to reduce this result to the non-equivariant case, where Wendl's methods can apply. Indeed, this theorem looks very similar to [Wen23d, Theorem 5.26]. However, there is a subtle difference: Wendl's result concerns genericity within the set of *all* compatible or tame almost complex structures, whereas we are looking for a co-meagre set of *equivariant* compatible resp. tame almost complex structures. Fortunately, the core argument of both proofs is the same — our workhorse theorem *can* re-use some of Wendl's machinery.

To set up the proof of Theorem 5.27, we need to go into some more detail about the setting Wendl used to prove Petri's condition. Our general set-up is similar to [Wen23d, §5.1–5.3], except that we need suitable adaptations to take the equivariance in our setting into account.

In all this discussion, we assume $E = N_u \subset u^*TM$ and $F = \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u)$ for some holomorphic curve $u\colon \Sigma \to M$ with $u \in \widetilde{\mathcal{M}}_{\mathcal{U}}^{A,H}(J)$. (Usually, we also assume $u$ lies in some iso-symmetric stratum $\mathcal{M}_{\mathcal{U},1}^{A,H}(J)$.) In particular, $H$ acts linearly on $\Gamma(E)$ and $\Gamma(F)$. We fix an $A$-invariant metric on $\Sigma$ and a $G$-invariant bundle metric on $TM$; this induces $H$-invariant $L^2$-pairings on $\Gamma(E)$ and $\Gamma(F)$, for every curve $u$. Let $D^*$ *always* denote the formal adjoint of some Cauchy–Riemann type operator $D\colon \Gamma(E) \to \Gamma(F)$, with respect to these pairings. When we speak about Petri's condition for $D$, we always use this formal adjoint $D^*$. Since the $L^2$-pairings on $\Gamma(E)$ and $\Gamma(F)$ are $H$-invariant, by Lemma 4.84, $D^*$ is also $H$-equivariant.

**Recall.** For any smooth vector bundle $E \to \Sigma$, a point $z \in \Sigma$ and $k \in \mathbb{Z}$, denote by $(\Gamma_z(E))^k \subset \Gamma_z(E)$ the space of germs of smooth sections at $z$ whose derivatives of order up to $k-1$ all vanish at $z$. The space of *$k$-jets of sections at $z$* is defined as

$$J_z^k E := \Gamma_z(E)/(\Gamma_z(E))^{k+1}.$$

A linear Cauchy–Riemann type operator $D\colon \Gamma(E) \to \Gamma(F)$ induces a map $\Gamma_z(E) \to \Gamma_z(F)$ between germs of sections at $z \in \Sigma$; this map descends to a map $J_z^k E \to J_z^{k-1} F$ between the $k$-jet spaces at $z$.

Denote by $\widehat{\mathscr{D}}_z(E, F)$ the real vector space of germs of Cauchy–Riemann type operators $\Gamma(E) \to \Gamma(F)$, and let $\widehat{\mathscr{D}}_z^k(E, F) \subset \mathrm{Hom}(J_z^k E, J_z^{k-1} F)$ be the real vector space of induced maps on $k$-jet spaces. Refining this, let $\widehat{\mathscr{D}}_z^H(E, F)$ denote the real vector space of germs of $H$-equivariant Cauchy–Riemann type operators $\Gamma(E) \to \Gamma(F)$, and $\widehat{\mathscr{D}}_z^{H,k}(E,F) \subset \mathrm{Hom}(J_z^k E, J_z^{k-1} F)$ be the real vector space of induced maps on $k$-jet spaces.

### 5.4.1. A science-fiction proof

To motivate the main ideas of our argument (including the use of restricted normal Cauchy–Riemann operators), let us start with a simplified version. The expert reader can safely skip this sub-section. Fix a point $z \in \Sigma$. As a first approximation of the real definition, consider the following. We begin with considering the subset

$$\mathcal{V}^k := \{(D,t) \in \widehat{\mathscr{D}}_z^{H,k}(E,F) \times (J_z^k E \otimes J_z^k F) \mid t \in \ker D \otimes \ker D^*\}.$$

Then, let us augment this definition in two ways. Firstly, we prescribe the *rank* of $t$: given a pair of real vector spaces $V$ and $W$, we say an element $t \in V \otimes W$ has rank $r$ if $t = \sum_{j=1}^r v_j \otimes w_j$ for two linearly independent sets $\{v_1, \dots, v_r\} \in V$ and $\{w_1, \dots, w_r\} \in W$. Secondly, we include the order of vanishing of $t$: altogether, we consider the subset

$$\mathcal{V}_{r,\ell}^k := \{(D,t) \in \mathcal{V}^k \mid \mathrm{rank}(t) = r, t \notin (J_z^k E \otimes J_z^k F)^\ell\}$$

for given integers $k, r$ and $\ell$ (with $\ell \in \{1, \dots, k\}$). Sometimes, it will be useful to also consider the following variant, for $D \in \widehat{\mathscr{D}}_z^{H,k}(E,F)$:

$$\mathcal{V}_{r,\ell}^k(D) := \{t \in (J_z^k E \otimes J_z^k F) \mid (D,t) \in \mathcal{V}_{r,\ell}^k\}.$$

For all $k \in \mathbb{Z}$, the local Petri map $\Pi\colon \Gamma_z(E) \otimes \Gamma_z(F) \to \Gamma_z(E \times F)$ descends to a linear map

$$\Pi^k\colon J_z^k E \otimes J_z^k F \to J_z^k(E \otimes F),$$

which preserves the vanishing orders. As the projection $\mathcal{V}_{r,\ell}^k \to J_z^k E \otimes J_z^k F, (D,t) \mapsto t$ is smooth,[8] $\Pi^k$ gives rise to a smooth map

$$\Pi_{r,\ell}^k\colon \mathcal{V}_{r,\ell}^k \to J_z^k(E \otimes F), (D,t) \mapsto \Pi^k(t)$$

whose zero set

$$\mathscr{P}_{r,\ell}^k := (\Pi_{r,\ell}^k)^{-1}(0) = \{(D,t) \in \mathcal{V}_{r,\ell}^k \mid \Pi^k(t) = 0\}$$

we will call the *universal Petri moduli space*. (This is not the fully correct definition yet, but the real definition will be very similar.)

We would like to prove a result similar to Wendl's "workhorse lemma" [Wen23d, Proposition 5.25]. That Proposition is the main step towards Petri's condition being generic among Cauchy–Riemann type operators. It is a statement about $\mathscr{P}_{r,\ell}^k \subset$

[8] It turns out $\mathcal{V}_{r,\ell}^k$ is a smooth finite-dimensional manifold: we omit the details, but refer the reader to the analogous argument in the proof of Proposition 5.38.

$\mathcal{V}^k_{r,\ell}$ being a $C^\infty$*-sub-variety*[9] with a suitable lower bound on its co-dimension. The following *looks* like a suitable analogue of Wendl's proposition.

**Pseudo-Proposition 5.28.** For every $\ell \in \mathbb{N}$, there exists a constant $C_\ell > 0$ such that for all integers $k \geq \ell$ and $r \in \mathbb{N}$, the universal Petri moduli space $\mathscr{P}^k_{r,\ell} \subset \mathcal{V}^k_{r,\ell}$ is a $C^\infty$-sub-variety of co-dimension at least $C_\ell k^2$.

**Remark 5.29.** This result looks almost the same as Wendl's [Wen23d, Theorem 5.25]. There are two subtle differences. Firstly, the notations $\mathscr{P}^k_{r,\ell}$ and $\mathcal{V}^k_{k,\ell}$ have analogous, but different meaning: in this thesis, it refers to a space defined using germs $\widehat{\mathscr{D}}^{H,k}_z(E, F)$ of $H$-equivariant Cauchy–Riemann type operators at $z$ — whereas Wendl's setting involves the germs $\widehat{\mathscr{D}}^k_z(E, F)$ of *all* Cauchy–Riemann type operators.

The second difference is already visible in the notation: in Wendl's setup, the spaces $\mathscr{P}^k_{r,\ell}$ and $\mathcal{V}^k$ depend on choices of *geometric data* used to define the formal adjoint $D^*$ of $D$: namely, germs at $z$ of bundle metrics on $E$ and $F$ and a volume form on $\Sigma$. To reflect this dependence, these data even appear explicitly in Wendl's notation.

In our context, the data defining the formal adjoint $D^*$ is chosen slightly differently, as we want to ensure that $D^*$ is also $H$-equivariant. Instead of choosing bundle metrics on $E$ and $F$, we choose (invariant) Riemannian metrics on $\Sigma$ and $M$, and consider the induced $L^2$-pairings on $\Gamma(E)$ and $\Gamma(F)$: we make on arbitrary (but global) choice, and only work with that. Hence, there is no mention of geometric data in this thesis' notation. Lest the reader worry about this difference creating a subtle incompatibility with Wendl's results: by Remark 5.25, Petri's local conditions are independent of the chosen geometric data, so fixing data this way does not cause any issues.

Let us illustrate the main idea of this section with a "science fiction proof" of Pseudo-Proposition 5.28, under the hypothetical assumptions that $\widehat{\mathscr{D}}^{H,k}_z(E, F)$ equals the space of all germs $\widehat{\mathscr{D}}^k_z(E, F)$ for all $z \in \Sigma$, and $D^N_u$ equals $D^{N,\text{restr}}_u$.

*Science fiction proof of Pseudo-Proposition 5.28.* Assume $\widehat{\mathscr{D}}^{H,k}_z(E, F) = \widehat{\mathscr{D}}^k_z(E, F)$ holds for all $z \in \Sigma$, and $D^N_u$ equals the restricted operator $D^{N,\text{restr}}_u$.

Let $\ell \in \mathbb{N}$ be arbitrary. By [Wen23d, Proposition 5.25], there exists a constant $C_\ell > 0$ such that for all integers $k \geq \ell$ and all $r \in \mathbb{N}$, the space $\mathscr{P}^k_{r,\ell}(g, h, \mu) \subset \mathcal{V}^k(g, h, \mu)$ (in Wendl's notation) is a $C^\infty$-sub-variety of co-dimension at least $C_\ell k^2$. Because of Remark 5.25 and our "science fiction" hypothesis, in fact Wendl's $\mathscr{P}^k_{r,\ell}(g, h, \mu)$ equals our space $\mathscr{P}^k_{r,\ell}$ and Wendl's space $\mathcal{V}^k(g, h, \mu)$ equals our $\mathcal{V}^k_{r,\ell}$. In other words, there is nothing left to prove. □

[9] We do not specify the precise definition of $C^\infty$-sub-varieties: the interested reader may consult [Wen23d, Appendix C] or pretend they are smooth submanifolds. A $C^\infty$-subvariety of a smooth Banach manifold need not be a Banach submanifold (though every Banach submanifold is a $C^\infty$-subvariety), but it is "almost as nice" in a precise sense. For instance, the Sard–Smale theorem still applies to $C^\infty$-subvarieties.

