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Integral models of Shimura varieties beyond parahoric level

jt. with Pappas p fixed, and $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$.

§ Motivation

Let (G, X) Sh-datum, let level $K = K^p K_p$. Get

$S(G,X)_K \;/\; E \subset \bar{\mathbb{Q}}$. Let $E = E_v$.

If K parahoric have (in many cases)

- canonical integral model $\mathcal{S}_K$ over O_E
- LMD $\varphi: \mathcal{S}_K \longrightarrow [\mathcal{G} \backslash M_{\mathcal{G},\mu}]$ smooth,

where LM $M_{\mathcal{G},\mu}$ good properties, e.g. normal, CM, **flat + proj.** $/ O_E$.

Problem: What happens for deeper level, e.g. if K has depth zero, i.e. ex parahoric K_0 s.t. $R_u(K_0) \subset K \subset K_0$? Analogue of rep's of depth zero.

Here: First step, namely: let $K_0 = \mathcal{G}(\mathbb{Z}_p)$ parahoric, consider

$K_1 = \ker\left(\mathcal{G}(\mathbb{Z}_p) \to \bar{\mathcal{G}}(\mathbb{F}_p) \to (\bar{\mathcal{G}}_{red})_{ab}(\mathbb{F}_p)\right)$,

or inverse image of $T(\mathbb{F}_p)$, for $T \subset (\bar{\mathcal{G}}_{red})_{ab}$.

Get $Sh_{K_1} \to Sh_K$ étale $T(\mathbb{F}_p)$-cover.

Want to extend to $T(\mathbb{F}_p)$-cover

$$\mathcal{S}_{K_1} \longrightarrow \mathcal{S}_K$$

s.t. $\mathcal{S}_K = \mathcal{S}_{K_1} / T(\mathbb{F}_p)$ + "LMD".

Example: $G = GL_2$, $K_0 = \Gamma_0(p)$. Iwahori. Let $K_1 = \Gamma_1(p)^{bal} = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod p \right\}$.

Then $\mathcal{S}_{K_0} = \{ (E, \alpha, G) \mid G \text{ order } p \}$

$\mathcal{S}_{K_1} = \{ (E, \alpha, G, P, P') \mid P \text{ gen. of } G,\; P' \text{ gen. of } E[p]/G \}$

In ss point, $\mathcal{S}_{K_1} \to \mathcal{S}_{K_0}$ given étale-loc by

$$\operatorname{Spec} \mathbb{Z}_p[U,V]/(U^{p-1}V^{p-1} - p) \longrightarrow \operatorname{Spec} \mathbb{Z}_p[X,Y]/(XY - p)$$
$$X \mapsto U^{p-1}, \quad Y \mapsto V^{p-1}.$$

Observation: cover defined by monomials in parameters upstairs

Idea: Use toric schemes to construct covers of LM $M_{\mathcal{G},\mu}$.

§2 The divisor map

Let $\mathcal{G}/\mathbb{Z}_p$ parahoric. Let $T = (\overline{\mathcal{G}}_{red})_{ab}$ and

$T_{\mathcal{G}}$ = unique torus lift over $\mathbb{Z}_p$.

Thm: Let $(G, \mu, \mathcal{G})$ be LM-triple $/\mathbb{Q}_p$. Assume G tame + $p \nmid \pi_1(G_{der})$. There ex! pair $(P_{\mathcal{G},\mu}, s_{\mathcal{G},\mu})$ consist. $\mathcal{G}$-equiv. $T_{\mathcal{G}}$-torsor over $M_{\mathcal{G},\mu}$ + $\mathcal{G}$-equivar. trivializ. of $P_{\mathcal{G},\mu}, E$ s.t. foll. holds:

For character $\chi : T_{\mathcal{G}, \breve{\mathbb{Z}}_p} \longrightarrow G_{m, \breve{\mathbb{Z}}_p}$ $\mapsto$ $G_{m, O_{\breve{E}}}$-torsor P_χ over $M_{\mathcal{G},\mu, O_{\breve{E}}}$, corresp. to line bdle $\mathcal{L}_\chi$. Has merom. section s_χ, ~~defining~~ with divisor D_χ w. support in special fiber. Then

$$D_\chi = \sum_{\mu' \in \Lambda_{\mu,\mathcal{G}}} e \langle \mu', \chi \rangle \cdot Z_{\mu'}.$$

Here $e = e_{E/\mathbb{Q}_p}$ and $M_{\mathcal{G},\mu} \otimes \mathbb{F}_p = \bigcup_{\mu'} Z_{\mu'}$ irred. comp.

