


Integral Hecke correspondences

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joint work in progress w.

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§ 1 Formulation of the problem

Let

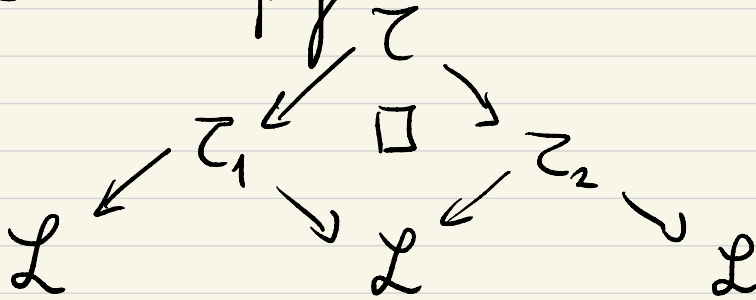
$$\mathcal{L} = \{ \Lambda \subset \mathbb{C} \mid \text{lattice} \}$$

A lattice correspondence $\bar{\sigma}_f$

$$\mathcal{Z} \rightarrow \mathcal{L} \times \mathcal{L}$$

s.t. p_1 and p_2 finite fibers
(multi-valued maps from $\mathcal{L}$ to
itself).

Can multiply



Take formal $\mathbb{Z}$ -combinations m

$\text{Corr}(L) = \mathbb{Z}\text{-algebra (commutative!)}$

To Z associate linear operator on

functions : for $F: L \rightarrow \mathbb{C}^m$

$$Z_*(F)(\Lambda) = \sum_{\Lambda'} \text{mult}_Z(\Lambda, \Lambda') F(\Lambda').$$

Apply this to modular forms:

Let $f \in M_k(\Gamma)$ mod F_f with

$$F_f(\Lambda(\omega_1, \omega_2)) = \omega_2^{-k} \cdot f\left(\frac{\omega_1}{\omega_2}\right).$$

Then

$$Z_*(F_f) = F_{Z_*(f)} \quad , \quad Z_*(f) \in M_k.$$

Get

$$\text{Corr}(L) \rightarrow \text{End}(M_k(\Gamma)).$$

Heccke operators.

3 steps:

- Correspondence
- action on coefficients
- linear operator

Modern point of view: Let S_K be
tower of modular curves,

$$S_K(\mathbb{C}) = GL_2(\mathbb{Q}) \backslash (X \times GL_2(\mathbb{A}_f) / K),$$

where

$$X = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) = \pm \text{ half plane}$$

K open compact subgp.

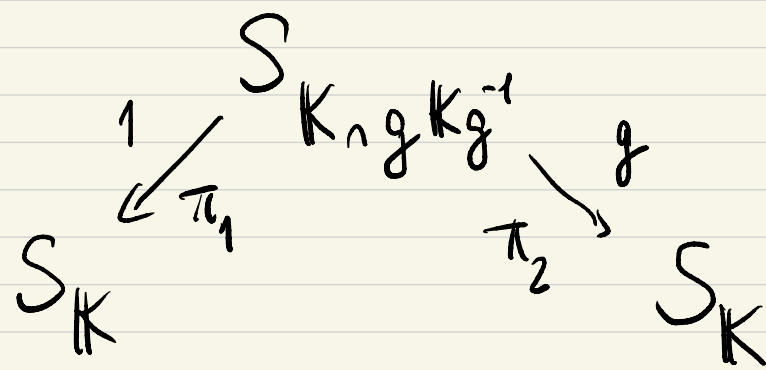
$$= \coprod_i \Gamma_i \backslash X$$

Connection to beginning: if $K = K_0 = GL_2(\hat{\mathbb{Z}})$,

$$S_{K_0} = \mathcal{L} / \mathbb{C}^\times.$$

Whole tower defined / $\mathbb{Q}$.

Let $g \in G(A_f)$. Get geometric Flecke corresp. by



(finite fibers). This gives 1st step.

2nd step: Let $\omega_{k,K}$ = line bundle of modular forms of wt k on S_K .