For posterity, let us note that this assumption is satisfied if $u$ has trivial stabiliser $G_u$: then $D_u^N$ and $D_u^{N,\mathrm{restr}}$ agree by definition. By Observation 3.12, all but finitely many points $z \in \Sigma$ satisfy $G_u = G_{u(z)}$, hence also have trivial stabilisers. The following argument proves that our assumption holds for all such points. Hence, Pseudo-Proposition 5.28 holds *if* all stabilisers $G_{u(z)}$ are trivial.

Let $\mathcal{U} \subset \Sigma$ be an open subset with compact closure. Fix one $H$-equivariant operator $D_{\mathrm{fix}} \in \mathcal{CR}(E)$ and denote

$$\mathcal{CR}(E;\mathcal{U}, D_{\mathrm{fix}}) := \{D \in \mathcal{CR}(E) \mid D = D_{\mathrm{fix}} \text{ on } \Sigma \setminus \mathcal{U}\}.$$

Denote by $\mathcal{CR}_H(E;\mathcal{U}, D_{\mathrm{fix}})$ the space of $H$-equivariant Cauchy–Riemann type operators in $\mathcal{CR}(E;\mathcal{U}, D_{\mathrm{fix}})$. Recall that in this section, we always have $E = N_u$.

**Lemma 5.30.** *Let $u \in \widetilde{\mathcal{M}}_{\mathcal{U}}^{A,H}(J)$ such that $[u] \in \mathcal{M}_{\mathcal{U},l}^{A,H}(J)$. Suppose $z \in \Sigma$ is an injective point of $u$ as in Lemma 4.75 and $G_{u(z)}$ is trivial. Then, for each germ $D_z \in \mathrm{Hom}(\Gamma_z(N_u), \Gamma_z(F))$ of some Cauchy–Riemann type operator $D \in \mathcal{CR}(N_u;\mathcal{U}, D_{\mathit{fix}})$ at $z$, there exists an $H$-equivariant Cauchy–Riemann type operator $D^H \in \mathcal{CR}_H(N_u;\mathcal{U}, D_{\mathit{fix}})$ with germ $D_z$ at $z$.*

Its proof uses the following basic observation.

**Observation 5.31.** Let $D_1, D_2 \colon \Gamma(E) \to \Gamma(F)$ be real Cauchy–Riemann type operators and $\psi \colon \Sigma \to \mathbb{R}$ be a smooth function. The convex combination $D := \psi D_1 + (1 - \psi) D_2$ is a real Cauchy–Riemann type operator.

*Proof.* We need to show that $D(fs) = (\overline{\partial} f)s + fDs$ for all $s \in \Gamma(E)$ and $f \in C^\infty(\Sigma, \mathbb{R})$. Indeed, for any $s$ and $f$, we compute

$$\begin{aligned} D(fs) &= \psi D_1(fs) + (1-\psi) D_2(fs) = \psi((\overline{\partial} f)s + fD_1 s) + (1-\psi)((\overline{\partial} f)s + fD_2 s) \\ &= (\psi + 1 - \psi)(\overline{\partial} f)s + \psi f D_1 s + (1-\psi) f D_2 s = (\overline{\partial} f)s + f(\psi D_1 s + (1-\psi) D_2 s) \\ &= (\overline{\partial} f)s + fDs. \end{aligned}$$

□

*Proof of Lemma 5.30.* Write $H = \{h_0 = \mathrm{id}, h_1, \dots, h_n\} = \{(\mathrm{id}, \mathrm{id}), (\phi_1, g_1), \dots, (\phi_n, g_n)\}$; denote $A_H := \pi_1(H) = \{\mathrm{id}, \phi_1, \dots, \phi_n\}$. Since $G_{u(z)}$ is trivial, the stabiliser $G_u$ also is trivial, in particular $\phi_i \neq \mathrm{id}$ for all $i > 0$. Stronger yet, we have $\phi(z) \neq z$ for all $i > 0$: otherwise, one could have

$$u(z) = (h_i^{-1} \cdot u)(z) = g_i^{-1} \cdot u(\phi_i(z)) = g_i^{-1} \cdot u(z),$$

hence $g_i^{-1} \in G_{u(z)}$, contradiction. Choose an open neighbourhood $U \subset \Sigma$ of $z$ such that the sets $\{\phi_i(U)\}$ are pairwise disjoint. Choose a smooth function $\psi \colon U \to [0, 1]$ such that $\psi \equiv 1$ on a neighbourhood of $z$ and $\operatorname{supp} \psi \subset U$. Extend $\psi$ to $A_H \cdot U$ by $\psi(\phi_i(x)) := \psi(x)$; let us denote this extension by $\psi$ again.[10] Observe that $\psi$ is $A_H$-invariant.

[10]Since the sets $\phi_i(U)$ are disjoint, this extension is well-defined without having to prove anything.

We construct the operator $D^H$ in two steps. We begin by writing down a Cauchy–Riemann type operator $D_0$ whose germ at each $\phi_i(z)$ is induced by the germ $D_z$. To wit, we consider the operator $D_0$ defined by, for any $s \in \Gamma(E)$ and $z' \in \Sigma$,

$$D_0 s(z') := \begin{cases} \psi[d\psi_{g_i} \circ Ds(\phi_i^{-1}(z'))] + (1-\psi)Ds(z') & \text{if } z' \in \phi_i(U) \\ Ds(p) & \text{if } z' \in \Sigma \setminus \operatorname{supp}(\psi). \end{cases}$$

By Observation 5.31, this defines a real Cauchy–Riemann type operator. By construction, the germ $(D_0 s)_{\phi_i(z)}$ of $D_0 s$ at $\phi_i(z)$ is given as

$$(D_0)_{\phi_i(z)} s = (d\psi_{g_i} \circ Ds(\phi_i^{-1}(p)))_{\phi_i(z)} = d\psi_{g_i} \circ (D_z)s.$$

This completes the first step of the construction.

Next, we average $D_0$ to obtain an $H$-equivariant operator $D^H$ whose germ at $z$ is $D_z$. Consider the operator $D^H \colon \Gamma(E) \to \Gamma(F)$ defined by

$$D^H(s)(z') := \frac{1}{|H|} \sum_{h \in H} d\psi_{g_i}^{-1} \circ D_0 s(\phi_i(z')),$$

for $s \in \Gamma(E)$ and $z' \in \Sigma$. Applying Observation 5.31 finitely many times, we deduce that $D^H$ is a real Cauchy–Riemann type operator; it is clearly $H$-equivariant by construction. Moreover, its germ at $z$ is given by

$$\begin{aligned}(D^H)_z s &= \frac{1}{|H|} \sum_{h \in H} d\psi_{g_i}^{-1} \circ (D_0)_{\phi_i(z)}(s) = \frac{1}{|H|} \sum_{h \in H} d\psi_{g_i}^{-1} \circ (d\psi_{g_i} \circ (D_z)s) \\ &= \frac{1}{|H|} \sum_{h \in H} (D_z)s = D_z s,\end{aligned}$$

thus $D^H$ is an equivariant Cauchy–Riemann type operator with the desired properties. □

As mentioned above, pseudo-Proposition 5.28 holds at almost all points, if $u$ has trivial stabiliser. While nice to know, this only helps so much, precisely because the stabiliser $G_u$ need not be trivial in general. This imposes a constraint on the germs $\widehat{\mathscr{D}}_z^{H,k}(E,F)$, and is the reason we consider the restricted normal Cauchy–Riemann operators. Lemma 5.30 *can* be generalised to curves with non-trivial stabiliser, but this requires changing the definitions of $\mathcal{V}_{r,\ell}^k$ and $\mathscr{P}_{r,\ell}^k$.

### 5.4.2. The universal Petri moduli space

We end the digression of the previous section, and return to fully rigorous mathematics. As mentioned already, the correct way to encode the additional constraint imposed by the stabiliser group $G_u$ is to consider the sub-bundle $E^K := N_{u^K}$ of $E = N_u$. Let us recall the relevant notation from Section 5.1.

**Recall.** We have $u \in \mathcal{M}^{A,H}_{\mathcal{U},1}(J)$; the fixed point set $M^K := \mathrm{Fix}(G_u) \subset M$ is a symplectic and almost complex manifold. $u$ co-restricts to a holomorphic curve $u^K\colon \Sigma \to M^K$, and there are injective bundle maps $E^K := N_{u^K} \to E$ and $F^K := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E^K) \to F$. The restricted normal Cauchy–Riemann operator $D_u^{N,\mathrm{restr}}\colon \Gamma(E^K) \to \Gamma(F^K)$ is the normal Cauchy–Riemann operator of $u^K$. Let $K$ denote the group $G_u$ if $G_u$ is non-trivial, and $H$ otherwise. Then $\Gamma(E^K)$ and $\Gamma(F^K)$ are $K$-invariant, and $D_u^{N,\mathrm{restr}}$ is a $K$-equivariant Fredholm operator.

The above inclusion descends to germs of sections at any $z \in \Sigma$, and to $k$-jets of sections, yielding an injective linear map

$$\Phi\colon J^k_z E^K \otimes J^k_z F^K \to J^k_z E \otimes J^k_z F.$$

There is another way to look at $D_u^{N,\mathrm{restr}}$: as $D_u^N$ is $H$-equivariant, it must send $\Gamma(E^K) \subset \Gamma(E)$ to $\Gamma(F^K)$[11]. In fact, this holds for any $H$-equivariant Cauchy–Riemann operator.

**Lemma 5.32.** *An $H$-equivariant Cauchy–Riemann type operator $D\colon \Gamma(E) \to \Gamma(F)$ maps $\Gamma(E^K)$ to $\Gamma(F^K)$, hence induces a Cauchy–Riemann type operator $\Gamma(E^K) \to \Gamma(F^K)$.*

*Proof.* Recall that $T_{u^K} M^K = \mathrm{Fix}(G_u) \subset T_u M$, by Lemma 4.56. Let $\eta \in \Gamma(E^K)$ be arbitrary. We will show that $D(d\iota(\eta))$ is $G_u$-invariant. Indeed, for all $g \in G_u$, $z \in \Sigma$ and $Y \in T_z\Sigma$ we have

$$g \cdot ((D\eta)_z Y) = (g \cdot D\eta)_z Y = D(g \cdot \eta)_z Y = (D\eta)_z Y,$$

using the $G_u$-invariance of $\eta$ as well as the $H$-equivariance of $D$.[12] □

In other words, there is a well-defined linear map $\Phi^*\colon \mathcal{CR}_H(E) \to \mathcal{CR}(E^K)$. This map is the proper context for the definition of the restricted normal Cauchy–Riemann operators: observe that $\Phi^*(D_u^N)$ is precisely the restricted normal Cauchy–Riemann operator $D_u^{N,\mathrm{restr}}$. The map $\Phi^*$ descends to germs of operators at any $z \in \Sigma$, and to $k$-jets at $z$, yielding a linear map

$$\Phi^*\colon \widehat{\mathscr{D}}^{H,k}_z(E,F) \to \widehat{\mathscr{D}}^{k}_z(E^K,F^K). \tag{5.4}$$

**Remark 5.33.** The maps $\Phi$ and $\Phi^*$ are compatible, in the sense that $D(\Phi(t)) = (\Phi^* D)(t) \in J^k_z F^K$ for all $D \in \widehat{\mathscr{D}}^{H,k}_z(E,F)$ and $t \in J^k_z E^K$. In particular, $t \in \ker \Phi^*(D)$ iff $\Phi(t) \in \ker D$.

