Integral Hecke correspondences

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joint work in progress w.

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§1 Formulation of the problem

Let

$$\mathcal{L} = \{ \Lambda \subset \mathbb{C} \mid \text{lattice} \}.$$

A lattice correspondence $\overset{=}{\text{Df}}$

$$\mathcal{T} \longrightarrow \mathcal{L} \times \mathcal{L}$$

s.t. p_1 and p_2 finite fibers

(multi-valued maps from $\mathcal{L}$ to itself).

Can multiply

$$\begin{tikzcd}
 & & T \arrow[dl] \arrow[dr, squiggly] \arrow[d, phantom, "\square"] & & \\
 & T_1 \arrow[dl] \arrow[dr, squiggly] & {} & T_2 \arrow[dl] \arrow[dr, squiggly] & \\
\mathcal{L} & & \mathcal{L} & & \mathcal{L}
\end{tikzcd}$$

Take formal $\mathbb{Z}$-combinations $\rightsquigarrow$

$\mathrm{Corr}(\mathcal{L})$ = $\mathbb{Z}$-algebra (commutative!)

To T associate <u>linear operator</u> on functions: for $F: \mathcal{L} \longrightarrow \mathbb{C}$ $\rightsquigarrow$

$$T_*(F)(\Lambda) = \sum_{\Lambda'} \mathrm{mult}_T(\Lambda, \Lambda') F(\Lambda').$$

Apply this to modular forms:

Let $f \in M_k(\Gamma) \mapsto F_f$ with

$$F_f(\Lambda(\omega_1, \omega_2)) = \omega_2^{-k} \cdot f\left(\frac{\omega_1}{\omega_2}\right).$$

Then

$$\tau_*(F_f) = F_{\tau_*(f)}, \quad \tau_*(f) \in M_k.$$

Get

$$\mathrm{Corr}(\mathcal{L}) \longrightarrow \mathrm{End}(M_k(\Gamma)).$$

<u>Hecke operators</u>.

3 steps:

- Correspondence
- action on coefficients
- linear operator

<u>Modern point of view</u>: Let $S_{\mathbb{K}}$ be tower of modular curves,

$$S_{\mathbb{K}}(\mathbb{C}) = GL_2(\mathbb{Q}) \backslash (X \times GL_2(\mathbb{A}_f)/\mathbb{K}),$$

where

$$X = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) = \pm \text{ half plane}$$

$\mathbb{K}$ open compact subgp.

$$= \coprod_i \Gamma_i \backslash X$$

<u>Connection to beginning</u>: if $\mathbb{K} = \mathbb{K}_0 = GL_2(\hat{\mathbb{Z}})$,

$$S_{\mathbb{K}_0} = \mathcal{L} / \mathbb{C}^\times.$$

Whole tower defined $/\mathbb{Q}$.

Let $g \in G(\mathbb{A}_f)$. Get geometric Hecke corresp. to g

$$\begin{tikzcd} & S_{\mathbb{K} \cap g \mathbb{K} g^{-1}} \arrow[dl, "1"', "\pi_1"] \arrow[dr, "g", "\pi_2"'] & \\ S_{\mathbb{K}} & & S_{\mathbb{K}} \end{tikzcd}$$

(finite fibers). <u>This gives 1st step.</u>

<u>2nd step:</u> Let $\omega_{k,\mathbb{K}}$ = line bdle of modular forms of wt k on $S_{\mathbb{K}}$.

Then

$$\pi_2^*(\omega_k) \xrightarrow{\sim} \pi_1^*(\omega_k)$$

But π_1 finite + flat $\Rightarrow$ trace map

$$\pi_{1*}\pi_1^*(\omega_k) \longrightarrow \omega_k$$

equivalently, using <u>extraordinary pullback</u>, have identification

$$\pi_2^*(\omega_k) \to \pi_1^*(\omega_k) \cong \pi_1^!(\omega_k).$$

<u>cohomological Hecke corresp.</u> $(S_{\mathbb{K}}, \omega_k) \to (S_{\mathbb{K}}, \omega_k)$

<u>3rd step</u>: Cohom. corr. induces linear map on coho

$$T_g : H^i(S_{\mathbb{K}}, \omega_k) \longrightarrow H^i(S_{\mathbb{K}}, \omega_k).$$

<u>Hecke operator</u>.

<u>Remark</u>: $H^i(S_{\mathbb{K}}, \omega_k)$ can be infinite-dim.

Replace by $H^i(\tilde{S}_{\mathbb{K}}, \tilde{\omega}_k)$.

By linear extension get homo of algebras

$$\mathcal{H}(G(\mathbb{A}_f)/K) \to \mathrm{End}\left(H^i(\tilde{S}_K, \tilde{\omega}_k)\right).$$

<u>Remark</u>: Let $k \geq 1$. Then $H^i = 0$ for $i > 0$.

