# Talk: Faltings and the future of the Verlinde formula.

(*) Let $k = \bar{k}$, char $k = 0$. Let $X/k$ proj. smooth curve.

**Classical Verlinde formula**: Let $G/k$ be simple, simply conn., semi-simple group $\leadsto$ $\mathrm{Bun}_G$ = stack of $G$-bdles on $X$.

**Verlinde formula**: Let $\mathcal{L}/\mathrm{Bun}_G$ dominant line bdle. Then

$$\dim H^0(\mathrm{Bun}_G, \mathcal{L}) = \text{concrete formula}.$$

(proved by Faltings for $G$ of classical type or of type $G_2$). (**)

Concrete formula is in terms of numerical invariants of $\mathcal{L}$ through uniformization.

(*) Connes. between Faltings and Scholze: what F. is working on – commissioned by me.

Disclaimer: I'm an outsider and didn't spend enough time on literature. Also: ess. no historical remarks.

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Important special case: $G = SL_n$. Then $Bun_G$ = moduli stack of v.b. of rank $n$ w. trivial determ.

On $Bun_G$, have $\mathcal{L}_0$ = "det of coho". Then for $c > 0$, $\mathcal{L}_0^{\otimes c}$ is dominant, and $\dim H^0(Bun_G, \mathcal{L}_0^c)$ is „classical" VF: $\longrightarrow$

VF for $G = SL_r$, $\mathrm{Bad} = \emptyset$.

$$\boxed{\dim = \left(\frac{r}{r+c}\right)^g \cdot \sum_{\substack{T \subset [1, r+c] \\ |T| = r}} \prod_{\substack{t \in T \\ u \notin T}} \left| 2 \sin \frac{t-u}{r+c} \right|^{g-1}}$$

(Zagier).

In many contexts, need to replace $\mathrm{Bun}_G$ by $\mathrm{Bun}_{\mathcal{G}}$ (Pappas-Rapoport), e.g. unitary gps. 2

<u>Def</u>: A <u>parahoric BT-gp scheme</u> over $X$ $\underset{\text{def}}{=}$ a smooth gp scheme $\mathcal{G}/X$ with connected fibers s.t.

- $\mathcal{G}_K$ is abs. simple, simply conn., semi-simple gp $/ K = k(X)$
- $\mathcal{G}_x = \mathcal{G} \times_X \mathrm{Spec}(\mathcal{O}_x)$ is a BT-gp scheme $/\mathcal{O}_x$, $\forall x$

<u>Example</u>: $\mathcal{G}$ = smooth gp scheme $/X$ s.t. $\mathcal{G}(x)$ is simply conn, semi-simple gp over $k$.

$\rightsquigarrow \mathrm{Bun}_{\mathcal{G}}$ (e.g. $\mathcal{G} = SL(\mathcal{V}_{/X})$).

<u>Thm 1</u> (Heinloth, conj. by PR): (i) Let $x \in X$. Set

$$\mathcal{F}_x = L\mathcal{G}_x / L^+\mathcal{G}_x$$

be partial affine flag variety. Then have __uniform morphism__

$$p_x : \mathcal{F}_x \longrightarrow \mathrm{Bun}_{\mathcal{G}}$$

s.t. $$\mathrm{Bun}_{\mathcal{G}} = \mathcal{F}_x / \Gamma_{X \setminus \{x\}}(\mathcal{G}).$$

Here $\Gamma_{X \setminus \{x\}}(\mathcal{G})$ is ind-gp scheme $/k$ with

$$\Gamma_{X \setminus \{x\}}(\mathcal{G})(k) = \Gamma(X \setminus \{x\}, \mathcal{G}).$$

(ii) There is a SES

$$0 \to \bigoplus_{x \in X(k)} X^*(\mathcal{G}(x)) \longrightarrow \mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}}) \xrightarrow{ch} c \cdot \mathbb{Z} \to 0$$

Here ch is given by $\mathcal{L} \mapsto ch(p_x^*(\mathcal{L}))$,

where $ch : \mathrm{Pic}(\mathcal{F}_x) \to \mathbb{Z}$ is the __central charge__ map (indep't of $x$).