We also need to study the behaviour of the formal adjoint $D^*$ of $D \in \mathcal{CR}_H(E)$ w.r.t. these sub-bundles. Recall that the formal adjoint $D^*$ was defined using choices of an $A$-invariant metric on $\Sigma$ and a $G$-invariant bundle metric on $TM$. Since $M^K \subset M$ is a smooth submanifold, the latter induces a bundle metric on $TM^K$, hence $L^2$-pairings

[11] or, for the pedantic, to the image of $\Gamma(F^K)$ under the map $F^K \to F$

[12] Observe that $g \in G$ satisfies $g \in G_u$ if and only if $(\mathrm{id}, g) \in H$. Thus, we may regard $G_u$ as a subgroup of $H$.

on $\Gamma(E^K)$ and $\Gamma(F^K)$. By construction, these are compatible with the pairings on $\Gamma(E)$ and $\Gamma(F)$, w.r.t. the inclusions $E^K \hookrightarrow E$ and $F^K \hookrightarrow F$. (Informally speaking, "the pairings on $\Gamma(E^K)$ and $\Gamma(F^K)$ are the restrictions of the corresponding pairings on $\Gamma(E)$ and $\Gamma(F)$".) In particular, the formal adjoint $\Phi^*(D)^*$ of the operator $\Phi^*(D)\colon \Gamma(E^K) \to \Gamma(F^L)$ is exactly the operator induced by $D^*$ itself. In formulas, letting $\Psi^*\colon \widehat{\mathscr{D}}_z^{H,k}(F,E) \to \widehat{\mathscr{D}}_z^k(F^K, E^K)$ denote the analogous map to $\Phi^*$, we have $\Phi^*(D)^* = \Psi^*(D^*)$.

Since $D$ and $D^*$ both descend to $E^K$ and $F^K$, we can also study Petri's condition for the operators $\Phi^*(D)$. The careful reader will note that our setting involves two local Petri maps, one for $E$ and $F$, the other for $E^K$ and $F^K$. (By abuse of notation, we will denote both by $\Pi^k$; it will be clear from context, which one is meant.) Fortunately, these are compatible in a precise sense.

**Lemma 5.34.** *For all $t \in J_z^k E^K \otimes J_z^k F^K$, we have $\iota(\Pi^k(t)) = \Pi^k(\Phi(t))$.*

*Proof.* The analogous statement $\iota \circ \Pi = \Pi \circ \Phi$ holds for the global Petri map. Passing to germs of sections at $z$ and $k$-jets yields the claim. □

**Corollary 5.35.** *Suppose $D \in \widehat{\mathscr{D}}_z^{H,k}(E,F)$ and $t \in \ker \Phi^*(D) \otimes \ker \Phi^*(D)^*$. Then $\Pi^k(\Phi(t)) = 0$ if and only if $\Pi^k(t) = 0$.*

*Proof.* By Remark 5.33, we have $\Phi(t) \in \ker(D) \otimes \ker D^*$. Now apply Lemma 5.34. □

The map $\Phi$ will be extremely useful: while the left hand side consists of (jets of germs of) $H$-equivariant operators, the right hand side has no such equivariance constraint any more — so we can apply Wendl's non-equivariant theory. To make use of this observation, it is crucial to observe that $\Phi$ is surjective, for suitable points $z$. The following result makes this precise, and is the proper generalisation of Lemma 5.30.

**Lemma 5.36.** *If $z \in \Sigma$ is an injective point of $u$ as in Lemma 4.75, the map $\Phi$ from Equation (5.4) is surjective.*

*Proof.* Let $D_z \in \widehat{\mathscr{D}}_z^k(E^K, F^K)$ be arbitrary; choose some Cauchy–Riemann type operator $D\colon \Gamma(E^K) \to \Gamma(F^K)$ whose germ at $z$ is $D_z$. We aim to find some operator $D^H \in \widehat{\mathscr{D}}_z^{H,k}(E,F)$ whose germ $(D^H)_z$ at $z$ satisfies $\Phi((D^H)_z) = D_z$. We imitate the proof of Lemma 5.30; unlike in that proof, we need to make auxiliary choices to extend $D$ to $\Gamma(E)$. The $G_u$-equivariance of $F^K$ will ensure the well-definedness of the result.

Write $H = \{h_1, \ldots, h_n\}$ with $h_i = (\phi_i, g_i)$ for all $i$; denote $A_H := \pi_1(H) := \{\phi_1, \ldots, \phi_n\}$. Unlike in Lemma 5.30, the elements $g_i$ are *not* all distinct: instead, for each $\phi \in A_H$, the group $H$ contains precisely $|G_u|$ distinct elements of the form $(\phi, g)$, i.e. the multi-set $\{g_1, \ldots, g_n\}$ contains each element exactly $|G_u|$ times. Observe further that $\phi_i \neq \mathrm{id}$ implies $\phi_i(z) \neq z$: otherwise, one would have have

$$u(z) = (h_i^{-1} \cdot u)(z) = g_i^{-1} \cdot u(\phi_i(z)) = g_i^{-1} \cdot u(z),$$

hence $g_i^{-1} \in G_{u(z)} = G_u$ and $(\mathrm{id}, g_i) \in H$, contradicting $\phi_i \neq \mathrm{id}$.[13]

Choose an open $G_u$-invariant neighbourhood $U \subset \Sigma$ of $z$ such that the sets $\{\phi_i(U)\}$ are pairwise either identical or disjoint. Choose a smooth $G_u$-invariant function $\Psi_0 \colon U \to [0,1]$ such that $\Psi_0 \equiv 1$ on a neighbourhood of $z$ and $\operatorname{supp} \Psi_0 \subset U$. Extend $\Psi_0$ to a function $\Psi$ on $A_H \cdot U$ by $\Psi(\phi_i(x)) := \Psi_0(x)$. This is well-defined since $\Psi_0$ was $G_u$-invariant: if $x \in \phi_i(U) \cap \phi_j(U)$, by construction $\phi_i(U) = \phi_j(U)$, hence $\phi_i(U) = \phi_i(\phi_i^{-1} \circ \phi_j(U))$, thus $\phi_i^{-1} \circ \phi_j(U) = U$ and $(\phi_i \circ \phi_j^{-1})(z) \in U$ follows. By construction, this implies $(\phi_i \circ \phi_j^{-1})(z) = z$, i.e. $g_i \circ g_j^{-1} \in G_{u(z)} = G_u$. Since $\Psi_0$ is $G_u$-invariant, this proves well-definedness. By construction, $\Psi$ is $A_H$-invariant.

The operator $D$ defines a Cauchy–Riemann type operator $\Gamma(E^K) \to \Gamma(F)$. Choose some extension $D'$ of this operator to a Cauchy–Riemann type operator $\Gamma(E) \to \Gamma(F)$. Let $D_{\mathrm{arb}} \colon \Gamma(E) \to \Gamma(F)$ be *any* Cauchy–Riemann type operator.[14] As the next step, we consider the operator $D_0 \colon \Gamma(E) \to \Gamma(F)$ defined by, for any $s \in \Gamma(E)$ and $z' \in \Sigma$,

$$D_0 s(z') := \begin{cases} \psi[d\psi_{g_i} \circ D's(\phi_i^{-1}(z'))] + (1-\psi) D_{\mathrm{arb}} s(z') & \text{if } z' \in \phi_i(U) \\ D_{\mathrm{arb}} s(z') & \text{if } z' \in \Sigma \setminus \operatorname{supp}(\psi). \end{cases}$$

This definition is well-defined: if $\phi_i(U) = \phi_j(U)$, we have $g_i \circ g_j^{-1} \in G_u$ (as shown above), and hence compute

$$d\psi_{g_i}(D's(\phi_i(x))) = d\psi_{g_i}(D's(\phi_j(x))) = d\psi_{g_i} \circ (d\psi_{g_i \circ g_j^{-1}} D's(\phi_j(x))) = d\psi_{g_j}(D's(\phi_j(x))),$$

hence the first branch in the definition of $D_0$ is independent of the index $i$. Both branches of the definition agree on their intersection, by construction. Both defining sets are open, so the joint operator is continuous and smooth. Since $\operatorname{supp}(\psi) \subset \bigcup_i \phi_i(U)$, this defines $D_0$ everywhere. Applying Observation 5.31 finitely many times, $D_0$ is indeed a Cauchy–Riemann type operator. This completes the first step of the construction.

Next, we average $D_0$ to obtain an $H$-equivariant operator $D^H$ whose germ at $z$ is $D_z$. Consider the operator $D^H \colon \Gamma(E) \to \Gamma(F)$ defined by

$$D^H(s)(z') := \frac{1}{|H|} \sum_{h \in H} d\psi_{g_i}^{-1} \circ D_0 s(\phi_i(z')),$$

for $s \in \Gamma(E)$ and $z' \in \Sigma$. Applying Observation 5.31 finitely many times, we deduce that $D^H$ is a real Cauchy–Riemann type operator; it is clearly $H$-equivariant by construction.

It remains to verify that $\Phi^*((D^H)_z) = D_z$. Indeed, let us compute the germ of $D^H$ at $z$. The germ of $D_0$ at $\phi_i(z)$ is given by

$$(D_0)_{\phi_i(z)} s = (d\psi_{g_i} \circ D's(\phi_i^{-1}(z')))_{\phi_i(z)} = d\psi_{g_i} \circ (D'_z) s,$$

[13] At this step, we are using that $u$ is simple: if $g \cdot u = g \cdot u \circ \phi$ for some $(\phi, g) \in A \times G$, we have $u = u \circ \phi$ since $G$ acts by diffeomorphisms, and simplicity of $u$ implies $\phi = \mathrm{id}$.

[14] For instance, we could choose $D_{\mathrm{arb}} = D'$: we merely use a different name to stress that the choice of $D_{\mathrm{arb}}$ will not matter at all.

and we deduce that

$$
\begin{aligned}
(D^H)_z s &= \frac{1}{|H|}\sum_{h\in H} d\psi_{g_i}^{-1}\circ (D_0)_{\phi_i(z)}(s) = \frac{1}{|H|}\sum_{h\in H} d\psi_{g_i}^{-1}\circ (d\psi_{g_i}\circ (D'_z)s) \\
&= \frac{1}{|H|}\sum_{h\in H}(D'_z)s = D'_z s.
\end{aligned}
$$

In particular, $\Phi^*((D^H)_z) = \Phi^*(D'_z) = D_z$, completing the proof. □

Let us now state the main result of this section. The statement might be a big surprising, as it combines germs of operators $\widehat{\mathscr{D}}_z^{H,k}(E,F)$ with sections $t \in J_z^k E^K \otimes J_z^k F^K$ (so $D \in \widehat{\mathscr{D}}_z^{H,k}(E,F)$ is applied to $\Phi(t) \in J_z^k E \otimes J_z^k F$). The main reason is that Wendl's workhorse result applies to the space $\widehat{\mathscr{D}}_z^k(E^K, F^K)$ and sections $t \in J_z^k E^K \otimes J_z^k F^K$, and provides a lower bound for the co-dimension of the universal Petri moduli space in terms of the rank and vanishing order of $t$.

