① Bhatt's lemma:

Def.: An object $M \in D_{qc}(X)$ is CM $\underset{\text{Df.}}{=}$ $\forall x \in X$,

$$R\Gamma_x(M) \in D(\mathcal{O}_{X,x})^{\geq \dim \mathcal{O}_{X,x}}.$$

Let X be bi-equidimensional, noeth., let ω_X be normalized dualizing sheaf, hence get involutive functor

$$\mathbb{D} : D_{coh}(X) \longrightarrow D_{coh}(X),$$

$$\mathbb{D}(M) = R\mathcal{H}om(M, \omega_X).$$

Proposition 1: Let $M \in D_{coh}(X)$. Then

$$M \text{ is CM} \iff \mathbb{D}(M) \in D_{coh}^{\leq -\dim X}(X)$$

Example: Let X bi-equidimensional, noeth. Let $\mathcal{O}_X \in D_{coh}(X)$ (degree zero). Then X Cohen-Macaulay scheme $\iff$ $\mathcal{O}_X$ CM complex

Proposition 2: Let $f: X \to Y$ separated morph. of f.t. between bi-equidim. noeth. schemes of same dimension and s.t. Y has dualizing complex. If $M \in D_{coh}(Y)$ is

CM, then $f^!(M) \in D_{coh}(X)$ is CM.

Proof: Write $M = \mathbb{D}(M')$, where $M' \in D^{\leq -\dim \cancel{X} Y}(Y)$. Then

$$f^!(M) = f^!(\mathbb{D}_Y(M')) = \mathbb{D}_X(f^*(M'))$$

But $M' \in D^{\leq -\dim Y}(Y) \Rightarrow f^*(M') \in D^{\leq -\dim X}(X)$.

Now apply Prop 1.

Prop. 3: Let $M \in D_{coh}(X)$ be CM, let $U \subset X$ be large open. Then $H^0(X, M) \longrightarrow H^0(U, M)$ is an isomorphism.

Proof: M CM $\Rightarrow$ $\forall\, Z \subset X$ of codim $\geq c$, have $R\Gamma_Z(M) \in D^{\geq c}(X)$. Long exact sequ.

$$H^i(X, M) \xrightarrow{\sim} H^i(X \setminus Z, M),$$

$$\forall\, i < c-1$$

Now note that $c(Z) \geq 2$ for $Z = X \setminus U$.

Applications: Let $f : X \to Y$ bi-equiv., noeth., same dim ~~over~~ $\mathcal{O} \in$ DVR

Assume Y is Cohen-Macaulay.

Lem: If $U \subset X$ and $V \subset Y$ open s.t. $f(U) \subset V$ and U large.

1) • if U, V smooth / base. Then get $\mathcal{O}_U \to f^! \mathcal{O}_V$ and hence fund. class $\mathcal{O}_X \to f^! \mathcal{O}_Y$ (see Coll. talk)

2) • if $U \to V$ finite + flat, have trace map

$$f_* \mathcal{O}_U \to \mathcal{O}_V \rightsquigarrow \mathcal{O}_U \to f^! \mathcal{O}_V,$$

hence fund. class.

1') • if U, V regular, also works as follows: $U \to V$ is lci, hence $\omega_{U/V}$ is perfect complex of flat amplitude in $[-1, 0]$ of virtual dimension zero. Hence can be represented by $[\mathcal{E}_{-1} \xrightarrow{d} \mathcal{E}_0]$ loc. free of same rank

Hence $\det(d)$ = global section of

$$\Lambda^{max}(\mathcal{F}_0) \otimes \Lambda^{max}(\mathcal{F}_{-1})^{\otimes(-1)} = \cancel{\det}(\omega_{U/V}) \simeq f^!(\mathcal{O}_V).$$

(Need to check independence of choices!)

<u>Remark</u>: In case we are in the intersection of case 1') and 2), get same fund. class (Illusie, Pilloni).

