Part 2 : Integral Hecke correspondences

§1 The problem

Let (G, X) Shimura datum, with reflex field $\mathbb{E} \subset \overline{\mathbb{Q}}$. Fix p and $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$.

Let $E = \mathbb{E}_\nu$, let $(S_{\mathbb{K}})/E$ be SV base changed to E. Let $G = G \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

Let $\mu \in \mathcal{M}_X(G)$ defined $/E$ and let

$M_\mu = \text{centr}(\mu) \subset G_E$.

For a rep'n (V, ρ) of M_μ over $E \rightsquigarrow$ (under mild cond. on center of G): automorph vb $\mathbb{V} = (\mathbb{V}_{\mathbb{K}})$ with identific.

$$\pi^{*}_{\mathbb{K}', \mathbb{K}}(\mathbb{V}_{\mathbb{K}}) = \mathbb{V}_{\mathbb{K}'}.$$

For $g \in G(\mathbb{A}_f)$ get <u>Hecke corresp.</u>

$$
\begin{tikzcd}
 & S_{\mathbb{K} \cap g\mathbb{K}g^{-1}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
S_{\mathbb{K}} & & S_{\mathbb{K}} ,
\end{tikzcd}
$$

and <u>cohom. corresp.</u>

$$\pi_1^*(\mathbb{V}_{\mathbb{K}}) \longrightarrow \pi_2^*(\mathbb{V}_{\mathbb{K}}) = \pi_2^!(\mathbb{V}_{\mathbb{K}}),$$

and hence <u>Hecke operator</u>

$$
\begin{tikzcd}
T_g : R\Gamma(S_{\mathbb{K}}, \mathbb{V}_{\mathbb{K}}) \arrow[r] & R\Gamma(S_{\mathbb{K}'}, \pi_1^*(\mathbb{V}_{\mathbb{K}})) \arrow[r] & R\Gamma(\pi_2^! \mathbb{V}_{\mathbb{K}}) \arrow[ll, bend left, "\text{adj.}"]
\end{tikzcd}
$$

<u>Remark</u>: If $S_{\mathbb{K}}$ proper, then finiteness. Otherwise replace by $R\Gamma(S_{\mathbb{K}}^{tor}, \mathbb{V}_{\mathbb{K}}^{tor})$.

<u>Goal</u>: Let $\mathbb{K} = K^p . K$, where $K \subset G(\mathbb{Q}_p)$ hs

Assume have "natural" model $\mathcal{S}_{\mathbb{K}} / \mathcal{O}_E$ (e.g. smooth). Also assume "natural" extension $\mathcal{V}_{\mathbb{K}}$.

Then want integral ext. of Hecke corresp./coho. corresp., to get action of $\mathcal{H}$ on $R\Gamma(\mathcal{S}_{\mathbb{K}}, \mathcal{V}_{\mathbb{K}})$. Should be optimal.

<u>Example</u>: $G = GL_2$, let $g = \begin{pmatrix} p & \\ & 1 \end{pmatrix}$. <u>Hecke corr.</u>

$$\begin{tikzcd} & \mathcal{S}_{\Gamma_0(p)} \arrow[dl] \arrow[dr] & \\ \mathcal{S} & & \mathcal{S} \end{tikzcd}$$

Then $\omega = \omega_{\mathcal{E}/\mathcal{S}}$ is autom. sh. <u>Coho. corr.</u>

$$\pi_2^*(\omega^{\otimes k}) \longrightarrow \pi_1^!(\omega^{\otimes k}). \quad (\text{sic!})$$

For $k \geq 2$, $R\Gamma(\mathcal{S}_{\mathbb{K}}^{tor}, \omega^{\otimes k}) = H^0(\mathcal{S}_{\mathbb{K}}^{tor}, \omega^{\otimes k}) =$

$M_k(\mathbb{K})$ space of modular forms.

Hecke operator T_g acts on q-expansion as

$$T\left(\sum a_n q^n\right) = \sum_{n\geq 0} \left(p\, a_{np}\, q^n + p^k a_n q^n\right)$$

To get optimal operator, divide by p.

§ 2 The main result in the Siegel case

"<u>Thm</u>": Let $(G, X) = (GSp_{2g}, X)$. Let (V, ϱ) irrep of M, of highest wt. κ ($M = GL_g \times G_m / \langle \pm 1 \rangle$).

Assume $\kappa + \varrho$ regular (for $g=1$, this excludes $k=1$ above). Let $\lambda \in X_*^+$, let $A_\lambda \in \mathcal{H}$ be "a" corresp. atomic element. Then

$p^{\langle \lambda, w_0(\kappa) - \varrho \rangle} \cdot A_\lambda$ "extends integrally" as cohom. corresp.

Notation: $\infty(\kappa) = (\kappa+\rho)_{\mathrm{dom}}$. infin. character

The exponent $\in \mathbb{Z}$, ≥ 0 if $Z(G)$ anisotropic

Remark: If $\lambda_{ad} = \omega_j^\vee$ (minuscule cocharacter), this is proved in [FP]; they conjectured this result (but without regularity condition)

Without regularity cond., the result fails in general – but we are looking for refinement.