Then

$$\pi_2^*(\omega_k) \xrightarrow{\sim} \pi_1^*(\omega_k)$$

But π_1 finite + flat implies trace map

$$\pi_{1*} \pi_1^*(\omega_k) \rightarrow \omega_k$$

equivalently, using extraordinary pullback,
have identification

$$\pi_2^*(\omega_k) \rightarrow \pi_1^*(\omega_k) \stackrel{!}{=} \pi_1^!(\omega_k).$$

cohomological Hecke corresp. $(S_K, \omega_k) \rightarrow (S_{K'}, \omega_{k'})$

3rd step: Cohom. corr. induces linear
map on coho

$$T_g: H^i(S_K, \omega_k) \rightarrow H^i(S_{K'}, \omega_{k'}).$$

Hecke operator.

Remark: $H^i(S_K, \omega_k)$ can be infinite-dim.

Replace by $H^i(\tilde{S}_K, \tilde{\omega}_k)$.

By linear extension get homo of algebras

$$\mathcal{H}(G(A_f)/K) \rightarrow \text{End}(\mathcal{H}^i(\tilde{S}_K, \tilde{\omega}_K))$$

Remark: Let $k \geq 1$. Then $\mathcal{H}^i = 0$ for $i > 0$.

Any elt. of $M_k(\Gamma)$ has q -expansion

$$\text{Let } q = (1, \dots, \binom{p}{1}, 1, \dots).$$

Then

$$T_p\left(\sum_{n \geq 0} a_n q^n\right) = \sum_{n \geq 0} \left(p a_{pn} q^n + p^k a_n q^{pn}\right)$$

Note: Not optimal!

General Shimura varieties: Let $(G, X) =$

Shimura datum $\mapsto \mathbb{E} \subset \mathbb{C}$ reflex field

$$S_K = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$$

1st step: Define geom. Hecke corresp. to

$g \in G(\mathbb{A}_f)$: clear

2nd step: Replace ω_K by any automorphic vol. $\mathbb{V}$, get

$$\pi_1^*(\mathbb{V}) \xrightarrow{\sim} \pi_2^*(\mathbb{V}) = \pi_2^!(\mathbb{V}).$$

3rd step: get

$$\alpha: \mathcal{H}(G(\mathbb{A}_f), K) \rightarrow \text{End}(H^i(\tilde{S}_K, \tilde{\mathbb{V}}_K))$$

$i=0, \dots, d.$

Problem: Fix p , $\bar{Q} \hookrightarrow \bar{Q}_p$. Let

$$E = E_v, \quad G = G \otimes_{\mathbb{Q}} \mathbb{Q}_p, \quad K_0 = K^p \cdot K_0,$$

where $K_0 \subset G(\mathbb{Q}_p)$ hyperspecial. Then

have S_{K_0} smooth model of $S_{K_0} \otimes_{\mathbb{Q}_E} E / \mathcal{O}_E$
 $\mathcal{V}_0$ natural vb. extending $\mathcal{V}_{K_0}$.

Question: Does α extend to

$$\alpha: \mathcal{H}(G(\mathbb{A}_f), K_0) \rightarrow \text{End}(\text{RP}(\tilde{S}_{K_0}, \tilde{\mathcal{V}}_0))$$

Note: $\mathcal{H}(G(\mathbb{A}_f), K_0) = \mathcal{H}(G(\mathbb{A}_f^p), K^p) \otimes \mathcal{H}(G(\mathbb{Q}_p), K_0)$

Problem splits up in 3 parts:

1st step: Extend roof integrally

$$S_K \swarrow \searrow \\ S_{K_0} \quad S_{K_0}$$

Cannot be done in naive way: have good integral models only for $K = K^p \cdot K$, where $K \subset \mathbb{Q}_p$ parahoric

2nd step: Lift to coho. correspondence of $\mathcal{V}_0$.

3rd step: Deduce estimate on p -adic absolute values of eigenvalues of Hecke operators on $H^i(\tilde{S}_K, \tilde{\mathcal{V}}_K)$.