And $\langle\, , \rangle$ pairing, coeff. $\in \mathbb{Z}$.

Remark: $Z_{\mu'}$ are Weil divisors. Part of assertion is that RHS is Cartier.

Example: $G = GL_2$, $\mathcal{G} = \Gamma_0(p)$. Then LM repres. functor

$$\begin{tikzcd} R\oplus R \arrow[r, "\begin{pmatrix} p & \\ & 1\end{pmatrix}"] & R\oplus R \arrow[r, "\begin{pmatrix} 1 & \\ & p\end{pmatrix}"] & R\oplus R \\ \mathcal{L}_0 \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "\delta_0"] & \mathcal{L}_1 \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "\delta_1"] & \mathcal{L}_0 \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

Here $T_{\mathfrak{g}} = \mathbb{G}_m^2$. Then $P_{\mathfrak{g},\mu} = (\mathcal{L}_0^{-1}\mathcal{L}_1, \mathcal{L}_1^{-1}\mathcal{L}_0)$ and $s = (\delta_0, \delta_1)$.

From thm. deduce the <u>divisor map</u> (split torsor in generic)

$$P_{\mathfrak{g},\mu} \otimes_{\mathcal{O}_E} E \longrightarrow T_{\mathfrak{g},E}$$

(G-equiv. for trivial action on target; fibers $\simeq \mathcal{F}(G,\mu)$).

$$M_{\mathfrak{g},\mu} \otimes E \longrightarrow [T_{\mathfrak{g},E} \backslash T_{\mathfrak{g},E}]$$

§ 3 Toric schemes

To simplify, take $\mathcal{G}$ = Iwahori. Let

$$\sigma_\mu = \{ \sum_{\mu' \in \Lambda_\mu} r_{\mu'} \cdot \mu' \ , \ r_{\mu'} \geq 0 \} \subset X_*(T_{\mathcal{G}}) \otimes \mathbb{R}$$

cone gen. by rays μ'.

Remark: Consider $T_{\mathcal{G}, ad}$ in "adjoint group." Then image $\sigma_{\mu, ad}$ defines fan in $X_*(T_{\mathcal{G}, ad}) \otimes \mathbb{R}$ (take cones over rays). If μ projects non-triv. to all simple factors, this toric variety is projective! Uses: let (R, V) irreduc. root system, let $\mu \in V^*$ non-triv. Then $\mathrm{Conv}(W \cdot \mu)$ contains 0 in ~~ext~~ interior. New toric variéties!

But we want affine toric variety, i.e. σ_μ non-degenerate. This holds e.g. if G_{der} 1-conn and G_{ab} unramif ($+ \mu$ minuc)

or if G local Hodge type.

or if $\mu_{ab} \in X_*(G_{ab})$ not torsion.

We will assume this. Hence get toric scheme

$$T_{\mathcal{G}, O_{E_0}} \hookrightarrow Y_{\mathcal{G}, \mu} .$$

Here

$$\Gamma(Y_{\mathcal{G}, \mu}, \mathcal{O}) = O_{E_0} \overbrace{[\mathcal{X}^\chi \mid \langle \mu', \chi \rangle \geq 0, \forall \mu' \in \sigma_\mu]}^{S_\mu}$$

For certain satur. sub-semigps $S \subset S_\mu$ get Y_S.

The divisor map extends to

$$\Delta : [\mathcal{G} \backslash M_{\mathcal{G},\mu}] \longrightarrow [T_{\mathcal{G}} \backslash Y_{\mathcal{G},\mu}].$$

Have conjectural purely comb. description of underlying map on points

$$\mathrm{Adm}(\mu) \longrightarrow \mathrm{Face}(\sigma_\mu).$$

(Q. Yu has used this).