(2) Transition morphisms and local models

Recall principle of LM: Let (G, X) Sh-datum, let $\mathbb{K} = \mathbb{K}^p \cdot K$, where $K \subset G(\mathbb{Q}_p)$ parahoric (or quasi-paral)

Let $\mathcal{S}_{\mathbb{K}}$ be integral model $/ O_E$. To $(G, \{\mu\}, K)$ have LM $\mathbb{M}_{\mathbb{K}} = \mathbb{M}(G, \mu, K)$ = projective scheme over O_E, w. action of $\mathcal{G} \otimes O_E$.

+ LMD

$$\begin{tikzcd} & \tilde{\mathcal{S}}_{\mathbb{K}} \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ \mathcal{S}_{\mathbb{K}} & & \mathbb{M}_K \end{tikzcd}$$

where π = p.h.s. under $\mathcal{G}_K \otimes_{\mathbb{Z}_p} O_E$ and

φ smooth morph. of <u>same</u> relative dim, equivariant.

In brief

$$\mathcal{S}_{\mathbb{K}} \longrightarrow [\mathbb{M}_K / \mathcal{G}_{O_E}]$$ stack, with finitely many pts.

In particular, $\forall x \in \mathcal{S}_{\mathbb{K}}(\mathbb{F})$ ex $y \in \mathbb{M}_K(\mathbb{F})$ s.t.
$$O^{sh}_{\mathcal{S}_{\mathbb{K}}, x} \simeq O^{shK}_{\mathbb{M}_K, y}.$$

Now let $K' \subset K$, get

$$\begin{array}{ccc} \mathcal{S}_{\mathbb{K}'} & \longrightarrow & [\mathbb{M}_{K'} / \mathcal{G}'_{\mathcal{O}_E}] \\ \downarrow & & \downarrow \\ \mathcal{S}_{\mathbb{K}} & \longrightarrow & [\mathbb{M}_K / \mathcal{G}_{\mathcal{O}_E}] \end{array}$$

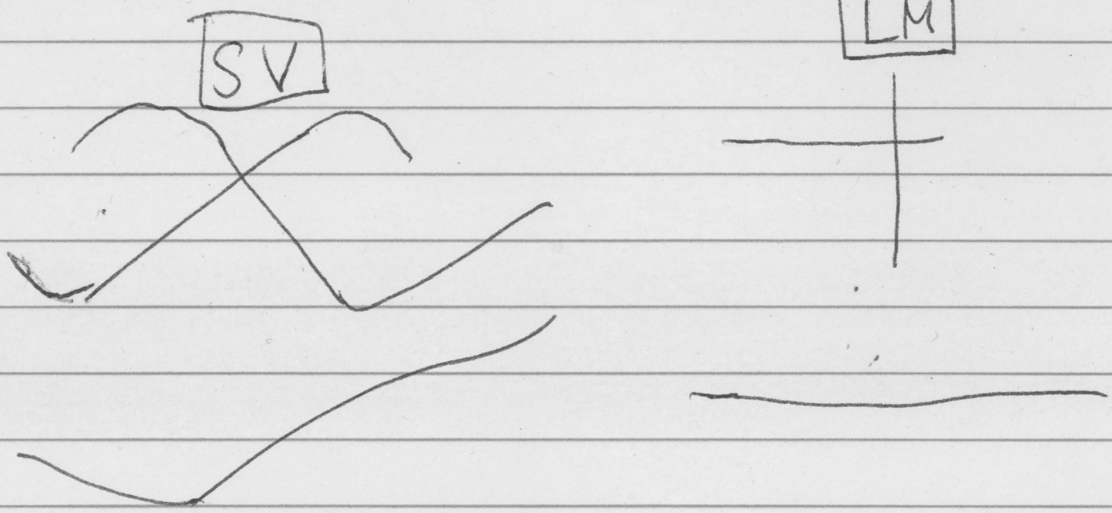
Z not true that morph. RHS "models" LHS

$$\cancel{\begin{array}{ccc} \hat{\mathcal{O}}_{\mathcal{S}_{\mathbb{K}},x} & \longrightarrow & \hat{\mathcal{O}}_{\mathcal{S}_{\mathbb{K}'},x'} \\ & & \wr \\ \hat{\mathcal{O}}_{\mathbb{M}_{K'},y} & \longrightarrow & \hat{\mathcal{O}}_{\mathbb{M}_{K'},y'} \end{array}}$$