Explanation: Let

$H_{\mathbb{Z}_p}^i \subset H^i(\tilde{S}_K, \tilde{V}_K) \otimes \bar{\mathbb{Q}}_p$ be $\bar{\mathbb{Z}}_p$ -lattice
stable under $\mathbb{Z}_p$ -algebra $\mathcal{H}$. Let

$$\chi: \mathcal{H} \rightarrow \bar{\mathbb{Q}}_p^*$$

eigenvalue occ. in H^i . Then χ
factors through $\bar{\mathbb{Z}}_p$ and hence

$$|\chi(h)| \leq 1, \quad \forall h \in \mathcal{H}.$$

Two kind of lattices:

$$\bullet H^i(\tilde{S}_K, \tilde{V}_K) \rightarrow H^i(\tilde{S}_K, \tilde{V}_K)$$

$$\bullet \text{ If } V = L \otimes O, \text{ then}$$

$$H^i(\tilde{S}_K, L) \rightarrow H^i(\tilde{S}_K, \tilde{V}_K).$$

(Vincent Lafforgue, Fakhruddin - Pilloni)

§2 A new basis of spherical Hecke algebra

Notation: $G/\mathbb{Q}_p$, K_0 , $A \subset T \subset B$

$\mathcal{H}(G, K_0)$ has minimal system of generators
 $\{T_\lambda \mid \lambda \text{ indecomposable in } X_*(A)^+\}$.

If λ adjoint, then λ indecomposable
 iff $\lambda = \omega_i^\vee$, fundamental coweight,
 and

$$\mathcal{H} = \mathbb{Z}_p[T_{\omega_1^\vee}, \dots, T_{\omega_m^\vee}] \text{ polynomial algebra}$$

Theorem: Fix Iwahori $I \subset K_0$. There
 ex another minimal system of generators
 $\{A_\lambda \mid \lambda \text{ as before}\}$

s.t. A_λ of form

$$z K_r \bar{z}^1 = K_0$$

$$A_\lambda = \mathbb{1}_{K_0 K_1} \cdot \mathbb{1}_{K_1 K_2} \cdots \mathbb{1}_{K_{r-1} K_r} \cdot T_{z(\lambda)}$$

where $K_0, K_1, \dots, K_r$ std. maximal parabol.

Furthermore, the base change matrix from

T_λ to A_λ is upper triangular.

If G classical gp, then $r=1$ or 2 .

Example: Let $G = Sp(V, \langle, \rangle)$. Fix maximal selfdual periodic lattice chain

$$\Lambda_{-m} \subset \Lambda_{-m+1} \subset \cdots \subset \Lambda_0 = \overset{\vee}{\Lambda}_0 \subset \Lambda_1 = \overset{\vee}{\Lambda}_{-1} \subset \cdots$$

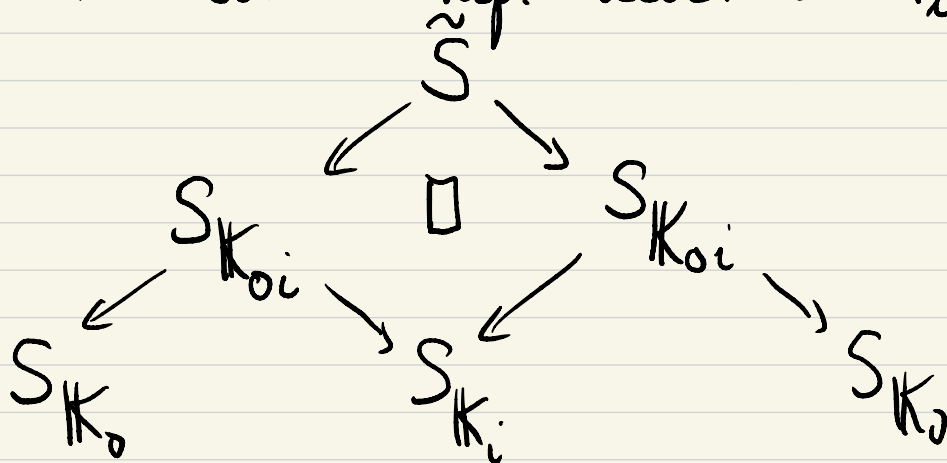
Then atomic basis given by

$$A_i = \mathbb{1}_{K_0 K_i} \cdot \mathbb{1}_{K_i K_0} \quad K_i = \text{Stab}(\lambda_i)$$