Any elt. of $M_k(\Gamma)$ has <u>q-expansion</u>

Let $g = (1, \dots, \begin{pmatrix} p & \\ & 1 \end{pmatrix}, 1, \dots)$.

Then

$$T_p\left(\sum_{n \geq 0} a_n q^n\right) = \sum_{n \geq 0}\left(p a_{pn} q^n + p^k a_n q^{pn}\right)$$

<u>Note</u>: Not optimal!

<u>General Shimura varieties</u>: Let $(G, X) =$ Shimura datum $\rightsquigarrow$ $\mathbb{E} \subset \mathbb{C}$ <u>reflex field</u>

$$S_{\mathbb{K}} = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/\mathbb{K}$$

<u>1st step</u>: Define geom. Hecke corresp. to $g \in G(\mathbb{A}_f)$: clear

<u>2nd step</u>: Replace ω_k by any automorphic vb. $\mathbb{V}$, get

$$\pi_1^*(\mathbb{V}) \xrightarrow{\sim} \pi_2^*(\mathbb{V}) = \pi_2^!(\mathbb{V}).$$

<u>3rd step</u>: Get

$$\alpha : \mathcal{H}(G(\mathbb{A}_f), \mathbb{K}) \longrightarrow \mathrm{End}(H^i(\tilde{S}_{\mathbb{K}}, \tilde{\mathbb{V}}_{\mathbb{K}}))$$

$$i = 0, \dots, d.$$

Problem: Fix p, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let

$$E = \mathbb{E}_v, \quad G = G \otimes_{\mathbb{Q}} \mathbb{Q}_p, \quad \mathbb{K}_0 = \mathbb{K}^p \cdot K_0,$$

where $K_0 \subset G(\mathbb{Q}_p)$ hyperspecial. Then have $\mathcal{S}_{\mathbb{K}_0}$ smooth model of $S_{\mathbb{K}_0} \otimes_E E$
$/\mathcal{O}_E$

$\mathcal{V}_0$ natural v.b. extending $\mathbb{V}_{\mathbb{K}_0}$.

Question: Does α extend to

$$\alpha : \mathcal{H}(G(\mathbb{A}_f), \mathbb{K}_0) \to \mathrm{End}(R\Gamma(\tilde{\mathcal{S}}_{\mathbb{K}_0}, \tilde{\mathcal{V}}_0))$$

Note: $\mathcal{H}(G(\mathbb{A}_f), \mathbb{K}_0) = \mathcal{H}(G(\mathbb{A}_f^p), \mathbb{K}^p) \otimes \mathcal{H}(G(\mathbb{Q}_p), K_0)$

Problem splits up in 3 parts:

1st step: Extend roof integrally

$$\begin{tikzcd} & S_{\mathbb{K}} \arrow[dl] \arrow[dr] & \\ S_{\mathbb{K}_0} & & S_{\mathbb{K}_0} \end{tikzcd}$$

Cannot be done in naive way: have good integral models only for $\mathbb{K} = \mathbb{K}^p . K$, where $K \subset G(\mathbb{Q}_p)$ parahoric

2nd step: Lift to coho. correspondence of $\mathcal{V}_0$.

3rd step: Deduce estimate on p-adic absolute values of eigenvalues of Hecke operators on $H^i(\tilde{S}_{\mathbb{K}}, \tilde{\mathbb{V}}_{\mathbb{K}})$.

Explanation: Let

$$H^i_{\overline{\mathbb{Z}}_p} \subset H^i(\tilde{S}_{\mathbb{K}}, \hat{\mathbb{V}}_{\mathbb{K}}) \otimes \overline{\mathbb{Q}}_p \text{ be } \overline{\mathbb{Z}}_p\text{-lattice}$$

stable under $\mathbb{Z}_p$-algebra $\mathcal{H}$. Let

$$\chi : \mathcal{H} \longrightarrow \overline{\mathbb{Q}}_p$$

eigenvalue occ. in H^i. Then χ factors through $\overline{\mathbb{Z}}_p$ and hence

$$|\chi(h)| \leq 1, \qquad \forall\, h \in \mathcal{H}.$$

Two kind of lattices:

- $H^i(\tilde{S}_K, \tilde{\mathcal{V}}_K) \to H^i(\tilde{S}_{\mathbb{K}}, \tilde{\mathbb{V}}_{\mathbb{K}})$
- If $\mathbb{V} = \mathbb{L} \otimes \mathcal{O}$, then

$$H^i(\tilde{S}_{\mathbb{K}}, \mathbb{L}) \longrightarrow H^i(\tilde{S}_{\mathbb{K}}, \tilde{\mathbb{V}}_{\mathbb{K}}).$$

(Vincent Lafforgue, Fakhruddin – Pilloni)

§2 A new basis of spherical Hecke algebra

<u>Notation</u>: $G/\mathbb{Q}_p$, K_0, $A \subset T \subset B$

$\mathcal{H}(G, K_0)$ has minimal system of generators

$\{T_\lambda \mid \lambda \text{ indecomposable in } X_*(A)^+\}$.

