PB, ~~Oct.~~ Sept. 24

Talk: AFL for whole Hecke algebra

First, about the name "FL" (Langlands): statement about spherical Hecke alg., arising from comparison of trace formulas. Many variants, here is one:

Notation: $p \neq 2$, F/F_0 unr. quadr.

Let $W_0 = F/F_0$-hermit. of dim. n, split

$$U_{W_0} \otimes_{F_0} F \simeq GL_n/F \rightsquigarrow {}^L U_{W_0} \longrightarrow {}^L GL_n$$

dually $\rightsquigarrow$

$$b : \mathcal{H}_{GL_n} \longrightarrow \mathcal{H}_{U_{W_0}}$$

closed, max. dim.

FL: Let $g \in U_{W_0}(F_0)_{rs}$. For $\varphi' \in \mathcal{H}_{GL_n}$, let $\varphi = b(\varphi')$. Then

$$SO_g(\varphi) = \begin{cases} SO_{\gamma\sigma}(\varphi') & \text{if } \gamma \in GL_n(F)_{\sigma\text{-}rs} \text{ matches } g \\ 0 & \text{if no such } \gamma. \end{cases}$$

Proof proceeds in 2 steps:

1.) For $\varphi' = \mathbb{1}$ (Kottwitz)

2.) general case via global methods.

Today: • replace TF by RTF, ← talk Raphael

• conjug. action of G on itself replaced by more complic.

• FL replaced by JR-FL.

for $\varphi' = \mathbb{1}$

Fix $n \geq 1$. Let

W_0 = split F/F_0 - hermit. of dim. $n+1$ (sic!)

$W_0^\flat = \langle u_0 \rangle^\perp$, where $\|u_0\| = 1$.

$G_{W_0} = U_{W_0^\flat} \times U_{W_0}$ (homog.)

$G' = GL_{n/F} \times GL_{n+1}/F$ over F_0.

Group actions:

$H = U_{W_0^\flat} \times U_{W_0^\flat}$ on G_{W_0} : obvious

$H' = GL_{n/F} \times (GL_n \times GL_{n+1}/F_0)$ on G' : less obvious.

⇝ $G_{W_0, rs}$ and G'_{rs}. (closed, of max. dim.)

Thm (FL): Let $\gamma \in G'_{rs}$. Then

$$O_\gamma(\mathbb{1}_{K'}) = \begin{cases} O_g(\mathbb{1}_K) & \text{if } g \in G_{W_0, rs} \text{ matches } \gamma \\ 0 & \text{if no such } g. \end{cases}$$

$\omega(\gamma) \cdot \int \varphi'(h'\gamma)\eta(h')dh'$

<u>matching</u>: in same orbit in $G'(F_0) = G_{W_0}(F)$ under H'.

Final proof by B-P ('21), resp. Z. Zhang: based on W. Zhang

<u>Now assume that γ in 2^{nd} case.</u> Let W_1 = non-split, u_1

<u>Fact</u>: If γ in 2^{nd} case, then ex. $g \in G_{W_1, rs}$ matches γ

<u>Thm (AFL)</u>: Let $\gamma \in G'_{rs}$ match $g \in G_{W_1, rs}$. Then

$$-\frac{1}{2}\,\partial O_{\gamma}(\mathbb{1}_{K'}) = \langle g\Delta, \Delta\rangle_{\mathcal{N}_{n,n+1}} \cdot \log q$$

Here LHS = special value of deriv. (introduce $s \in \mathbb{C}$).

<u>Explanation of RHS</u>: Let

$\mathcal{N}_n$ = formal moduli space of $(X, \iota, \lambda, \rho)$, where

X = formal O_F-module, $ht_{F_0}(X) = 2n$

ι : action of O_F s.t. Lie X has sgn $(1, n-1)$

λ : princ. polariz. compat. with ι

ρ : framing with framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ $/\overline{\kappa_F}$.

Then

$\mathcal{N}_n$ = formal scheme $/O_{\breve{F}}$, smooth of rel. dim. $n-1$.

<u>Example</u>: $n=1$. Then $\mathcal{N}_1 = \mathrm{Spf}\, O_{\breve F}$, with universal object $\mathcal{E}/O_{\breve F}$. $n=2$: $\mathcal{N}_2 = \mathrm{Spf}\, O_{\breve F}[[t]]$
$= LT_2$

Finally,

$$\mathcal{N}_{n,n+1} = \mathcal{N}_n \times \mathcal{N}_{n+1} \quad \text{regular of dim. } 2n,$$
w. action of G_{W_1}.

$\frac{1}{2}$ - dimension!

$$\Delta = \Gamma_\delta = \text{graph of } \delta : \mathcal{N}_n \hookrightarrow \mathcal{N}_{n+1}, \; X \mapsto X \times \bar{\mathcal{E}}.$$

$$\langle A, A' \rangle = \chi(\mathcal{N}_{n,n+1}, O_A \otimes^L O_{A'}).$$

<u>Global proof</u> by W. Zhang, M-Z, Z. Zhang

<u>Local</u>: $n=1$ W.Z.