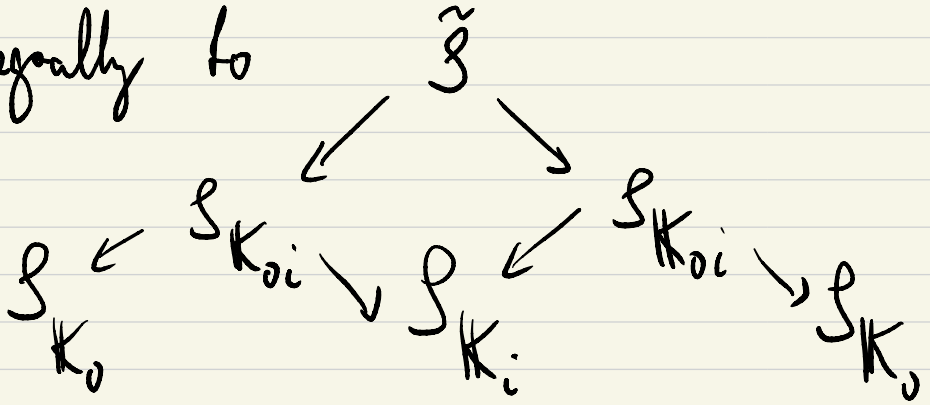
For adjoint group: the same, except
replace A_m by

$$A'_m = \mathbb{1}_{K_0 K_m} \cdot T_{\tau(m)}$$

($\omega_m^\vee$ minuscule fundamental coweight).
Geometric Hecke corresp. assoc. to A_i
is



Key point: this diagram extends
integrally to



§3 Cohomological integral Hecke corresp. ($X = \emptyset$).

Fact 1: Let $f: X \rightarrow Y$ morph.
of $\mathcal{O}$ -schemes of f.t. Assume X, Y
are smooth / $\mathcal{O}$ of same dimension. Then
natural map (fundamental class of f)
 $\mathcal{O}_X \rightarrow f^*(\mathcal{O}_Y)$.

Construction: exact sequence

$$\Omega_{Y/S}^1 \otimes_{\mathcal{O}_Y} \mathcal{O}_X \xrightarrow{df} \Omega_{X/S}^1 \rightarrow \Omega_{X/Y}^1 \rightarrow 0$$

Get $\det(df): \Omega_{Y/\mathcal{O}}^d \otimes_{\mathcal{O}_X} \mathcal{O}_X \rightarrow \Omega_{X/\mathcal{O}}^d$,

hence

$$\mathcal{O} \rightarrow \omega_{X/\mathcal{O}} \otimes \omega_{Y/\mathcal{O}}^{\otimes (-1)} \stackrel{!}{=} f^!(\mathcal{O}_Y)$$

(Grothendieck duality)

Remark: Extends to case where X, Y regular.

Fact 2: Let $f: X \rightarrow Y$ over $\mathcal{O}$.

Assume $U \subset X$ and $V \subset Y$ opens with

U, V regular of same dimension, with

$f(U) \subset V$, and $U \subset X$ large.

Also assume Y Cohen-Macaulay.

Then $\mathcal{O}_U \rightarrow f^!(\mathcal{O}_Y)|_U$ extends uniquely to $\mathcal{O}_X \rightarrow f^!(\mathcal{O}_Y)$.

(uses lemma of Bhett).