If G adjoint, then λ indecomposable

iff $\lambda = \omega_i^\vee$, fundamental coweight,

and

$$\mathcal{H} = \mathbb{Z}_p[T_{\omega_1^\vee}, \ldots, T_{\omega_m^\vee}] \quad \text{polynomial algebra}$$

<u>Theorem</u>: Fix Iwahori $I \subset K_0$. There ex another minimal system of generators

$\{A_\lambda \mid \lambda \text{ as before}\}$

s.t. A_λ of form $\quad \tau K_r \tau^{-1} = K_0$

$$A_\lambda = \mathbb{1}_{K_0 K_1} \cdot \mathbb{1}_{K_1 K_2} \cdot \ldots \mathbb{1}_{K_{r-1} K_r} \cdot T_{\tau(\lambda)}$$

where $K_0, K_1, \ldots, K_r$ std. maximal parah.

Furthermore, the base change matrix from T_λ to A_λ is upper triangular.

If G classical gp, then $r = 1$ or 2.

<u>Example</u>: Let $G = Sp(V, \langle\, , \rangle)$. Fix maximal selfdual periodic lattice chain

$$\Lambda_{-m} \subset \Lambda_{-m+1} \subset \ldots \subset \Lambda_0 = \Lambda_0^\vee \subset \Lambda_1 = \Lambda_{-1}^\vee \subset \ldots$$

Then atomic basis given by

$$A_i = \mathbb{1}_{K_0 K_i} \cdot \mathbb{1}_{K_i K_0} \qquad K_i = \mathrm{Stab}(\Lambda_i)$$

<u>For adjoint group</u>: the same, except replace A_m by

$$A_m' = \mathbb{1}_{K_0 K_m} \cdot T_{\tau(m)}$$

($\omega_m^\vee$ minuscule fundamental coweight).

Geometric Hecke corresp. assoc. to A_i is

$$\begin{tikzcd}
& & \tilde{S} \arrow[dl] \arrow[dr] & & \\
& S_{\mathbb{K}_{0i}} \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\square"] & & S_{\mathbb{K}_{0i}} \arrow[dl] \arrow[dr] & \\
S_{\mathbb{K}_0} & & S_{\mathbb{K}_i} & & S_{\mathbb{K}_0}
\end{tikzcd}$$

<u>Key point</u>: this diagram extends integrally to

$$\begin{tikzcd}
 & & \tilde{\mathcal{S}} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{S}_{\mathbb{K}_{0i}} \arrow[dl] \arrow[dr] & & \mathcal{S}_{\mathbb{K}_{0i}} \arrow[dl] \arrow[dr] & \\
\mathcal{S}_{\mathbb{K}_0} & & \mathcal{S}_{\mathbb{K}_i} & & \mathcal{S}_{\mathbb{K}_0}
\end{tikzcd}$$

§3 Cohomological integral Hecke corresp.

($\Lambda = \mathcal{O}$).

Fact 1: Let $f: X \to Y$ morph. of $\mathcal{O}$-schemes of f.t. Assume X, Y are smooth $/\mathcal{O}$ of same dimension. Then natural map (fundamental class of f)

$$\mathcal{O}_X \to f^!(\mathcal{O}_Y).$$

Construction: exact sequence

$$\Omega^1_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{O}_X \xrightarrow{df} \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

Get $\det(df) : \Omega^d_{Y/\mathcal{O}} \otimes \mathcal{O}_X \to \Omega^d_{X/\mathcal{O}}$;

hence

$$\mathcal{O} \to \omega_{X/\mathcal{O}} \otimes \omega_{Y/\mathcal{O}}^{\otimes(-1)} \doteq f^!(\mathcal{O}_Y)$$

(Grothendieck duality)

<u>Remark:</u> Extends to case where X, Y regular.

<u>Fact 2</u>: Let $f : X \to Y$ over $\mathcal{O}$.

Assume $U \subset X$ and $V \subset Y$ opens with U, V regular of same dimension, with $f(U) \subset V$, and $U \subset X$ <u>large</u>.

Also assume Y Cohen-Macaulay.

Then $\mathcal{O}_U \to f^!(\mathcal{O}_Y)_{|U}$ extends uniquely to $\mathcal{O}_X \to f^!(\mathcal{O}_Y)$.
(uses lemma of Bhatt).

