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Talk: Toric varieties & integral models of Shimura varieties

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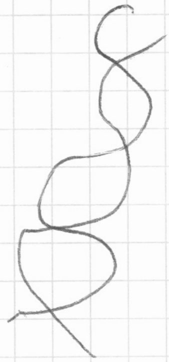
§. Motivation.

Fix p. Consider modular curve with $\Gamma_0(p)$-level over $\operatorname{Spec} \mathbb{Z}_p$, $\mathcal{S}_{\Gamma_0(p)}$.

Level $\Gamma_0(p)$ at p $= \left\{ \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p \right\}$. Represents foll. moduli pb. $= \{ (\mathcal{E}, G, \alpha) \mid (\mathcal{E}, \alpha)$ ell. with level α prime to p, $G \subset \mathcal{E}[p]$ gp scheme of order $p \}$.

Famous picture:



At ss $x \in \mathcal{S}_{\Gamma_0(p)}(\bar{\mathbb{F}})$, $\hat{\mathcal{O}}_x \simeq \breve{\mathbb{Z}}_p[[X,Y]]/(X \cdot Y - p)$. regular.

~~This extends to all Shim-var.~~ More canonical via LM.

Let $M_{\Gamma_0(p)} = bl_0(\mathbb{P}^1_{\mathbb{Z}_p})$. Have LMD

$$\begin{tikzcd} & \tilde{\mathcal{S}}_{\Gamma_0(p)} \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ \mathcal{S}_{\Gamma_0(p)} & & M_{\Gamma_0(p)} \end{tikzcd}$$

where π phs and φ smooth of same dim.

$\forall\, x \in \mathcal{S}(\bar{\mathbb{F}})$ ex. $y \in M(\bar{\mathbb{F}})$ s.t. $\hat{\mathcal{O}}_x \simeq \hat{\mathcal{O}}_y$.

Extends to all Shim.-var., all parahoric levels.

Want to look at smaller level, $\Gamma_1(p) = \{ \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod p \}$.

Represents foll. moduli pb. $\{ (\mathcal{E}, \alpha, G, P, P') \mid$

P generator of G, P' generator of $\mathcal{E}[p]/G \}$.

Then $\mathcal{S}_{\Gamma_1(p)} / \mathbb{Z}_p[\zeta_p]$ and

$* ss$: $\hat{\mathcal{O}}_{\mathcal{S}_{\Gamma_1(p)}, x} \cong \breve{\mathbb{Z}}_p[\zeta_p][[U,V]]/\cancel{U \cdot V} - \pi$

$X \mapsto U^{p-1}$, $Y \mapsto V^{p-1}$, $p = \pi^{p-1}$.

In particular, $\mathcal{S}_{\Gamma_1(p)} \to \mathcal{S}_{\Gamma_0(p)}$ finite + flat, normal

Would like to generalize this to all Shim.-var., e.g.

Siegel case and $\Gamma_1(p)$ = pro-unip. radical of $\Gamma_0(p)$ = Iwahori

Idea: Take roots of monomials $\rightsquigarrow$ toric varieties.

§ Our toric varieties

Situation: Let G = reductive alg. gp / $k = \bar{k}$, let $T \subset G$ maximal torus. Want torus var. for T having to do with G.

- DeConcini-Procesi: Let G adjoint. Then

$$X_*(T)_{\mathbb{R}} = \bigcup_{w \in W} w \bar{C},$$

gives complete fan $\rightsquigarrow$ toric variety Y ~~projective~~ complete

(numerous papers)

- Our variant: Let $\mu \in X_*(T)$. Form

"Whythoff's construction"

$$P_\mu = \mathrm{Conv}(W_0\mu) \qquad \text{"coweight polytope"},$$

$$\Sigma_\mu = \text{Cone over } P_\mu.$$

Picture

If G adjoint, get again complete torus compact.

never seen before (?)

But we want an affine toric variety.

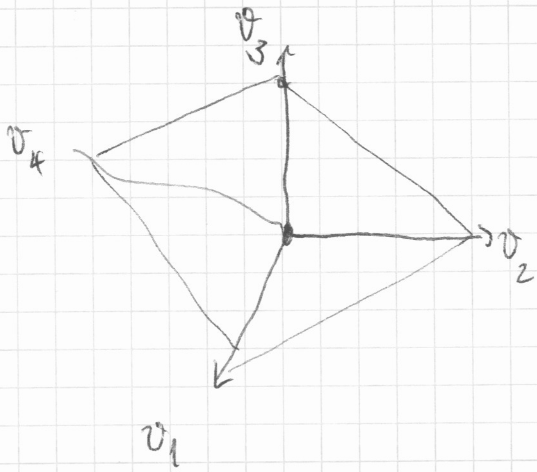
Observation: Consider $\mu_{ab} \in X_*(G_{ab})$. If this is non-torsion, then

$$\sigma_\mu = \sum_w \mathbb{R}_+ w\mu$$

is strictly convex cone, get $T \subset Y_\sigma$ affine torus emb.