Can apply this to $S_{|K'} \rightarrow S_{|K}$:

- all integral models are CM
- let $U_K^{\text{extr}} \subset S_K$ $\bar{\mathcal{D}}_f$ complement of all non-extreme Kottwitz-Rapoport

strata in special fiber. Then

- U_K^{extr} smooth / $\mathcal{O}_E$ ($\stackrel{?}{=} \text{smooth locus?}$)
- $U_{K'}^{\text{extr}} \rightarrow U_K^{\text{extr}}$

Hence get $\mathcal{O}_{S_{K'}} \rightarrow f^!(\mathcal{O}_{S_K})$. □

Apply to atomic generator

$$A_\lambda = \mathbb{1}_{K_0 K_1} \cdot \mathbb{1}_{K_1 K_2} \cdot \dots \cdot \mathbb{1}_{K_{r-1} K_r} \cdot T_{\lambda(N)}$$

get cohom. correspondences

$$(S_{K_0}, \sigma) \rightarrow (S_{K_1}, \sigma) \rightarrow \dots \rightarrow (S_{K_r}, \sigma)$$

$$(S_{K_0}^{12}, \sigma)$$

Obtain linear operator

$$A_{\lambda *}: \mathcal{R}\Gamma(\mathcal{S}_{K_0}, \sigma) \rightarrow \mathcal{R}\Gamma(\mathcal{S}_{K_0}, \sigma)$$

as composition.

Conjecture: (i) A_{λ_*} only depends on λ
(not on atomic presentation).

(ii) These endomorphisms commute and
define $\mathcal{H}(G, K_0) \rightarrow \text{End}(\mathcal{R}\Gamma(S_{K_0}, \mathcal{O}))$

(iii) A_{λ_*} optimal, i.e. not divisible
by p .

Theorem: The composed cohomological
correspondence

$$(S_{K_0}, \mathcal{O}) \rightarrow (S_{K_0}, \mathcal{O})$$

is optimal.

§4 Non-trivial coefficients (A classical)

Let autom. vb. $\mathbb{V}$ corresp. to irreduc.

rep'n of $M = M_\mu$ over E : hence

have $\kappa \in X^*(T)^{+M}$. Consider

atomic generator $A_i = A_{\lambda_i} = \frac{1}{\lambda_i} \frac{1}{K_0 K_i} \frac{1}{K_i K_0} T_{\varepsilon_i}$

Then have "natural integral extensions"

$\mathcal{V}_0$ of $\mathbb{V}_{K_0}$ resp. $\mathcal{V}_i$ of $\mathbb{V}_{K_i}$

plus natural maps

$$\pi_0^*(\mathcal{V}_0) \rightarrow \pi_i^*(\mathcal{V}_i), \quad \pi_i^*(\mathcal{V}_i) \rightarrow \pi_0^*(\mathcal{V}_0)$$

Compose with fundamental class

$$\pi_i^*(\mathcal{V}_i) \rightarrow \pi_i^*(\mathcal{V}_i) \otimes \pi_i^!(\mathcal{O}) = \pi_i^!(\mathcal{V}_i)$$

$$\pi_0^*(\mathcal{V}_0) \rightarrow \pi_0^*(\mathcal{V}_0) \otimes \pi_0^!(\mathcal{O}) = \pi_0^!(\mathcal{V}_0)$$

get cohom. correspondences

$$(\mathcal{S}_{K_0}, \mathcal{V}_0) \rightarrow (\mathcal{S}_{K_i}, \mathcal{V}_i) \rightarrow (\mathcal{S}_{K_0}, \mathcal{V}_0)$$

Projected theorem: Consider composed coh.

correspondence $(\mathcal{S}_{K_0}, \mathcal{V}_0) \rightarrow (\mathcal{S}_{K_0}, \mathcal{V}_0)$

Let $\infty(k) = (k+Q)_{\mathbb{C}\text{-dom.}}$ infinit. char.

Set

$$e_i(k) = \langle \lambda_i, \infty(k) - Q \rangle + \langle \lambda_i, \beta(k) \rangle$$

Here $\beta(\kappa) = \text{pos. linear comb. of simple roots}$, hence $\text{error term} \geq 0$

Then above coho. corresp. is divisible
by $P^{e_i(\kappa)}$.

Note: If $K+Q$ regular, or if λ_i minuscule,
then $\beta(\kappa) = 0$ (last case leads to
conjecture of Fakhruddin-Pilloni).