Can apply this to $\mathcal{S}_{\mathbb{K}'} \to \mathcal{S}_{\mathbb{K}}$:

- all integral models are CM
- let $\mathcal{U}_{\mathbb{K}}^{\mathrm{extr}} \subset \mathcal{S}_{\mathbb{K}} \overset{=}{\underset{\mathrm{Df}}{}}$ complement of all non-extreme Kottwitz-Rapoport strata in special fiber. Then
  - $\mathcal{U}_{\mathbb{K}}^{\mathrm{extr}}$ smooth $/\mathcal{O}_E$ ($\overset{?}{=}$ smooth locus?)
  - $\mathcal{U}_{\mathbb{K}'}^{\mathrm{extr}} \longrightarrow \mathcal{U}_{\mathbb{K}}^{\mathrm{extr}}$.

Hence get $\mathcal{O}_{\mathcal{S}_{\mathbb{K}'}} \to f^!(\mathcal{O}_{\mathcal{S}_{\mathbb{K}}})$.

Apply to atomic generator

$$A_\lambda = \mathbb{1}_{K_0 K_1} \cdot \mathbb{1}_{K_1 K_2} \cdot \ldots \cdot \mathbb{1}_{K_{r-1} K_r} \cdot T_{\tau(\lambda)}$$

get cohom. correspondences

$$(\mathcal{S}_{\mathbb{K}_0}, \mathcal{O}) \to (\mathcal{S}_{\mathbb{K}_1}, \mathcal{O}) \to \ldots \to (\mathcal{S}_{\mathbb{K}_r}, \mathcal{O}) \cong (\mathcal{S}_{\mathbb{K}_0}, \mathcal{O})$$

Obtain linear operator

$$A_{\lambda *} : R\Gamma(\mathcal{S}_{\mathbb{K}_0}, \mathcal{O}) \to R\Gamma(\mathcal{S}_{\mathbb{K}_0}, \mathcal{O})$$

as composition.

<u>Conjecture</u>: (i) A_{λ_*} only depends on λ (not on atomic presentation).

(ii) These endomorphisms commute and define $\mathcal{H}(G, K_0) \to \mathrm{End}(R\Gamma(S_{\mathbb{K}_0}, \theta))$

(iii) A_{λ_*} optimal, i.e. not divisible by p.

<u>Theorem</u>: The composed cohomological correspondence

$$(S_{\mathbb{K}_0}, \theta) \longrightarrow (S_{\mathbb{K}_0}, \theta)$$

is optimal.

§4 Non-trivial coefficients (G classical)

Let autom. v.b. $\mathbb{V}$ corresp. to irreduc. rep'n of $M = M_\mu$ over E: hence have $\kappa \in X^*(T)^{+-M}$. Consider atomic generator $A_i = A_{\lambda_i} = \mathbb{1}_{K_0 K_i} \mathbb{1}_{K_i K_0} \cdot T_{\tau_i}$.

Then have "natural integral extensions" $\mathcal{V}_0$ of $\mathbb{V}_{K_0}$ resp. $\mathcal{V}_i$ of $\mathbb{V}_{K_i}$, plus natural maps

$$\pi_0^*(\mathcal{V}_0) \to \pi_i^*(\mathcal{V}_i), \quad \pi_i^*(\mathcal{V}_i) \to \pi_0^*(\mathcal{V}_0)$$

Compose with fundamental class

$$\pi_i^*(\mathcal{V}_i) \to \pi_i^*(\mathcal{V}_i) \otimes \pi_i^!(\mathcal{O}) = \pi_i^!(\mathcal{V}_i)$$

$$\pi_0^*(\mathcal{V}_0) \to \pi_0^*(\mathcal{V}_0) \otimes \pi_0^!(\mathcal{O}) = \pi_0^!(\mathcal{V}_0).$$

Get cohom. correspondences

$$(S_{\mathbb{K}_0}, \mathcal{V}_0) \to (S_{\mathbb{K}_i}, \mathcal{V}_i) \to (S_{\mathbb{K}_0}, \mathcal{V}_0).$$

<u>Projected theorem</u>: Consider composed coh. correspondence $(S_{\mathbb{K}_0}, \mathcal{V}_0) \to (S_{\mathbb{K}_0}, \mathcal{V}_0)$.

Let $\quad \alpha(\kappa) = (\kappa+\rho)_{G\text{-dom.}}$ infinit. char.

Set

$$e_i(\kappa) = \langle \lambda_i, \alpha(\kappa) - \rho \rangle + \langle \lambda_i, \beta(\kappa) \rangle$$

Here $\beta(\kappa)$ = pos. linear comb. of simple roots, hence error term ≥ 0

Then above coho. corresp. is divisible by $p^{e_i(\kappa)}$.

Note: If $\kappa+Q$ regular, or if λ_i minuscule, then $\beta(\kappa)=0$ (last case leads to conjecture of Fakhruddin–Pilloni).