Assumption satisfied when (G,μ) comes from Shimura G_{der} 1-conn., e.g. $G = GL_2$, GSp_{2n} etc.

Picture for $G = GSp_4$, $\mu = (1,1,0,0)$ minuscule (Fulton p. 17)



4 vertices

4 edges

1 full

§ Set-up of our theory.

Let $(\mathbf{G}, X)$ Shimura datum, let $G = \mathbf{G} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\{\mu\}$ of $G_{/E}$

For simplicity assume: G unr $G \otimes \breve{\mathbb{Q}}_p$ split, level

$\mathbb{K}^p \cdot K$, where $K = \text{Iwahori} = \mathcal{G}(\mathbb{Q}_p)$.

Get $\mathcal{G} \otimes \mathbb{F}_p \longrightarrow \bar{\mathcal{G}}^{red} \longrightarrow \bar{\mathcal{G}}^{red}_{ab} = \dot{T}$ torus over $\mathbb{F}_p$,

has unique lift over $\mathbb{Z}_p$ (induces a max. torus of G).

Have LMD of schemes over $\mathcal{O}_E$

$$\begin{tikzcd} & \tilde{\mathcal{S}}_{\mathbb{K}} \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ \mathcal{S}_{\mathbb{K}} & & M_\mu \end{tikzcd}$$

π phs under $\mathcal{G}_{\mathcal{O}_E}$, φ smooth map of $\mathcal{O}_E$-schemes.

In stack language,

$$\mathcal{S}_{\mathbb{K}} \longrightarrow [\mathcal{G}_{\mathcal{O}_E} \backslash M_\mu].$$

<u>Before stating the construction, 2 basic facts.</u>

1) <u>The stack</u> $[\mathbb{G}_m \backslash \mathbb{A}^1]$: section over $S \;\hat{=}$

$$\mathbb{G}_m - \begin{array}{c} L \longrightarrow \mathbb{A}^1 \\ \downarrow \\ S \end{array} = \{(\mathcal{L}, s) \mid s \in \Gamma(S, \mathcal{L})\}.$$

Note $\overset{\text{effective}}{\text{CarDiv}}(S) \longrightarrow [\mathbb{G}_m \backslash \mathbb{A}^1]$:

$$D \longmapsto (\mathcal{O}(D), \mathcal{O} \to \mathcal{O}(D)).$$

Structure of LM:

2) ${}^{\Gamma}M_\mu$ normal scheme, special fiber

GL_2

$$M_\mu \otimes \mathbb{F} = \sum_{\mu \in \{\mu\}_T} Z_\mu \qquad Z_\mu \text{ Weil divisors } (!).$$

Theorem: Ex!! $T_{\mathcal{O}_E}$-torsor $P_\mu \to M_\mu$, $\mathcal{G}_{\mathcal{O}_E}$-equiv.

plus G_E-equiv. section s_E over generic fiber s.t. foll. holds:

(*) $\forall \chi \in X^*(T)$ get via push-out $(\mathcal{L}_\chi, s_\chi)$ hence Cartier divisor D_χ on $M_\mu \otimes_{\mathcal{O}_E} \breve{\mathbb{Z}}_p$ with support in special fiber

Condition is:

$$D_\chi = \sum_\mu \langle \mu, \chi \rangle Z_\mu .$$

In particular, RHS is a Cartier divisor (!).

~~In stack language, get~~

$$\cancel{[\mathcal{G}_{\mathcal{O}_E} \backslash M_\mu] \longrightarrow} .$$

Consider generic fiber: $P_\mu \otimes_{\mathcal{O}_E} E = M_{\mu E} \times T_E \rightsquigarrow$

$$\sigma_E : P_\mu \otimes_{\mathcal{O}_E} E \longrightarrow T_E .$$

Proposition: This extends (uniquely) to $\mathcal{O}_E$-morphism

$$\sigma : P_\mu \longrightarrow Y_\sigma \qquad \sigma = \sigma_\mu$$

defines

$$\Delta : [\mathcal{G}_{\mathcal{O}_E} \backslash M_\mu] \longrightarrow [T_{\mathcal{O}_E} \backslash Y_\sigma]$$

Here make ass.: $\sigma \otimes \mathbb{R} = X_*(T)_{\mathbb{R}}$, μ_{ad} non-torsion.

§ The Lang map + the fundamental construction

Let T torus over $\mathbb{F}_p$. Then have surjective étale homomorph.

$$\mathcal{L}: T \to T, \quad x \mapsto \mathrm{Frob}(x) - x$$

On $X^*(T)$: $\lambda \mapsto p\sigma(\lambda) - \lambda$.

If T split (like for GL_2 or GSp_{2n}); $x \mapsto x^{p-1}$ (!).