For $G = GL_2$, $[T_{\mathcal{G}} \backslash Y_{S_\mu}] = [\mathbb{G}_m \backslash \mathbb{A}^1]^2$ and ext. is via

$$\delta_0 : \mathcal{L}_0 \to \mathcal{L}_1, \text{ resp. } \delta_1 : \mathcal{L}_1 \to \mathcal{L}_0$$

§ 4 The root stack + Conjecture

Consider Lang map

$$L : T_{\mathfrak{g}} \to T_{\mathfrak{g}} \quad , \quad x \mapsto \mathrm{Frob}(x) \cdot x^{-1} .$$

Get cartesian diagram

$$\begin{tikzcd} T_{\mathfrak{g}, \mathcal{O}_E} \arrow[r, hook] \arrow[d, "L"'] & \tilde{Y}_{S, \mathcal{O}_E} \arrow[d] \\ T_{\mathfrak{g}, \mathcal{O}_E} \arrow[r, hook] & Y_{S, \mathcal{O}_E} \end{tikzcd} \qquad \tilde{Y}_{S,\mathcal{O}_E} = \text{normaliz.}$$

Note: If $T_{\mathfrak{g}}$ split, then $L(x) = x^{q-1}$ (sic!)

We define the root stack

$$\begin{tikzcd} M^{\sqrt{S}}_{\mathfrak{g}, \mu} \arrow[r] \arrow[d] & {[T_{\mathfrak{g}} \backslash \tilde{Y}_S]} \arrow[d, "L"] \\ M_{\mathfrak{g}, \mu} \arrow[r] & {[T_{\mathfrak{g}} \backslash Y_S]} \end{tikzcd}$$

Explanation: If $[T_{\mathfrak{g}} \backslash Y_S] = [\mathbb{G}_m \backslash \mathbb{A}^1]$, then S-valued pt $\hat{=}$ $\mathbb{G}_m$-torsor P over S + equiv. $P \to \mathbb{A}^1$ $\Longleftrightarrow$ line bdle $\mathcal{L}$ + global section.

Cadman defines the n-th root stack via $[\mathbb{G}_m \backslash \mathbb{A}^1] \xrightarrow{n} [\mathbb{G}_m \backslash \mathbb{A}^1]$.

He mostly applies this to effective Cartier div. D and $\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}(D)$

Conjecture: Back to Shim-varieties and $\mathbb{K}_1 \subset \mathbb{K}$. $\underbrace{\text{Fix } S}$ There ex.

$\mathcal{O}_E$-int. model $\mathcal{S}_{\mathbb{K}_1,S}$ of $Sh(G,h)_{\mathbb{K}_1}$ s.t.

- Morph. $\mathcal{S}_{\mathbb{K}_1,S} \to \mathcal{S}_{\mathbb{K}}$ ext. map. in gen. fibers
- Action of $T_{\mathfrak{g}}(\mathbb{F}_p)$ extends and $\mathcal{S}_{\mathbb{K}} = T_{\mathfrak{g}}(\mathbb{F}_p) \backslash \mathcal{S}_{\mathbb{K}_1,S}$
- equiv. for $G(\mathbb{A}_f^p)$-action on towers

* $\exists$ 2-comm.

$$\begin{tikzcd}
\mathcal{S}_{\mathbb{K}_1,S} \arrow[r, "\psi_{1,S}"] \arrow[d] & {[\mathfrak{g} \backslash M_{\mathfrak{g}}^{\sqrt{S}}]} \arrow[d] \\
\mathcal{S}_{\mathbb{K}} \arrow[r, "\psi"] & {[\mathfrak{g} \backslash M_{\mathfrak{g}}]}
\end{tikzcd}$$

s.t. get iso

Scholze: $\psi_{1,S}$ smooth

$$[T_{\mathfrak{g}}(\mathbb{F}_p) \backslash \mathcal{S}_{\mathbb{K}_1,S}] = \text{fiber product.}$$

Remark: gives local descr. of $\mathcal{S}_{\mathbb{K}_1,S}$ in full. sense: Let $x \in \mathcal{S}_{\mathbb{K}}$ with $y \in M_{\mathfrak{g}}$. Then ex iso $T_{\mathfrak{g}}(\mathbb{F}_p)$-equiv.