No

$$\begin{array}{ccc} \hat{\mathcal{O}}^{sh}_{\mathbb{K}',x'} & \simeq & \hat{\mathcal{O}}^{sh}_{\mathbb{M}_{K'}} \\ \uparrow & & \uparrow \\ \mathcal{O}^{sh}_{\mathbb{K},x} & \simeq & \mathcal{O}^{sh}_{\mathbb{M}_K} \end{array}$$

<u>Example</u>: GL_2 : $K = K_0$ $K' = I$



<u>Here observation:</u> Let $\xi' \in \mathcal{U}^{ext}_{\mathbb{K}'}$ generic point, then $\xi = \pi(\xi') \in \mathcal{U}^{ext}_{\mathbb{K}}$ is again a generic point, and

$$[k(\xi'):k(\xi)]_i = \begin{cases} 1 \\ p \end{cases} = p^{\dim_{\eta'}(\mathbb{M}_{K'}/\mathbb{M}_K)}.$$

This is a general fact:

<u>Theorem</u>: Consider Siegel case or unitary case (GSp or GU).

Let $K = K_i$ max. parah., $K' = K_{0i}$. Let $\xi' \in \mathcal{U}_{\mathbb{K}'}^{\text{extr}}$ generic in special fiber, with image $\xi \in \mathcal{U}_{\mathbb{K}}^{\text{extr}}$. Then

- ξ is also generic (no contraction)
- Consider exact sequence of $\mathcal{O}_{\mathcal{U}_{K'},\xi'}$ - modules

$$\Omega^1_{\mathcal{U}_{\mathbb{K}}/\mathcal{O}} \otimes \mathcal{O}_{\xi'} \to \Omega_{\mathcal{U}_{\mathbb{K}'}/\mathcal{O},\xi'} \to \Omega_{\mathcal{U}_{\mathbb{K}'}/\mathcal{U}_{\mathbb{K}},\xi'} \to 0$$

Then

$$\ell(\Omega_{\mathcal{U}_{\mathbb{K}'}/\mathcal{U}_{\mathbb{K}},\xi'}) \geq \ell(\Omega_{\mathcal{U}_{\mathbb{K}'}/\mathcal{U}_{\mathbb{K}}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p) = \ell(\Omega_{\mathbb{M}_{K'}/\mathbb{M}_K,\eta'})$$

$$= \dim_{\eta'}(\mathbb{M}_{K'} \to \mathbb{M}_K).$$

And ~~Furthermore~~, the length in the middle equals $[k(\xi'):k(\xi)]_i$

Furthermore, have equality if ξ' lies in

μ-ordinary locus (always the case when G is split, but not in general).

Uses foll. elementary lemma.

<u>Lemma</u>: Let $A = \mathrm{DVR} / \mathbb{Z}_p$ with uniform. p. Let $A \to B$ local homo in DVR where again p is uniformizer. Assume $\kappa(B) \subset \kappa(A)^{1/p}$. Then $\Omega_{B/A} \simeq \Omega_{B/A} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ and

$$\ell(\Omega_{B/A}) = [\kappa(B) : \kappa(A)]_i$$

<u>Remarks</u>: One can show:

- for any generic $\xi \in U_{\mathbb{K}}$ ex! $\xi' \in U_{\mathbb{K}'}$ with image ξ s.t. trans. map is local iso. around ξ'. In this case

$$lg(\Omega_{\mathcal{S}_{\mathbb{K}'}/\mathcal{S}_{\mathbb{K}}, \xi'}) = 0 .$$

This is used in proof of optimality theorem for $V = \mathcal{O}$