<u>Numerical invariants of $\mathcal{L} \in \mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}})$:</u>

For $x$ have <u>degree map</u>

$$\deg_x : \mathrm{Pic}(\mathcal{F}_x) \xrightarrow{\sim} \bigoplus_{\mathrm{Vert}(a_x)} \mathbb{Z}$$

If $\mathcal{G}(x)$ is semi-simple, then RHS $= \mathbb{Z}$.

Then $\mathcal{L}$ <u>dominant</u> $\overset{=}{\text{if}}$ $\deg_x(\mathcal{L}) \in \bigoplus \mathbb{Z}_{\geq 0}$.

<u>Thm</u> 2 (Damiolini - Hong - Kumar (?), conj. by PR). Let $\emptyset \neq S \supset \mathrm{Bad}$. Let $\mathcal{L} \mid \mathrm{Bun}_{\mathcal{G}}$ dominant.

$$H^0(\mathrm{Bun}_{\mathcal{G}}, \mathcal{L}) = \Big[ \bigotimes_{x \in S} H^0(\mathcal{F}_x, p_x^*(\mathcal{L})) \Big]^{H^0(X \setminus S, \mathrm{Lie}\,\mathcal{G})} .$$

<u>Remarks</u>: i) The fact that $\mathrm{Lie}\,\mathcal{G}$ acts is non-trivial, lifts the action of $\bigoplus_{x \in S} \mathrm{Lie}(\mathcal{G}_{K_x})$.

ii) RHS does not change when $S$ replaced by $S' \supset S$.
( <u>propagation of vacua</u> )

<u>The problem is:</u> what is dim RHS ?

If $\mathcal{G}_K = G_0 \otimes_k K$ is constant, this is given by the <u>Verlinde formula</u> ($\to$ Beauville's paper). Uses that $H^0(\mathcal{F}_x, \mathcal{L}_x) =$ dual of integrable highest wt rep'n of central charge $c_x$ of central extension of $\mathfrak{g} \otimes K_x$.

Hence get formula in terms of $(G_0, \deg_x(\mathcal{L}), x \in S)$. $\longrightarrow (*)$ next page

<u>Generalization to non-constant $\mathcal{G}$</u>:

<u>Thm</u> 3 (Damiolini-Hong, PR): Let $\mathcal{G}/X$ parahoric BT. Then ex. Galois cover $X'/X$ w. Galois gp $\Gamma$ and a semi-simple $\mathcal{G}'/X'$ with lift of action of $\Gamma$ s.t.

$$\mathcal{G} = \pi_*(\mathcal{G}')^{\Gamma}.$$

Now want VF in terms of $\mathcal{G}'$ and $\Gamma$-action.

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Thm (Faltings): For $\mathfrak{g}_K = G_0 \otimes_k K$, with $G_0$ of type $A, B, C, D$ or $G_2$, have

$$\boxed{\dim = |T_c|^{g-1} \cdot \sum_{\mu \in P_c} \mathrm{Tr}_{V_{\underline{\lambda}}}\left(\exp 2\pi i \frac{\mu+\rho}{c+h^\vee}\right) \cdot \prod_{\alpha>0} \left| 2 \sin \pi \frac{\langle \alpha, \mu+\rho \rangle}{c+h^\vee} \right|^{2-2g}}$$

Here $S = \{x_1, \ldots, x_n\}$ and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ corresp. to $p_x^*(\mathcal{L})$.

Such a formula is given by Deshpande-Mukhopadhyay and by Hong-Kumar, under some hypothesis on action of $\Gamma$ on $\mathfrak{g}'$. Form of formula is

$$\dim = \sum_{\mu \in P_c^{\Gamma}} \frac{S_{\lambda_1,\mu}^{m_1} \cdot \ldots \cdot S_{\lambda_n,\mu}^{m_n}}{(S_{0,\mu})^{n+2g-2}}.$$

Here $\pi^{-1}(S) = \{x_1', \ldots, x_n'\}$, $m_i = e_{x_i}(\tilde{\pi})$, and „$S$" are the S-matrices (depend on roots/weights)

Want: " Simple formula".

Conjecture (Damiolini-Hong): $\dim \neq 0$.

Zu tun: Falls $\mathfrak{g}_x$ hs. relate $\deg(\mathcal{L}_x) \in \mathbb{Z}$ to central charge: Via $(a_0^\vee, \ldots, a_r^\vee)$.

$$c(\epsilon_i) = a_i^\vee \quad , \quad i = 0, \dots, \ell.$$