$$\mathcal{S}_{\mathbb{K}_1,S} \times_{\mathcal{S}_{\mathbb{K}}} \operatorname{Spec}(\mathcal{O}^{sh}_{\mathcal{S}_{\mathbb{K}},x}) \simeq \tilde{Y}_S \times_{Y_S} \operatorname{Spec}(\mathcal{O}^{sh}_{M_{\mathfrak{g}},y})$$

E.g., if $T_{\mathfrak{g}} = G_m^r$ split, then RHS = Spec of

$$\mathcal{O}^{sh}_{M,y}[u_1,\dots,u_r] / u_1^{p-1} - \delta^*(s_1), \dots, u_r^{p-1} - \delta^*(s_r)$$

where $s_1, \dots, s_r$ = min. gener. set of S. $\quad t \cdot u_i = \chi_i(t) \cdot u_i$

Properties of $\mathcal{S}_{\mathbb{K}_1}$ (if exists)

- $\mathcal{S}_{\mathbb{K}_1} \longrightarrow \mathcal{S}_{\mathbb{K}}$ finite + surj. Hence if $\mathcal{S}_{\mathbb{K}}$ flat, then $\mathcal{S}_{\mathbb{K}_1}$ is top. flat.
- If $S = S_\mu$, then $\mathcal{S}_{\mathbb{K}_1}$ is R1 (if $E/\mathbb{Q}_p$ unram.) If $S \neq S_\mu$, then $\mathcal{S}_{\mathbb{K}_1}$ not R1 (if $T_{\mathfrak{g}}$ split).
- We conjecture that $\tilde{Y}_{S_\mu} \longrightarrow Y_{S_\mu}$ flat iff Y_{S_μ} smooth. (holds if $T_{\mathfrak{g}}$ split, or in many PEL-cases (Altmann)). Then $\mathcal{S}_{\mathbb{K}_1} \longrightarrow \mathcal{S}_{\mathbb{K}_a}$ flat only if Y_{S_μ} smooth. If G_{ad} abs. simple then this holds only when $(G_{ad} \otimes \mathbb{Q}_p, \mu) \simeq (PGL_n, \omega_1^\vee)$ (Drinfeld case)
- Expect that $\mathcal{S}_{\mathbb{K}_1}$ outside Drinfeld case not normal. Often not even flat/$\mathbb{Z}_p$ (Marazza).

Z: sensitivity to center: for $(GL_n, \omega_{n-1}^\vee)$ bad!

Amusing fact behind class. result: Let (R, V) irred. root syst, with Weyl gp W. Then for proper psgp W_J have

$$\#(W/W_J) \geq \dim V + 1$$

w. equality only for type A_n and $J = \{s_1, \dots, s_{n-1}\}$ or $\{s_2, \dots, s_n\}$.

Variant of Conjecture: For LSV + ILSV in the sense of [SW].

§ 5 Confirmation of the conjecture.

Thm: Let $(B, V, (\,,\,), *, h, O_B, \mathcal{L})$ be integral PEL data.

Let $\mathcal{G} = \mathrm{Aut}_{O_B, (\,,\,)}(\mathcal{L})$ with common similitude factor in $\mathbb{Z}_p^*$.

Assumptions: $p \neq 2$, $G = \mathcal{G} \otimes \mathbb{Q}_p$ splits over tame ext. and

(*) $\bar{\mathcal{G}}$ connected and $\mathcal{G}^\circ$ is Iwahori for G°.

+ condition (S) (ess. $T_{\mathcal{G}}$ induced torus in specif. way).

Then Conj. holds for $Sh(G^\circ, h)_{\mathcal{G}^\circ(\mathbb{Z}_p)}$.

Example: Let $G = GU_2$ for $F/\mathbb{Q}$ s.t. p ramified and $G \otimes \mathbb{Q}_p$ anisotropic. Hence $K = K_p$ unique (Iwahori) but (*) fails. We cannot prove Conj. in this case!