While there is a splitting $J_z^k E \otimes J_z^k F = J_z^k E^K \otimes J_z^k F^K \oplus R$ (where $R$ is some choice of complement of $J_z^k E^K \otimes J_z^k F^K$, e.g. induced by the $L^2$-pairings on $\Gamma(E)$ and $\Gamma(F)$), the rank and vanishing order of a section $\tilde{t} \in J_z^k E \otimes J_z^k F$ are not directly related to the rank and vanishing order of its projection $t \in J_z^k E^K \otimes J_z^k F^K$. As such, a universal Petri moduli space defined in terms of rank and vanishing orders of sections $J_z^k E \otimes J_z^k F$ is not as useful, as there is no clear relation to results about $J_z^k E^K \otimes J_z^k F^K$.

**Definition 5.37.** *Consider the set*

$$
\begin{aligned}
\mathcal{V}_{r,\ell}^k(u,z) := \{(D,t) \in \widehat{\mathscr{D}}_z^{H,k}(E,F) \times (J_z^k E^K \otimes J_z^k F^K) \mid \\
\Phi(t) \in \ker D \otimes \ker D^*,\ \operatorname{rk}(t) = r, t \notin (J_z^k E^K \otimes J_z^k F^k)^l\},
\end{aligned}
$$

*a corrected version of $\mathcal{V}_{r,\ell}^k$. Correspondingly, the* universal Petri moduli space $\mathscr{P}_{r,\ell}^k(u,z)$ *is given by*

$$\mathscr{P}_{r,\ell}^k(u,z) := (\Pi^k)^{-1}(0) \subset \mathcal{V}_{r,\ell}^k(u,z).$$

Occasionally, for $D \in \widehat{\mathscr{D}}_z^{H,k}(E,F)$, we will also use the notation

$$\mathcal{V}_{r,\ell}^k(u,z;D) := \{t \in J_z^k E^K \otimes J_z^k F^K \mid (D,t) \in \mathcal{V}_{r,\ell}^k(u,z)\}.$$

The key result of this section is the following.

**Proposition 5.38** (Workhorse result)**.** *Suppose $u \in \widetilde{\mathcal{M}}_{\mathcal{U}}^{A,H}(J)$ with $[u] \in \mathcal{M}_{\mathcal{U},I}^{A,H}(J)$. Let $z \in \Sigma$ be an injective point of $u$ as in Lemma 4.75. For every $\ell \in \mathbb{N}$, there exists some constant $C_\ell > 0$ such that for all $k \geq \ell$ and $r \in \mathbb{N}$, the set $\mathscr{P}_{r,\ell}^k(u,z) \subset \mathcal{V}_{r,\ell}^k(u,z)$ is a smooth $C^\infty$-subvariety of codimension at least $C_\ell k^2$.*

*Proof.* As the first step, we prove that $\mathcal{V}_{r,\ell}^k(u,z)$ is a smooth manifold. The set

$$\mathcal{V}^k(u,z) := \{(D,t) \in \widehat{\mathscr{D}}_z^{H,k}(E,F) \times J_z^k E^K \otimes J_z^k F^K \mid t \in \ker \Phi^*(D) \otimes \ker \Phi^*(D)^*\}$$

is a smooth finite-dimensional manifold: the kernels of $\Phi^*(D)$ and $\Phi^*(D)^*$ depend smoothly on $D$, hence fit together to endow $\mathcal{V}^k(u,z) \to \widehat{\mathscr{D}}_z^{H,k}(E,F)$ with a smooth vector bundle structure. In particular, the total space $\mathcal{V}^k(u,z)$ is a smooth manifold. Refining this,

$$\mathcal{V}_r^k(u,z) := \{(D,t) \in \mathcal{V}^k(u,z) \mid \operatorname{rk}(t) = r\} \subset \mathcal{V}^k(u,z)$$

is a smooth submanifold of dimension

$$\dim_{\mathbb{R}} \mathcal{V}_r^k(u,z) = \dim_{\mathbb{R}} \widehat{\mathscr{D}}_z^{H,k}(E,F) + r\left(\operatorname{rk}_{\mathbb{C}} E^K + \operatorname{rk}_{\mathbb{C}} F^K\right)(k+1)(k+2) - r^2$$

because of the following lemma.

**Lemma 5.39** ([Wen23d, Equation (5.7)])**.** *For finite-dimensional vector spaces $V$ and $W$, the set of elements $t \in V \otimes W$ of rank $r$ is a smooth submanifold of dimension*

$$\dim\{t \in V \otimes W \mid \operatorname{rk}(t) = r\} = r(\dim V + \dim W) - r^2. \qquad \square$$

Lastly, each $\mathcal{V}_{r,\ell}^k(u,z) \subset \mathcal{V}_r^k(u,z)$ is an open subset, as non-vanishing is an open condition.[15] In particular, $\mathcal{V}_{r,\ell}^k(u,z)$ is a smooth manifold.

To prove that $\mathscr{P}_{r,\ell}^k(u,z)$ is a $C^\infty$-subvariety of $\mathcal{V}_{r,\ell}^k(u,z)$, we consider the spaces

$$\mathcal{V}_{r,\ell}^{k,\text{restr}}(u,z) := \{(D,t) \in \widehat{\mathscr{D}}_z^k(E^K,F^K) \times J_z^k E^K \otimes J_z^k F^K \mid t \in \ker D \otimes \ker D^*,\ \operatorname{rk}(t) = r, t \notin (J_z^k E^K \otimes J_z^k F^k)^l\}$$

and $\mathscr{P}_{r,\ell}^{k,\text{restr}}(u,z) := (\Pi^k)^{-1}(0) \subset \mathcal{V}_{r,\ell}^{k,\text{restr}}(u,z)$. Wendl proved [Wen23d, Proposition 5.25] that there exists a constant $C_\ell > 0$ such that for all $k \geq \ell$ and $r \in \mathbb{N}$, $\mathscr{P}_{r,\ell}^{k,\text{restr}}(u,z) \subset \mathcal{V}_{r,\ell}^{k,\text{restr}}(u,z)$ is a $C^\infty$-sub-variety of co-dimension at least $C_\ell k^2$. This statement is almost what we want to prove: it remains to relate it to our set-up.

This is where Lemma 5.36 comes in: it implies that $\Phi^*\colon \widehat{\mathscr{D}}_z^{H,k}(E,F) \to \widehat{\mathscr{D}}_z^k(E^K,F^K)$ is a surjective linear map. Choose a complement $R \subset \widehat{\mathscr{D}}_z^{H,k}(E,F)$ of $\ker \Phi^*$; consider the corresponding projection $\pi_{\ker \Phi^*}\colon \widehat{\mathscr{D}}_z^{H,k}(E,F) \to \ker \Phi^*$.

**Claim 1.** *The restriction $\Phi^*|_R\colon R \to \widehat{\mathscr{D}}_z^k(E^K,F^K)$ is a linear isomorphism, as is the map*

$$A\colon \widehat{\mathscr{D}}_z^{H,k}(E,F) \to \widehat{\mathscr{D}}_z^k(E^K,F^K) \times \ker \Phi^*, D \mapsto (\Phi^*(D), \pi_{\ker \Phi^*}(D)).$$

*Proof.* By construction, $\Phi^*|_R\colon R \to \widehat{\mathscr{D}}_z^k(E^K,F^K)$ is an injective linear map. It is surjective by Lemma 5.36. Let $\Psi\colon \widehat{\mathscr{D}}_z^k(E^K,F^K) \to R$ be its inverse, which is also linear. Then $A$ is a linear isomorphism, as

$$B\colon \widehat{\mathscr{D}}_z^k(E^K,F^K) \times \ker \Phi^* \to \widehat{\mathscr{D}}_z^{H,k}(E,F), (D_1,D_2) \mapsto \Psi(D_1) + D_2$$

[15] Regarding the optical difference between $t \in \ker \Phi^*(D) \otimes \ker \Phi^*(D)^*$ and $\Phi(t) \in \ker D \otimes \ker D^*$, we remark that these are equivalent by Remark 5.33.

is a linear inverse. Indeed, for all $(D_1, D_2) \in \widehat{\mathscr{D}}_z^k(E^K, F^K) \times \ker \Phi^*$, we compute

$$\begin{aligned} A(B(D_1, D_2)) &= A(\Psi(D_1) + D_2) \\ &= (\Phi^*(\Psi(D_1)) + \Phi^*(D_2), \pi_{\ker \Phi^*}(\Psi(D_1)) + \pi_{\ker \Phi^*}(D_2)) \\ &= (D_1 + 0, 0 + D_2) = (D_1, D_2) \end{aligned}$$

since $\Psi(D_1) \in R$. In the other direction, for $D \in \widehat{\mathscr{D}}_z^{H,k}(E^K, F^K)$, write $D = D_1 + D_2$ for $D_1 \in R$ and $D_2 \in \ker \Phi^*$. Then,

$$B(A(D)) = B(\Phi^*(D), \pi_{\ker \Phi^*}(D)) = (\Phi^*(D_1), D_2) = \Psi(\Phi^*(D_1)) + D_2 = D.$$

This completes the proof. △

**Claim 2.** *Then $\Phi^*$ induces a (smooth) diffeomorphism*

$$A\colon \mathcal{V}_{r,\ell}^k(u, z) \to \ker \Phi^* \times \mathcal{V}_{r,\ell}^{k,restr}(u, z), (D, t) \mapsto (\pi_{\ker \Phi^*}(D), (\Phi^*(D), t)), \tag{5.5}$$

*which maps $\mathscr{P}_{r,\ell}^k(u, z)$ to $\ker \Phi^* \times \mathscr{P}_{r,\ell}^{k,restr}(u, z)$.*

*Proof.* The map $A$ is well-defined by careful inspection of the definitions. It is smooth, as it is the restriction of the map

$$\begin{aligned} &\mathcal{V}^k \to \{(D, t) \in \widehat{\mathscr{D}}_z^k(E^K, F^K) \times J_z^k E^K \otimes J_z^k F^K \mid t \in \ker \Phi^*(D) \otimes \ker \Phi^*(D)^*\}, \\ &(D, t) \mapsto (\pi_{\ker \Phi^*}(D), (\Phi^*(D), t)), \end{aligned}$$

which is linear, thus smooth. The previous claim essentially furnishes an inverse: one easily verifies that the map

$$(D_1, t), D_2 \mapsto (\Psi(D_1) + D_2, t) = (B(D_1, D_2), t)$$

is smooth (analogously to $A$) and a two-sided inverse for $A$. Thus, $A$ is a diffeomorphism. Since $t \in \ker \Phi^* \otimes \ker \Phi^*(D)^*$ if and only if $\Phi(t) \in \ker D \otimes \ker D^*$ (by Remark 5.33), the claim follows. △

Therefore, $\mathscr{P}_{r,\ell}^k(u, z)$ is a $C^\infty$-sub-variety of $\mathcal{V}_{r,\ell}^k(u, z)$, diffeomorphic to $\ker \Phi^* \times \mathscr{P}_{r,\ell}^{k,\mathrm{restr}}(u, z)$. In particular, the co-dimension of $\mathscr{P}_{r,\ell}^k(u, z)$ in $\mathcal{V}_{r,\ell}^k(u, z)$ equals the co-dimension of $\mathscr{P}_{r,\ell}^{k,\mathrm{restr}}(u, z)$ in $\mathcal{V}_{r,\ell}^{k,\mathrm{restr}}(u, z)$. □

This proof also yields a formula for the dimension of $\mathcal{V}_{r,\ell}^k(u, z; D_u^N)$, which we explicitly note for later use.