"Corollary": Assume $\kappa+\rho$ regular. Then the $\mathcal{O}_E$-lattice

$$\mathrm{Im}\left(H^*(\mathcal{B}_{\mathbb{K}}, \mathcal{V}_{\mathbb{K}}) \to H^*(S_{\mathbb{K}}, \mathbb{V}_{\mathbb{K}})\right)$$

is invariant under all $p^{\langle \lambda, \infty(\kappa)-\rho \rangle} \cdot T_\lambda$.

In particular (V. Lafforgue), for any $\overline{\mathbb{Q}}_p$-char. $\chi : \mathcal{H} \to \overline{\mathbb{Q}}_p$ occ. in $H^*(S_{\mathbb{K}}, \mathbb{V}_{\mathbb{K}})$ have τ as general. eigenvalue.

estimate

$$\mathrm{val}(X) \leq \infty(\kappa) \quad \text{in} \quad X^{*+}.$$

Remark: This estimate also follows from [VL] + Faltings (BGG resolution).

The quotation marks mean: "desired result"

Explanation: Let $\lambda = \omega_r^\vee \in X_*^+$, with corresp. atomic elt.

$$A_\lambda = \mathbb{1}_{KK_r} \cdot \mathbb{1}_{K_rK}.$$

Form corresp. iterated Hecke corresp.

$$
\begin{tikzcd}[column sep=small]
 & & \tilde{\mathcal{S}} \arrow[dl] \arrow[dr] \arrow[dll, bend right, "\pi_1"'] \arrow[drr, bend left, "\pi_2"] & & \\
 & \mathcal{S}_{K_{0r}} \arrow[dl] \arrow[dr] & & \mathcal{S}_{K_{0r}} \arrow[dl] \arrow[dr] & \\
\mathcal{S}_{K_0} & & \mathcal{S}_{K_r} & & \mathcal{S}_K
\end{tikzcd}
$$

Let $m, n \gg 0$.

Over $\mathcal{S}_{K_r}$ have integral model $\mathcal{V}_r$ of $\mathbb{V}_{K_r}$ with

$$\pi_X^*(\mathcal{V}_0) \longrightarrow \pi_Y^*(\mathcal{V}_r), \quad \pi_Y^*(\mathcal{V}_r) \longrightarrow \pi_X^*(\mathcal{V}_0);$$

which induce in generic fiber $S_{K_{0r}}$

$$
\begin{array}{ccccc}
\pi_X^*(\mathbb{V}_{K_0}) & \xrightarrow{p^m} & \pi_Y^*(\mathbb{V}_{K_r}), & \pi_Y^*(\mathbb{V}_{K_r}) \xrightarrow{p^n} & \pi_X^!(\mathbb{V}_0), \\
\| & & \| & & \\
\mathbb{V}_{K_{0r}} & & \mathbb{V}_{K_{0r}} & \text{ditto} &
\end{array}
$$

Let $\mathcal{U}_{K_{0r}}$, resp. $\mathcal{U}_{K_r}$ ordinary locus: these are smooth $/ O_E$ + transition maps are finite flat. Hence trace map for finite flat maps define maps

$$\pi_y^*(\mathcal{V}_r)|_{\mathcal{U}_{0r}} \to \pi_y^!(\mathcal{V}_r)|_{\mathcal{U}_{0r}} \qquad \pi_y^*(\mathcal{V}_r)|_{\mathcal{U}_{0r}} \to \pi_y^!(\mathcal{V}_r)|_{\mathcal{U}}$$

Now $\mathcal{U}_{0r}$ large open in $\mathcal{S}_{K_{0r}}$. Hence

<u>Lemma</u>: There are unique extensions of these maps over $\mathcal{S}_{K_{0r}}$.

Proof based on e-mail of Bhatt: CM-complex in $D_{coh}(X)$.

Let $\tilde{\mathcal{U}} = \mathcal{U}_{0r} \times_{\mathcal{S}_r} \mathcal{U}_{0r} \subset \tilde{\mathcal{S}}$.

Propos.: Consider the composition over $\tilde{S}$,

$$d_{m,n} : \pi_1^*(\mathcal{V}_0) \longrightarrow \pi_2^!(\mathcal{V}_0).$$

Then ex!

$$d_{\tilde{U}} : \pi_1^*(\mathcal{V}_0)_{|\tilde{U}} \longrightarrow \pi_2^!(\mathcal{V}_0)_{|\tilde{U}}$$

s.t.

$$d_{m,n}{}_{|\tilde{U}} = p^{-\langle \lambda_r, \infty(K)-Q \rangle + m + n} \cdot d_{\tilde{U}}.$$

Furthermore, $d_{\tilde{U}}$ not divisible by p. (Optimality).

Still to show: The restriction map induces a map

$$\varrho : \mathrm{Hom}_{\tilde{\mathcal{S}}}(\pi_1^*(\mathcal{O}_0), \pi_2^!(\mathcal{O}_0)) \longrightarrow \mathrm{Hom}_{\tilde{\mathcal{U}}}(\quad " \quad)$$

which is injective, and also

$$\bar{\varrho} : \mathrm{Hom}_{\mathcal{O}_{\tilde{\mathcal{S}}}}(\pi_1^*(\mathcal{O}_0), \pi_2^!(\mathcal{O}_0))/p \rightarrow \mathrm{Hom}_{\tilde{\mathcal{U}}}(- " -)/p$$

is injective.

All this also works for unitary Sh.-var.

To do: Eliminate regularity hyp