Idea of proof: Assume divisor thm. ~~Obtain foll. statement.~~

~~[illegible]~~

Note that $T_{\mathcal{G}}(\mathbb{F}_p)$-torsor $Sh_{K_1} \to Sh_{\mathbb{K}}$ defines pair (Q, a),

where Q = $T_{\mathcal{G}}$-torsor over $Sh_{\mathbb{K}}$

a = section of $L_*(Q)$ over $Sh_{\mathbb{K}}$

(via $1 \to \underline{T_{\mathcal{G}}(\mathbb{F}_p)} \to T_{\mathcal{G}} \xrightarrow{L} T_{\mathcal{G}} \to 1$).

Consider foll. statement:

(†) $\exists$ $T_{\mathfrak{g}}$-torsor P over $\mathcal{S}_{\mathbb{K}}$ extending Q + iso of pairs

$$\beta : (L_*(P),\ a : T_{\mathfrak{g}}[1/p] \to L_*(P)[\tfrac{1}{p}]) \simeq \varphi^*(P_{\mathfrak{g},\mu}, s_{\mathfrak{g},\mu})$$

where $\varphi : \mathcal{S}_{\mathbb{K}} \longrightarrow [\mathfrak{g} \backslash M_{\mathfrak{g}}]$.

Note: indep't of S! Consider $L_* : S \subset \tilde{S}$.

<u>Then (†) implies Conj.</u>: Assume $T_{\mathfrak{g}}$ split. For $\chi \in S$ get

$$s_\chi : \mathcal{O}_{M_{\mathfrak{g}}} \longrightarrow \mathcal{L}_\chi \qquad \mathfrak{g}\text{-equiv. global section}$$

gives line bdles $\mathcal{M}_{\tilde{\chi}}$ for varying $\tilde{\chi} \in \tilde{S}$ +

Then (†) ~~implies~~ gives global sections

$$a_\chi : \mathcal{O}_{\mathcal{S}_{\mathbb{K}}} \longrightarrow \mathcal{M}_{L^*(\chi)}$$

~~Varying χ, get $\mathcal{S}_{\mathbb{K}} \to [T_{\mathfrak{g}} \backslash X_S]$, coincides w. divisor map.~~

Now <u>define</u> $\mathcal{S}_{\mathbb{K}_1}$ as ~~set~~ functor of sections $z_{\tilde{\chi}}$ of $\mathcal{M}_{\tilde{\chi}}$ for varying $\tilde{\chi} \in \tilde{S}$ s.t.

- $z_{L^*(\chi)} = a_\chi$
- multiplicative in $\tilde{\chi}_1, \tilde{\chi}_2$.

<u>How to prove (†)</u>: Is a problem of p-adic geometry.

We use OTR-theory of finite gp schemes, as follows.

Let G be Raynaud $\mathbb{F}_{p^d}$-gp scheme / S = flat/$\mathbb{Z}_p$.

Then get line bdles $(\mathcal{L}_i, \delta_{i-1}: \mathcal{L}_{i-1}^p \to \mathcal{L}_i, \gamma_i: \mathcal{L}_i \to \mathcal{L}_{i+1}^p)$

This defines T-torsor for $T = \mathrm{Res}_{\mathbb{F}_{p^d}/\mathbb{F}_p}(G_m)$ given as

$P = (\mathcal{L}_i^{-1})_{i \in \mathbb{Z}/d}$ with sections (δ_{i-1}) of $L_*(P) = (\mathcal{L}_{i-1}^{-p}\mathcal{L}_i)_i$

Pappas: Let G/S finite flat gp scheme, say S perfect. Then can one express

$\det(\mathrm{colie}_G)$

in terms of „Dieudonné module"?

For Raynaud gp schemes $(\mathrm{colie}_G)_i = [\mathcal{L}_{i-1}^p \to \mathcal{L}_i]$

Remark: In a few cases (much fewer than we thought), $\mathcal{S}_{K_1}$ represents moduli pb. formulated directly in terms of OTR-theory:

- Siegel case $\mathcal{S}_{K_0} = \{(A_0 \to A_1 \to \dots \to A_{2g-1} \to A_0, \lambda, \alpha)\}$

  Then $\mathcal{S}_{K_1}$ = add OT-gen. z_i of G_i s.t.

  (Haines-Stroh) $z_i z_{2g+1-i}$ const.

- Split unitary case: same without pairing

  (Haines-Rapo)

- Drinfeld case: add Raynaud gen. of $X[\Pi]$.