**Lemma 5.40.** *$\mathcal{V}_{r,\ell}^k(u, z; D_u^N)$ is a smooth manifold of dimension $4rm(k+1) - r^2$, where $m = \dim_{\mathbb{C}} M^{G_u} - 1$.*

*Proof.* The isomorphism (5.5) shows $\dim \mathcal{V}_{r,\ell}^k(u, z; D_u^N) = \dim \mathcal{V}_{r,\ell}^{k,\mathrm{restr}}(u, z; D_u^N)$. The latter dimension has been computed by Wendl[16] [Wen23d, Equation (5.19)] to be $4rm(k+1) - r^2$, where

$$m = \mathrm{rk}_{\mathbb{C}} E^K = \mathrm{rk}_{\mathbb{C}} N_u^K = \mathrm{rk}_{\mathbb{C}} u_K^* TM^K - 1 = \dim_{\mathbb{C}} M^{G_u} - 1.$$ □

[16]The assumptions 5.13 and 5.14 in [Wen23d] are satisfied in our setting, as we only work with normal Cauchy–Riemann operators [Wen23d, p. 180; Wen23d, Remark 5.15].

## 5.5. Completing the proof

Let us now complete the proof of Theorem 5.14. The structure of this argument largely parallels Wendl's argument [Wen23d, §5.4–6]. All new mathematical ideas have been mentioned in the previous sections already; this section is just putting them together. For this reason, we leave out some details whose ideas exactly parallel Wendl or match previous sections.

### 5.5.1. Proof of Theorem 5.27

Fix an open subset $\mathcal{U} \subset M$ with compact closure. Using countability of the isosymmetric strata, let us also fix $A$ and $H$. We need to exhibit a co-meagre subset $\mathcal{J}_{\text{reg}}$ of $\mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{\text{fix}})$ (resp. $\mathcal{J}^G(M,\omega;\mathcal{U},J_{\text{fix}})$) depending on $A$ and $H$ such that for all $J \in \mathcal{J}_{\text{reg}}$ and all $u \in \mathcal{M}(J;k,c)$, the restricted normal Cauchy–Riemann operator $D_u^{N,\text{restr}}$ satisfies Petri's condition to infinite order on an open and dense set of points in $u^{-1}(\mathcal{U})$.

Consider the subset

$$\mathcal{M}_+(J) \subset \mathcal{M}_{g,m+1}(C,J)$$

of simple curves $[(\Sigma,j,(\zeta_1,\dots,\zeta_{m+1}),u)]$ such that $[(\Sigma,j,(\zeta_i,\dots,\zeta_m),u)] \in \mathcal{M}_{\mathcal{U},1}^{A,H}(J)$ and the last marked point $\zeta_{m+1}$ is an injective point as in Lemma 4.75. By abuse of notation, we will write elements of $\mathcal{M}_+(J)$ as pairs $(u,\zeta)$, where $u \in \mathcal{M}_{\mathcal{U},1}^{A,H}(J)$ and $\zeta \in \Sigma$ is the last marked point. The subset $\mathcal{M}_+(J)$ inherits a topology from $\mathcal{M}_{g,m+1}(C,J)$: a sequence of pairs $(u_n,\zeta_n)$ converges iff $(u_n)$ is a convergent sequence in $\mathcal{M}(J)$ and $(\zeta_n)$ is a convergent sequence in $\Sigma$. At the end of the proof, we will see why this is useful: a hypothetical counterexample to the theorem gives rise to such a pair $(u,\zeta)$.

For each $k,r,\ell \in \mathbb{N}$ with $k \geq \ell$, we consider the space

$$\widehat{\mathcal{M}}^{k,r,\ell}(J) := \{(u,\zeta,t) \mid (u,\zeta) \in \mathcal{M}_+(J), u \in \mathcal{M}_{\mathcal{U},1}^{A,H}(J), t \in \mathcal{V}_{r,\ell}^k(u,\zeta;D_u^N)\}$$

and are particularly interested in the subset

$$\mathcal{M}^{k,r,\ell}(J) := \{(u,\zeta,t) \in \widehat{\mathcal{M}}^{k,r,\ell}(J) \mid \Pi^k(t) = 0\}$$

To study these better, recall the space $\mathcal{J}_\epsilon$ of $G$-equivariant perturbations of $J_{\text{ref}}$ from Chapter 4. Similarly to that chapter, we consider suitable universal moduli spaces

$$\begin{aligned}
\mathcal{U}_+^*(\mathcal{J}_\epsilon) &:= \{(u,\zeta,J) \mid J \in \mathcal{J}_\epsilon, (u,\zeta) \in \mathcal{M}_+(J)\},\\
\widehat{\mathcal{U}}^{k,r,\ell}(\mathcal{J}_\epsilon) &:= \{(u,\zeta,t,J) \mid J \in \mathcal{J}_\epsilon, (u,\zeta,t) \in \widehat{\mathcal{M}}^{k,r,\ell}(J)\}, \text{ and}\\
\mathcal{U}^{k,r,\ell}(\mathcal{J}_\epsilon) &:= \{(u,\zeta,t,J) \mid J \in \mathcal{J}_\epsilon, (u,\zeta,t) \in \mathcal{M}^{k,r,\ell}(J)\}.
\end{aligned}$$

Standard arguments (as in e.g. [MS12]) show that for $\epsilon \in \boldsymbol{\mathcal{E}}$ with sufficiently rapid decay, $\mathcal{U}_+^*(\mathcal{J}_\epsilon)$ is a Banach manifold such that the projection $\mathcal{U}_+^*(\mathcal{J}_\epsilon) \to \mathcal{J}_\epsilon, (u,\zeta,J) \mapsto$

$J$ is a smooth Fredholm map whose index is the virtual dimension of $\mathcal{M}_+(J)$. An analogous result holds for $\widehat{\mathcal{U}}^{k,r,\ell}(\mathcal{J}_\epsilon)$, as the additional $k$-jet datum $t$ varies in a smooth finite-dimensional manifold that depends smoothly on the $k$-jet of the operator $D_u^N$ at the immersed point $\zeta$; this in turn depends smoothly on $(u, \zeta, J) \in \mathcal{U}_+^*(\mathcal{J}_\epsilon)$.

It will be convenient to impose an extra condition defining an open subset of $\mathcal{U}^{k,r,\ell}(\mathcal{J}_\epsilon)$. Recall the constant $C_\ell > 0$ from Proposition 5.38; choose, once and for all, a value $C_\ell$ for every $\ell \in \mathbb{N}$.

**Definition 5.41.** *Given $J \in \mathcal{J}_\tau^G(M, \omega; \mathcal{U}, J_{fix})$ (resp. $J \in \mathcal{J}_\tau^G(M, \omega; \mathcal{U}, J_{fix})$) and $\epsilon \in \boldsymbol{\mathcal{E}}$, an element $(u, \zeta, t) \in \mathcal{M}^{k,r,\ell}(J)$ is called* $\epsilon$-regular *if $J \in \mathcal{J}_\epsilon$ and $(u, \zeta, t, J)$ has a neighbourhood $\mathcal{O} \subset \widehat{\mathcal{U}}^{k,r,\ell}(\mathcal{J}_\epsilon)$ such that $\mathcal{O} \cap \mathcal{U}^{k,r,\ell}(\mathcal{J}_\epsilon)$ is a $C^\infty$-sub-variety of $\widehat{\mathcal{U}}^{k,r,\ell}(\mathcal{J}_\epsilon)$ of co-dimension at least $C_\ell k^2$.*

By construction, $\epsilon$-regularity is an open condition. Lemma 5.21 implies that it is generally non-empty.

**Lemma 5.42.** *For all $r, \ell \in \mathbb{N}$ and all $k \geq \ell$, any given $(u, \zeta, t) \in \mathcal{M}^{k,r,\ell}(J_{ref})$ is $\epsilon$-regular for all $\epsilon \in \boldsymbol{\mathcal{E}}$ with sufficiently rapid decay.*

*Proof sketch.* We proceed similarly to [Wen23d, Lemma 5.31]. Observe that $J_{\text{ref}} \in \mathcal{J}_\epsilon$ for every $\epsilon \in \boldsymbol{\mathcal{E}}$. Given $(u, \zeta, t) \in \mathcal{M}^{k,r,\ell}(J_{\text{ref}})$, consider the Fréchet space

$$\mathcal{Y}_0 := \{Y \in \Gamma^G(\overline{\text{End}}_{\mathbb{C}}(TM, J_{\text{ref}})) \mid Y|_{M \setminus \mathcal{U}} \equiv 0, Y|_{\text{im}(u)} = 0, \omega(\cdot, Y \cdot) + \omega(Y \cdot, \cdot) = 0\},$$

which we may intuitively think of as the tangent space $T_{J_{\text{ref}}} \mathcal{J}^G(M, \omega; \mathcal{U}, J_{\text{fix}})$. For each $\epsilon \in \boldsymbol{\mathcal{E}}$, restrict this to the closed subspace

$$\mathcal{Y}_\epsilon := \{Y \in \mathcal{Y}_0 \mid \|Y\|_{C_\epsilon} < \infty\}$$

of $T_{J_{\text{ref}}} \mathcal{J}_\epsilon$ with the $C_\epsilon$-topology.

We apply Lemma 5.21: let $J_\zeta^{k-1,H}(\text{Hom}_{\mathbb{R}}(E, F))$ denote the space of all $(k-1)$-jets of germs of $H$-equivariant sections $A \in \Gamma^H(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, \text{End}_{\mathbb{R}}(N_u)))$ at $\zeta$. Then Lemma 5.21 provides a surjective linear mapping

$$\Psi_0 \colon \mathcal{Y}_0 \to J_\zeta^{k-1,H}(\text{Hom}_{\mathbb{R}}(E, F)), Y \mapsto J_\zeta^{k-1}(A_Y),$$

where $A_Y$ denotes (the germ near $\zeta$ of) the zeroth-order term determined by $Y$ according to the relation $A_Y \eta = \pi_N \circ \nabla_\eta Y \circ du \circ j$.[17]

[17] The careful reader may wonder about a small detail: Lemma 5.21 only applies to sections $H$-invariant sections $A$ whose support is contained in some open subset $S \subset u^{-1}(\mathcal{U})$ of "good" injective points. This is not an issue here, as we only care about *germs* of sections at $\zeta$: suppose $A$ is any $H$-equivariant section. Choose an $H$-invariant open subset $S \subset u^{-1}(\mathcal{U})$ of good injective points, as in Lemma 5.21. Choose an $H$-invariant bump function $\phi$ with $\phi \equiv 1$ near $\zeta$ and support contained in $S$; then $\phi A$ is another $H$-equivariant section, whose support is contained in $S$ and which defines the same germ at $\zeta$. Choosing $S$ as above is possible: $u^{-1}(\mathcal{U})$ is open and $H$-invariant, as is the set of injective points of $u$. Among these, only finitely many points are not "good", hence a suitable $H$-invariant neighbourhood of $\zeta$ contains only good injective points.