Lifts over $\mathbb{Z}_p$, get by normalization

$$\begin{tikzcd} T_{O_E} \arrow[r, hook] \arrow[d, "\mathcal{L}"'] & \tilde{Y}_\sigma \arrow[d] \\ T_{O_E} \arrow[r, hook] & Y_\sigma \end{tikzcd}$$

Fundamental construction (in stacks terms)

$$\begin{tikzcd} \mathcal{M}^1_\mu \arrow[r, "\Delta^1"] \arrow[d] \arrow[dr, phantom, "\boxtimes"] & {[T_{O_E} \backslash \tilde{Y}_\sigma]} \arrow[d, "\mathcal{L}"] \\ {[\mathcal{G}_{O_E} \backslash M_\mu]} \arrow[r, "\Delta"] & {[T_{O_E} \backslash Y_\sigma]} \end{tikzcd}$$

Theorem: Consider Siegel case $G = GSp_{2n}$. There ex. integral model $S_{\Gamma_1(p)} \to S_{\Gamma_0(p)}$ with $T(\mathbb{F}_p)$-action extending $S_{\Gamma_1(p)} \to S_{\Gamma_0(p)}$ s.th. in generic fiber s.t.

(i) $S_{\Gamma_0(p)} = S_{\Gamma_1(p)} / T(\mathbb{F}_p)$

(ii) $[S_{\Gamma_1(p)} / T(\mathbb{F}_p)] \simeq \mathcal{M}^1_\mu$ $S_{\Gamma_0(p)} \times_{[\mathcal{G}_{O_E} \backslash M_\mu]} \mathcal{M}^1_\mu$ (ex. such an iso)

<u>Theorem</u>: Consider Siegel case $G = GSp_{2n}$. There exists integral model $\mathcal{S}_{\Gamma_1(p)} \to \mathcal{S}_{\Gamma_0(p)}$ with $T(\mathbb{F}_p)$-action, extending $S_{\Gamma_1(p)} \to S_{\Gamma_0(p)}$ in generic fiber s.t.

- $\mathcal{S}_{\Gamma_0(p)} = \mathcal{S}_{\Gamma_1(p)} / T(\mathbb{F}_p)$
- In cartesian diagram of stacks

$$\begin{array}{ccc} * & \longrightarrow & \mathfrak{M}^1_\mu \\ \downarrow & & \downarrow \\ \mathcal{S}_{\Gamma_0(p)} & \longrightarrow & [\mathcal{G}_{O_E} \backslash M_\mu] \end{array},$$

the corner can be identif. with $[\mathcal{S}_{\Gamma_1(p)} / T(\mathbb{F}_p)]$.

In fact, in this case, can give a moduli description of $\Gamma_1(p)$, via Oort-Tate theory of group schemes of order p and their generators: the <u>miracle</u> is that the equations are <u>exactly</u> given by the Lang map construction.

<u>Concluding remarks</u>: This works more generally, if Iwahori. If replace Iwahori by parahoric, <u>conjectural</u>.

For Iwahori case, model is (R1). But the map $\mathcal{S}_{\Gamma_1(p)} \to \mathcal{S}_{\Gamma_0(p)}$ is almost never flat (in fact "only" for unramif. unitary gp and $X = X_{(n-1,1)}$.

Non-stacky formulation.

From $M \longrightarrow [T_{\mathfrak{g}} \backslash Y_\sigma]$, get

$$P_\mu \longrightarrow Y_\sigma .$$

Form

$$\begin{tikzcd} \tilde{P}_\mu \arrow[r] \arrow[d] & \tilde{Y}_\sigma \arrow[d] \\ P_\mu \arrow[r] & Y_\sigma \end{tikzcd}$$

Then from Lang map get action of $T_{\mathfrak{g}}(\mathbb{F}_p)$ on $\tilde{P}_\mu$.

Thm. $\Rightarrow$ $\mathcal{S}_{\Gamma_1(p)}$ <u>is smoothly equiv.</u> to $\tilde{P}_\mu$, <u>compat. w.</u>
$T_{\mathfrak{g}}(\mathbb{F}_p)$-<u>action</u>

<u>But careful here:</u>

(*) $S_{\Gamma_1(p)}$ is (R1) – but not normal, the map to $S_{\Gamma_0(p)}$ is not flat. Probably not even flat / $\mathbb{Z}_p$, if $n > 1$

Here $S_{\Gamma_0(p)}$ given by moduli pb:

$\Gamma_0(p)$: $\{ (A, \lambda, \alpha, G_\bullet) \mid$

$$G_\bullet = 0 \subset G_1 \subset G_2 \ldots \subset G_{2n} = A[p]$$

s.t. λ gives iso [illegible]

$$gr_i(G_\bullet) = gr_{2n+1-i}(G_\bullet)^\vee$$

$\Gamma_1(p)$: add generator P_i of $gr_i(G_\bullet)$ s.t.

$\langle P_i, P_{2n+1-i} \rangle$ indep. of i $\in \mu_p$.

Monomial equations are given by Oort-Tate presentation (Miracle! Also works for GU [illegible] – but only when signature is $(1, n-1)$ get good (flat, normal) models.