Since the co-domain of $\Psi_0$ is finite dimensional, Lemma 4.12 implies that $\Psi_0$ remains surjective when restricted to the subspace $Y_\epsilon$, for any $\epsilon \in \boldsymbol{\mathcal{E}}$ with sufficiently rapid decay.

Now, each $Y \in Y_\epsilon$ induces a 1-parameter family of almost complex structures $J_\tau := J_{\tau Y} \in \mathcal{J}_\epsilon$ defined via the exponential map (4.1), which match $J_{\text{ref}}$ along $u$ and satisfy $J_0 = J_{\text{ref}}$. This defines a smooth family $(u, \zeta, J_\tau)$ in $\mathcal{U}^*_+(\mathcal{J}_\epsilon)$ that deforms the normal Cauchy–Riemann operator of $u$ in the direction of $A_Y$. Therefore, the linearisation at $(u, \zeta, t, J_{\text{ref}})$ of the natural projection

$$\widehat{\mathcal{U}}^{k,r,\ell}(\mathcal{J}_\epsilon) \to \mathcal{V}^k_{r,\ell}(u,\zeta), (u,\zeta,t,J) \mapsto (D^N_u, t)$$

is surjective onto $T_{(D^N_u,t)}\mathcal{V}^k_{r,\ell}(u,\zeta)$ and the result follows from Proposition 5.38. □

We apply a Sard–Smale argument to the projection $\widehat{\mathcal{U}}^{k,r,\ell}(\mathcal{J}_\epsilon) \to \mathcal{J}_\epsilon$. To this end, consider the intersection $\mathcal{J}^{\text{reg}}_\epsilon \subset \mathcal{J}_\epsilon$ of the sets of regular values for all $k$, $r$ and $\ell$. Note that $\mathcal{J}^{\text{reg}}_\epsilon \subset \mathcal{J}_\epsilon$ is co-meagre again. Then, for each $J \in \mathcal{J}^{\text{reg}}_\epsilon \subset \mathcal{J}_\epsilon$, for all $r, \ell \in \mathbb{N}$, and $k \geq \ell$ the subset of $\epsilon$-regular elements in $\mathcal{M}^{k,r,\ell}(J) \subset \widehat{\mathcal{M}}^{k,r,\ell}(J)$ has co-dimension at least $C_\ell k^2$. On the other hand, the virtual dimension of $\widehat{\mathcal{M}}^{k,r,\ell}(J)$ only grows linearly in $k$, by Lemma 5.40. Thus, for fixed $\ell$ and $r$, for $k$ sufficiently large, the virtual dimension of the set of $\epsilon$-regular elements becomes negative. Thus, we have proven the following.

**Corollary 5.43.** *For every $\epsilon \in \boldsymbol{\mathcal{E}}$, there exists a co-meagre subset $\mathcal{J}^{reg}_\epsilon \subset \mathcal{J}_\epsilon$ such that for all $J \in \mathcal{J}^{reg}_\epsilon$ and any given $r, \ell \in \mathbb{N}$, the set of $\epsilon$-regular elements in $\mathcal{M}^{k,r,\ell}(J)$ is empty whenever $k$ is large enough.* □

Completing the proof of Theorem 5.27 is another application of Taubes' trick. Suppose the normal Cauchy–Riemann operator of $[u] \in \mathcal{M}^{A,H}_{\mathcal{U},1}(J)$ does not satisfy Petri's condition to infinite order on an open and dense set of points in $u^{-1}(\mathcal{U})$. By Lemma 4.75, there must exist an injective point $\zeta \in u^{-1}(\mathcal{U})$ of $u$ with $\zeta \notin \theta$ such that $D^{N,\text{restr}}_u$ does *not* satisfy Petri's condition to infinite order at $\zeta$. Then, $(u, \zeta) \in \mathcal{M}_+(J)$ by definition. By hypothesis, there exists a non-trivial element $t \in \ker D^{N,\text{restr}}_u \otimes \ker D^{N,\text{restr}*}_u$ such that $\Pi(t)$ has vanishing derivatives of all orders at $\zeta$. Then $t$ has finite vanishing order (as it's non-zero) and finite rank $r$. Consequently, $t \in \mathcal{V}^k_{r,\ell}(u, \zeta; D^N_u)$ for some $\ell$ and sufficiently large $k$. Since also $\Pi^k(t) = 0$, we deduce $(u, \zeta, t) \in \mathcal{M}^{k,r,\ell}(J)$.

Now Taubes' trick and Corollary 5.43 show that, for $J$ in a suitable co-meagre subset of $\mathcal{J}^G_\tau(M, \omega; \mathcal{U}, J_{\text{fix}})$ resp. $\mathcal{J}^G(M, \omega; \mathcal{U}, J_{\text{fix}})$, this cannot happen. We omit the details. □

### 5.5.2. Smoothness of the walls

*Proof sketch of Theorem 5.14.* Let us sketch how to prove Theorem 5.14: we have seen the overall idea several times now (though the details are slightly different). We write down a universal moduli space with a projection to a suitable Banach manifold of perturbed data; the regular values of the projection have the property stated in the theorem. The hard part is, of course, to prove that the universal moduli space is a smooth Banach manifold. This follows from the implicit function theorem, after proving that a suitable linearised operator is surjective; this is where the previous results about Petri's condition are needed.

For each choice of data $\mathcal{U}$, $A$, $H$, and $\mathbf{l}$ as in the theorem, we define a universal moduli space

$$\mathcal{U}(J,\mathbf{l}) := \mathcal{U}_{\mathcal{U}}^{A,H}(J,\mathbf{l}) := \{(u,J) \mid J \in \mathcal{J}_\epsilon, u \in \mathcal{M}_{\mathcal{U},\mathbf{l}}^{A,H}(J)\}.$$

For non-negative integers $k$ and $c$, we decompose this further into universal moduli spaces

$$\mathcal{U}(J,\mathbf{l};k,c) := \{(u,J) \in \mathcal{U}(J,\mathbf{l}) \mid u \in \mathcal{M}(J;k,c)\}.$$

We would be able to apply a Sard–Smale argument if we proved that $\mathcal{U}(J,\mathbf{l};k,c) \subset \mathcal{U}(J,\mathbf{l})$ was a Banach submanifold whose co-dimension equals the dimension of $\mathrm{Hom}_K(\ker D_u^{N,\mathrm{restr}}, \mathrm{coker}\, D_u^{N,\mathrm{restr}})$. We will prove this holds on a certain open subset; this will suffice since Petri's condition is generic.

**Definition 5.44.** *A curve $u \in \mathcal{M}(J;k,c)$ is called* Petri regular *if and only if $D_u^{N,restr}$ satisfies Petri's condition over $u^{-1}(\mathcal{U})$.*

We denote the set of Petri regular curves by $\mathcal{M}_\Pi(J;k,c) \subset \mathcal{M}(J;k,c)$ and define the corresponding universal moduli space

$$\mathcal{U}_\Pi(\mathcal{J}_\epsilon,\mathbf{l};k,c) \subset \mathcal{U}(\mathcal{J}_\epsilon,\mathbf{l};k,c)$$

to be the set of all pairs $(u,J) \in \mathcal{U}(\mathcal{J}_\epsilon,\mathbf{l};k,c)$ such that $u \in \mathcal{M}_\Pi(J;k,c)$.

**Remark 5.45.** By Theorem 5.27, there exists a co-meagre subset of $\mathcal{J}_\tau^G(M,\omega;\mathcal{U},J_{\mathrm{fix}})$ (resp. $\mathcal{J}^G(M,\omega;\mathcal{U},J_{\mathrm{fix}})$) for which $\mathcal{M}_\Pi(J;k,c) = \mathcal{M}(J;k,c)$.

**Lemma 5.46.** *For $\epsilon \in \boldsymbol{\mathcal{E}}$ with sufficiently rapid decay, $\mathcal{U}(\mathcal{J}_\epsilon,\mathbf{l})$ carries a smooth Banach manifold structure such that every $(u_0,J_0) \in \mathcal{U}(\mathcal{J}_\epsilon,\mathbf{l})$ has a neighbourhood $\mathcal{V} \subset \mathcal{U}(\mathcal{J}_\epsilon,\mathbf{l})$ with a smooth family of vector bundle isomorphisms*

$$u_0^*TM^K \overset{\cong}{\to} u^*TM^K$$

*for $u \in \mathcal{V}$, mapping $N_{u_0}$ isomorphically to $N_u$ and $N_{u_0^K}$ to $N_{u^K}$.*

*Sketch of proof.* Repeat the proof of [Wen23d, Lemma 6.4] mutatis mutandis — applied to the manifold $M^K$. This is where we use the fact that the topology of the generalised normal bundles $N_{u^K}$ does not change without each iso-symmetric stratum — to enable this argument, we had to prescribe the number and orders of $u^K$'s (or equivalently, $u$'s) critical points. □

**Lemma 5.47.** *The subset $\mathcal{U}_\Pi(\mathcal{J}_\epsilon,\mathbf{l};k,c) \subset \mathcal{U}(\mathcal{J}_\epsilon,\mathbf{l};k,c)$ is open.*

*Proof.* Lemma 5.46 implies that the operators $D_u^{N,\mathrm{restr}}$ and $(D_u^{N,\mathrm{restr}})^*$ can be understood as varying continuously with $(u, J) \in \mathcal{U}(\mathcal{J}_\epsilon, \mathbf{1})$, and the dimensions of their kernels are locally constant as long as $(u, J)$ moves only within $\mathcal{U}(\mathcal{J}_\epsilon, \mathbf{1}; k, c)$. Hence, the family of Petri maps defined on $\ker D_u^{N,\mathrm{restr}} \otimes \ker(D_u^{N,\mathrm{restr}})^*$ and then restricted to $u^{-1}(\mathcal{U})$ depends continuously on $(u, J) \in \mathcal{U}(\mathcal{J}_\epsilon, \mathbf{1}; k, c)$. Since their domains are finite dimensional, the injectivity of these maps is an open condition. □

The remainder of the proof is similar to that of Theorem 5.27. Let us merely outline the necessary steps.

- Define a notion of *$\epsilon$-regular* curves, for each $\epsilon \in \boldsymbol{\mathcal{E}}$. By definition, a curve $u \in \mathcal{M}(J; k, c)$ is $\epsilon$-regular if a suitable map into $\mathrm{Hom}_K(\ker D_u^{N,\mathrm{restr}}, \mathrm{coker}\, D_u^{N,\mathrm{restr}})$ is surjective.
- Using Lemma 5.21, prove that if $u$ is Petri regular, it is $\epsilon$-regular for all $\epsilon$ with sufficiently rapid decay.
- Apply a Sard–Smale argument and Taubes' trick. □

# 6. Conclusion and outlook

In this thesis, we have seen the first steps to a well-behaved equivariant transversality theory of closed holomorphic curves. Given a symplectic $G$-manifold, we saw how to decompose the moduli space of simple curves into countably many disjoint walls. If $G$ is finite and $2g + m \geq 3$, for generic equivariant $J$ all walls are smooth manifolds of explicit co-dimension in their respective stratum. There are various follow-up questions and future possibilities for study: let us comment on them in turn.

## 6.1. Stratifying multiply covered curves

The first obvious direction is to extend these results to cover (no pun intended) multiply covered curves as well. Candidate Definition 3.38 suggests how to define iso-symmetric strata in this case. Proving their smoothness (with explicit dimension) was straightforward. The next step is adapting the definition of the walls $\mathcal{M}(J; k, c)$ and proving that walls are generically smooth. This will entail combining the set-up for the strata with either Wendl's notion of twisted Cauchy–Riemann operators or Doan–Walpuski's rephrasing of twisted local systems. Since Petri's local condition is preserved under pullback by holomorphic branched covers, we expect the difficulty at this step to be mostly about finding the right definitions, not in proving smoothness.

**Objective 1.** *Prove that each iso-symmetric stratum decomposes into countably many walls, each of which is generically a smooth manifold.*

## 6.2. Computing the dimension of each wall

To apply this thesis' paradigm in applications, we would like better control over the dimension and co-dimension of each wall. (For instance, in the proof of the super-rigidity conjecture, any hypothetical counterexample is a curve which must live in a wall of co-dimension larger than its ambient stratum's dimension: this means the wall has negative dimension, thus is empty.)

This requires progress on two separate tasks. Firstly, we need to compute the co-dimension $\dim \mathrm{Hom}_K(\ker D_u^{N,\mathrm{restr}}, \mathrm{coker}\, D_u^{N,\mathrm{restr}})$ of each wall more explicitly, where $K$ is $H$ if $G_u$ is trivial, and $G_u$ otherwise.

**Objective 2.** *Compute the co-dimension of each wall $\mathcal{M}(J; k, c)$ in $\mathcal{M}_{\mathcal{U},I}^{A,H}(J)$ near $u \in \mathcal{M}(J; k, c)$, for instance in terms of Fredholm indices of suitable Cauchy–Riemann type* *operators.*

By Schur's lemma, any map of $K$-representations $\ker D_u^{N,\text{restr}} \to \operatorname{coker} D_u^{N,\text{restr}}$ must preserve the isotypical decomposition, so knowing the co-dimension requires computing the multiplicity of each irreducible $K$-representation in $\ker D_u^{N,\text{restr}}$ and $\operatorname{coker} D_u^{N,\text{restr}}$. Then, we would like to relate these multiplicities to Fredholm indices of suitable Cauchy–Riemann type operators.

Again, Wendl's paper can serve as a blueprint to follow. In their case, the normal Cauchy–Riemann operator $D_u^N$ of $u = v \circ \phi$ induced (via a choice of "minimal regular presentation" associated to the holomorphic branched cover $\phi$) a twisted Cauchy–Riemann operator. This twisted operator splits into summands corresponding the irreducible $H$-representations.

## 6.3. Beyond symplectic actions

Throughout this thesis, we considered equivariant holomorphic curves with respect to *symplectic* group actions. However, there are also some non-symplectic actions to which our methods could apply.

The simplest such case is a $\mathbb{Z}_2$-action induced by an *anti-symplectic involution*.

**Definition 6.1.** *An* anti-symplectic involution *on a smooth manifold* $(M, \omega)$ *is a smooth map* $\sigma\colon M \to M$ *such that* $\sigma^*\omega = -\omega$.

Anti-symplectic involutions occur naturally in some problems from celestial mechanics, such as the restricted three-body problem (and variants, such as Hill's lunar problem). One particular question worth studying is the existence of symmetric periodic orbits: if $(M, \xi = d\alpha)$ is a contact manifold and $\rho\colon M \to M$ an anti-contact involution (meaning $\rho^*\alpha = -\alpha$), does there exist a Reeb chord of $\alpha$ with ends on $\operatorname{Fix}(\rho)$? Kim [Kim22] proves that for certain tight contact 3-spheres, such symmetric periodic orbits always exist (and gives necessary and sufficient conditions when these bound a disk-like global surface of section). This result uses automatic transversality results, by virtue of working in dimension three. For higher-dimensional versions, new ideas are needed. The author expects that this thesis' methods can be useful.

Anti-symplectic involutions are also studied in *real* Gromov–Witten theory (e.g. [GI21; GZ23], [GI22] and [GZ]). In this case, studying $G$-invariant almost complex structures is not quite the correct approach: if $J \in \mathcal{J}(M, \omega)$ were invariant under an anti-symplectic involution $\phi$ of $(M, \omega)$, an easy compution shows that the corresponding Riemannian metric $g := g_J = \omega(\cdot, J\cdot)$ were to satisfy $\phi^* g = -g$. This is impossible since $\phi^* g$ is still a Riemannian metric, in particular positive definite. Instead, the correct approach is to require $g$ to be invariant, corresponding to the condition $\phi^* J = -J$. Real Gromov–Witten theory has another feature, requiring modifications of this thesis' setup: these invariants study $J$-holomorphic curves defined on a *real* Riemann surface, i.e. endowed with an anti-symplectic involution $\sigma$, such that $u \circ \sigma = \phi \circ u$ (i.e., holomorphic curves which are equivariant w.r.t. the induced $\mathbb{Z}_2$-actions on the domain and target).

Generalising beyond anti-symplectic involutions, a reasonable setting could be the following.

**Definition 6.2.** *A smooth (left or right) $G$-action on a symplectic manifold $(M, \omega)$ is called* anti-symplectic *if and only if there exists a set $S \subset G$ generating $G$ such that $s^*\omega = -\omega$ for all $s \in S$. This implies that each $g \in G$ acts either symplectically or anti-symplectically.*

The following generalisation also seems within reach; this setting includes e.g. a commuting symplectic and anti-symplectic action.

**Definition 6.3** (Ambi-symplectic group action)**.** *An* ambi-symplectic *group action on a symplectic manifold $(M, \omega)$ is a pair $(\psi, \epsilon)$ consisting of a group action $G \to \mathrm{Diff}(M)$, $g \mapsto \psi_g$ by diffeomorphisms and a group homomorphism $\epsilon\colon G \to \{\pm 1\}$ such that $\psi_g^*\omega = \epsilon(g)\omega$ for all $g \in G$.*

In all these cases, the non-emptiness and contractability of the equivaraint version of $\mathcal{J}_\tau(M, \omega)$ and $\mathcal{J}(M, \omega)$ follow similarly as in this document, as there exists a $G$-invariant Riemannian metric. The author expects the remaining ideas in this document to apply similarly.

## 6.4. Beyond finite groups

While the set-up in Chapters 2 and 3 has been fairly general (applying to proper symplectic actions of a smooth Lie group on $M$), the subsequent results about smoothness of the iso-symmetric strata and walls required working with finite $G$ and curves on a stable domain.

As we already commented briefly in Section 4.7, the reason for this is mostly technical and less conceptual, owing to the fact that the $A$-action is not smooth, and therefore an infinite-dimensional analogue of the slice theorem is not available. Therefore, proving smoothness of the iso-symmetric strata if $A \times G$ is an infinite group requires new ideas, beyond what is presented in this thesis.

Some indication in this direction was given by Observation 3.58 and Lemma 4.105: we may in fact avoid using adapted Teichmüller slices altogether for defining the iso-symmetric strata. (By Observation 3.58, the current description is useful for proving that the stabiliser $H$ of the $A \times G$-action is compact — but this does not require adapted Teichmüller slices and does not concern the core of the smoothness argument either.) Following this insight, a new approach to smoothness of the iso-symmetric strata is using the *global deformation operator* of $u$ instead: this shall be the subject of a future article.

Our current argument that all walls are smooth applies whenever the group $H$ is finite. (In particular, it does not require $A \times G$ to be finite — it suffices if each simple curve $u$ has finite stabiliser.) If the stabiliser $H$ has positive dimension, more care is required to prove Petri's condition. For instance, if $H$ has positive dimension, the requirement of $H$-invariance is non-local — and it may seem this invalidates Wendl's proof of Petri's condition (as that proof uses germs of Cauchy–Riemann operators and geometric data, which are inherently local). Fortunately, Wendl's argument can be strengthened: if all $H$-orbits are totally real (i.e., they do not contain any non-trivial complex subspace), a tweak of the argument proves that Petri's condition is still generic. If the $G$-action is *Hamiltonian*, every $G$-orbit (hence every $H$-orbit) is isotropic [Pel17, Remark 4.5], and in particular totally real for any tame almost complex structure $J$.

As a final word of caution, expecting a theorem which applies to *any* proper symplectic $G$-action is too ambitious: we need to impose suitable conditions on the action, to ensure the space $\mathcal{J}_\tau^G(M,\omega)$ is sufficiently large (for instance, infinite-dimensional). For instance, if $G$ acts transitively on $M$, any $G$-equivariant almost complex structure $J$ on $M$ is determined by prescribing its value at a given point $p \in M$. The space of complex structures on $T_pM$ is finite-dimensional, hence so are the spaces $\mathcal{J}_\tau^G(M,\omega)$ and $\mathcal{J}^G(M,\omega)$. These spaces can get even smaller: consider the action of the unitary group $\mathrm{U}(n)$ on complex projective space $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$. With respect to this action, there exists a *unique* $\mathrm{U}(n)$-equivariant compatible almost complex structure on $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$.
In both cases, this thesis' argument has no hope of applying: we need our space of perturbations to contain bump functions with sufficiently small support, which implies infinite-dimensionality.

At the time of writing, proving smoothness of iso-symmetric strata and walls for all proper Hamiltonian actions of abelian Lie groups seems plausible.

**Objective 3.** *Extend the smoothness of iso-symmetric strata and walls to $2g + m < 3$ or $G$ being compact, for instance for all proper Hamiltonian actions of abelian Lie groups.*

## 6.5. Equivariant punctured holomorphic curves

So far, we have only dealt with closed holomorphic curves. This fact, however, was not essential to our argument — I do not expect any substantial obstacle in generalising these results to punctured holomorphic curves (for equivariant *compatible* almost complex structures). The decomposition into iso-symmetric strata and walls and their smoothness should only require natural modifications (such as, modifying the spaces of equivariant almost complex structures considered to require admissibility in the sense of SFT).

**Objective 4.** *Generalise the definition of iso-symmetric strata and walls to punctured holomorphic curves, and prove that they are smooth manifolds for generic admissible equivariant compatible almost complex structures.*

We should mention Singh's in-progress work [Sin24], which generalises Wendl's stratification to punctured multiply covered curves. Thus, this objective is about generalising Singh's and this work, to *equivariant* punctured holomorphic curves.

Actually applying this framework to punctured curves requires clearing one further obstacle: the index of a punctured holomorphic curve depends on the Conley–Zehnder indices of the Reeb orbits it is asymptotic to. Hence, computing the co-dimension of these walls of (possibly multiply covered) punctured curves requires understanding the Conley–Zehnder indices of multiply covered Reeb orbits. For Reeb orbits which are neither hyperbolic nor elliptic, this can be non-trivial — for instance, there is no easy closed formula relating the index of the multiply covered orbit to the index of the underlying simple orbit. Singh has also investigated this question, for doubly covered curves such that each asymptotic Reeb orbit $\gamma_i$ is covered of order $2^{k_i}$ for some positive integer $k_i$ [Sin24]. We would like a theory requiring neither of these assumptions.

## 6.6. Applications

This thesis' ideas have the potential to apply in any setting with a symplectic $G$-action where holomorphic curves are a useful tool. Let us sketch two examples.

**Equivariant super-rigidity** As the reader has heard sufficiently often by now, this thesis is based on Wendl's solution of the super-rigidity conjecture: a natural question is whether a similar statement in the equivariant setting.

**Objective 5.** *Let $(M, \omega)$ be a closed symplectic G-manifold with* $\dim M \geq 6$. *Investigate whether generic G-equivariant compatible almost complex structures are super-rigid.*

By the definition of super-rigidity, this includes genericity of the property "all simple curves are embedded with disjoint image". For the purpose of the subsequent objective, we would like to prove the following, slightly stronger version (which is a standard fact in the non-equivariant case).

**Objective 6.** *Prove that for all G-equivariant $J$ but a subset of co-dimension at most two, all simple $J$-holomorphic curves are embedded and have disjoint images.*

Once this second step has been established, attacking Objective 5 requires the dimension computation from Objective 2. If the indices align, it should then be a straight-forward short computation.

**Equivariant Gromov invariant** Genericity of super-rigid compatible almost complex structures was used by Bai and Swaminathan [BS23] to define an analogue of Taubes' Gromov invariant [Tau96]: their invariant counts embedded closed $J$-holomorphic curves of prescribed genus and homology class, for super-rigid $J$ in a Calabi–Yau 3-fold. A priori, this count depends on the choice of $J$; producing an invariant independent of $J$ requires adding suitable correction terms to account for *wall-crossing phenomena*.

Most types of wall-crossing can be excluded from consideration by a dimension counting argument: the set of corresponding almost complex structures has codimension at least two, hence a generic 1-parameter family $\{J_t\}_{t\in[0,1]}$ of almost complex structure will not intersect them. If the class $A$ is twice a primitive class, this restricts the possible bifurcations to investigate to a small list. In each of these cases, Bai and Swaminathan construct a local Kuranishi model for the parametrised moduli space of holomorphic curves, and use this to find a suitable correction term.

While a non-trivial theorem in its own right, this analysis requires genericity of super-rigid almost complex structures to boot: without, the number of embedded curves one is counting need not be finite. Thus, a positive answer to Objective 5 would unlock progress on the following, more speculative, proposition.

**Task 7.** *Assuming Objective 5 holds, develop an equivariant version of Taubes' Gromov invariant and prove independence of $J$ via a suitable wall-crossing analysis.*

# A. Appendix

## A.1. A finite-dimensional toy model for the iso-symmetric strata

In this appendix, we consider a finite-dimensional toy model to motivate the local models for the iso-symmetric strata. Consider a smooth vector bundle $E \to M$ over a smooth (finite-dimensional) manifold $M$. Suppose the compact Lie group $G$ acts smoothly on $E$ via linear bundle isomorphisms $A_g$, over diffeomorphisms $\alpha_g \in \mathrm{Diff}(M)$ on the base.

Consider the zero set $\mathcal{M} := \sigma^{-1}(0) \subset M$ of a smooth $G$-equivariant section $\sigma \in \Gamma^G(E)$. For generic equivariant $\sigma$, this is generally not a manifold: because of equivariance, $\sigma$ is generally not transverse to the zero section. Indeed, the best we can hope for is a clean intersection condition. More specifically, we decompose $\mathcal{M}$ into iso-symmetric strata $\mathcal{M}^H(\sigma) := \{x \in \mathcal{M} \mid G_x \cong H\}$ for closed subgroups $H \leqslant G$ and show that each stratum (for generic equivariant $\sigma$) is a smooth manifold.

For a closed subgroup $H \leqslant G$, consider the orbit type $M^H := \{x \in M \mid G_x \cong H\}$. Since $G$ acts smoothly and properly on $M$, the orbit type $M^H \subset M$ is a smooth submanifold [DK00, Lemma 2.6.4(ii)]. In particular, the restriction $E' = E|_{M^H} \to M^H$ is a smooth vector bundle.

Denote $\sigma^H := \sigma|_{M^H} \colon M^H \to E|_{M^H}$ and observe $\mathcal{M}^H(\sigma) = \sigma^{-1}(0) \cap M^H = (\sigma^H)^{-1}(0)$. Hence, it suffices to show that $\sigma^H$ is transverse to the zero section in $E'$. Looking closely, that can *never* be true (for non-trivial $H$): the equivariance of $\sigma^H$ implies it takes values in a smaller sub-bundle. However, co-restricting $\sigma$ to that sub-bundle achieves transversality. More precisely, we want to show that $\sigma^H$ takes values in the sub-bundle

$$E^H := \{v \in E' \mid G_v \cong H\} = \{v \in E_x \colon x \in M^H, G_x = G_v \cong H\}.$$

The correct definition of $E^H$ is a bit subtle. We want to consider the subset $\{v \in E \mid G_v \cong H\}$, as the natural analogue of the orbit type $M^H$. Because of one addition wrinkle, we need to be slightly more careful.

The issue is that $G$ acts on both the total space $E$ and the base $M$, and these actions are not the same. The projection $\pi \colon E \to M$ is equivariant ($\pi(g \cdot v) = g \cdot \pi(v)$ for all $v \in E$ and $g \in G$); this implies $G_v \leqslant G_{\pi(v)}$ for all $v \in E$. However, the stabilisers need not be equal: the base point $\pi(v)$ could have larger stabiliser (w.r.t. the $G$-action on $M$) than $v$ (w.r.t. the $G$-action on $E$)!

**Example A.1** (Trivial example)**.** Consider the $\mathbb{Z}_2$-action on the trivial line bundle $\mathbb{R} \times M$ over any manifold $M$, by $\phi \colon \mathbb{R} \times M \ni (r, p) \mapsto (-r, p) \in \mathbb{R} \times M$. This action covers the trivial $\mathbb{Z}_2$-action on $M$. The zero section in $E$ has stabiliser $\mathbb{Z}_2$; every other $v \in E$ has trivial stabiliser. Thus, $v \in E \setminus M$ has trivial stabiliser, while its base point has stabiliser $\mathbb{Z}_2$.

Fortunately, this issue does not occur when restricting to the sub-bundle $E'$.

**Lemma A.2.** $\{v \in E\colon G_v \cong H\} \cap E|_{M^H} = \{v \in E_x\colon x \in M^H, G_x = G_v \cong H\}$

*Proof.* Inclusion "$\supset$" is obvious. "$\subset$": If $v \in E$ with $G_v \cong H$ and $x \in M^H$, we have $H \cong G_v \subset G_x \cong H$, hence $G_x = G_v$. □

We have already called $E^H$ a "sub-bundle" several times.

**Lemma A.3.** *The subset $E^H \subset E'$ is a smooth sub-bundle of $E'$.* □

**Lemma A.4.** *$\sigma^H$ takes values in $E^H$, hence is a smooth section $\sigma^H\colon M^H \to E^H$.*

*Proof.* Let $x \in M^H$ be arbitrary, then $v = \sigma(x)$ satisfies $v = \sigma(x) = \sigma(g \cdot x) = g \cdot \sigma(x) = g \cdot v$ for all $g \in G_x$, by $G$-equivariance of $\sigma$. This implies $G_x \leqslant G_v$. On the other hand, we have $G_v \leqslant G_{\pi(v)}$ for all $v \in E$, as $g \cdot v = v$ implies $x = \pi(v) = \pi(g \cdot v) = g \cdot \pi(v) = g \cdot x$. In total, we conclude $G_x = G_v$ for all $v \in \mathrm{im}(\sigma)$. By definition of $M^H$, we have $G_x \cong H$. □

Finally, the result we are looking for is the following.

**Proposition A.5.** *Fix a closed subgroup $H \leqslant G$. For generic $\sigma \in \Gamma^G(E)$, the restriction $\sigma^H$ is transverse to the zero section of $E^H$. In particular, $\mathcal{M}^H(\sigma)$ is a smooth manifold.*

*Proof sketch.* This is an exercise in the Sard–Smale theorem.

1. We consider the universal moduli space $\mathcal{U} := \{(x,\tau) \in M^H \times \Gamma^G(E^H) \mid \tau(x) = 0\}$, where we (by abuse of notation) write $\Gamma^G(E^H)$ for a suitable Banach completion of the space of smooth $G$-equivariant sections of $E^H$.

2. $\mathcal{U}$ is smooth: consider the map $F\colon M^H \times \Gamma^G(E^H) \to E^H, (x,\tau) \to \tau(x)$. Its linearisation at $(x,\tau) \in \mathcal{U}$ is given by $dF_{(x,\tau)}(X,Y) = Y(x) + d\tau(X)$, for $X \in T_xM^H$ and $Y \in \Gamma^G(\tau^*TE^H)$.[1] Clearly, $dF$ is surjective: given any $Y_0 \in T_{\tau(x)}E^H$, choose any equivariant $Y$ with $Y(x) = Y_0$, then $dF(0,Y) = Y_0$.[2] Hence, by the implicit function theorem, $\mathcal{U} = F^{-1}(0)$ is locally a smooth manifold.

[1] Note: we are adding the summands, not composing them. This is akin to the directional derivative, where we take the scalar product of the gradient with the direction vector. We do not apply the chain rule directly: while $\sigma(x(t))$ is a composition of functions, both $\sigma$ and $x$ are part of the input domain! (The directional derivative is computed using a chain rule; that is besides the point.)

[2] In every vector bundle, there exists a smooth section with prescribed value at each fixed point: one can construct one in a local trivialisation using an cut-off function. This construction can be made equivariant: choose a $G$-equivariant cut-off function $\phi\colon U \to \mathbb{R}$ and consider the section $s\colon U \to V \times \mathbb{R}^n, u \mapsto (u, \phi(u)v_0)$.

3. The projection map $\pi\colon \mathcal{U} \to \Gamma^G(E^H), (x,\tau) \to \tau$ is smooth and $\pi^{-1}(\tau)$ corresponds to $\tau^{-1}(0)$. By the Sard–Smale theorem, regular values of $\pi$ are generic. Hence, for a co-meagre set of $\tau$, the set $\pi^{-1}(\tau) = \tau^{-1}(0)$ is a smooth manifold.
4. The set of $\sigma$ such that $\sigma^H$ lies in such a co-meagre subset is co-meagre.
5. The intersection of co-meagre sets is co-meagre; hence for a co-meagre set of $\sigma$ *all* $\mathcal{M}^H(\sigma)$ are smooth manifolds.

□

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