<u>Problem</u>: What are the statements if unit elts replaced by arbitr. $\varphi' \in \mathcal{H}_{G'}$? $\quad b : \mathcal{H}_{G'} \to \mathcal{H}_{G_{W_0}}$

<u>Thm</u> (S. Leslie '23): Let $\gamma \in G'_{rs}$. Let $\varphi' \in \mathcal{H}_{G'}$ and $\varphi = b(\varphi')$.

$$O_\gamma(\varphi') = \begin{cases} O_g(\varphi) & \text{if } \gamma \text{ matches } g \in G_{W_0, rs} \\ 0 & \text{if no.} \end{cases}$$

uses global methods.

<u>Joint w. Chao Li, Wei Zhang</u>: AFL.

<u>Conj. (AFL)</u>: Let $\gamma \in G'_{rs}$ match $g \in G_{W_1, rs}$. Then

$$-\frac{1}{2}\, \partial O_\gamma(\varphi') = \langle g\Delta, T_\varphi(\Delta) \rangle_{\mathcal{N}_{n,n+1}} \cdot \log q .$$

<u>Thm (LRZ)</u>: Conjecture holds for $n = 1$.

Proof is local, uses theory quasi-canon. divisors on

Then Δ divisor! $\mathcal{N}_{1,2} = \mathcal{N}_2$: imitates local proof of AFL (W. Zhang)

came up ATC

<u>Main issue</u>: How to define Hecke correspondence, i.e.

$\mathbb{Z}$-alg. homo

$$\mathcal{H}_{G_{W_0}} \longrightarrow \mathrm{Corr}(\mathcal{N}_{n,n+1}).$$

<u>Explain this on example of Shimura varieties:</u>

Consider $Sh(G,X)_{\mathbb{K}} = S_{\mathbb{K}}$ over E.

Lots of correspondences, assoc. to elts $g \in G(\mathbb{A}_f)$:

$$(*) \quad \begin{tikzcd} & S_{\mathbb{K} \cap g\mathbb{K}g^{-1}} \arrow[dl] \arrow[dr, "R_g"] & \\ S_{\mathbb{K}} & & S_{\mathbb{K}} . \end{tikzcd}$$

$E = E_\nu$

Now fix p and $\nu | p$. Assume $\mathbb{K} = \mathbb{K}^p \cdot K$,

where $K \subset G(\mathbb{Q}_p)$ parahoric.

Then (in great generality) have integral models $\mathcal{S}_{\mathbb{K}}$ that are "natural".

<u>Problem</u>: Find diagram

$$\begin{tikzcd} & \mathcal{T} \arrow[dl] \arrow[dr] & \\ \mathcal{S}_{\mathbb{K}_0} & & \mathcal{S}_{\mathbb{K}_0} \end{tikzcd}$$

K_0 hs.

with generic fiber $(*)$.

<u>Obstacle</u>: $K_0 \cap g K_0 g^{-1}$ not parahoric in general.

<u>Idea</u>: construct $\mathcal{T}$ as an iterate of natural corresp. between $\mathcal{S}_{\mathbb{K}}$ for parahoric $\mathbb{K}$.

This indeed works in general (jt. w. U. Görtz, X. He).

$\mathcal{H}_{K_0} = \langle T_\lambda \rangle$ Let G/F unramified, with $K_0 \subset G(F)$ hs.

<u>Def</u>: Let $\lambda \in X_*(A)^+$. An <u>atomic elt. for λ</u> $\underset{Df}{=}$ is an elt. of $\mathcal{H}_{K_0}$ of the form

max
∨
Standard parah.
(Iwahori)

$$A = \mathbb{1}_{K_0 K_1} \cdot \mathbb{1}_{K_1 K_2} \cdot \ldots \cdot \mathbb{1}_{K_{r-1} K_r} \cdot T_\tau ,$$

where $\tau = \tau(\lambda) \in \pi_1(G)_p$, $\tau^{-1} K_r \tau = K_0$.

To atomic elt. get integral correspondence by iteration:

$$\begin{tikzcd}[column sep=small]
 & \mathcal{S}_{K_0\cap K_1} \arrow[dl] \arrow[dr] & & \arrow[dl] & \cdots & \arrow[dr] & \mathcal{S}_{K_{r-1}\cap K_r} \arrow[dl] \arrow[dr] & \\
\mathcal{S}_{K_0} & & \mathcal{S}_{K_1} & & \cdots & & & \mathcal{S}_{K_r} \simeq \mathcal{S}_{K_0}
\end{tikzcd}$$

<u>Theorem:</u> (He) Assume $\mathcal{H}_{K_0}$ of polynomial type, i.e.

$$\mathcal{H}_{K_0} = \mathbb{Z}\left[T_{\lambda_i} \mid \lambda_i \in X_*(A)^+ \text{ indec.}\right].$$

Then $\forall\, \lambda_i$ ex. atomic elt. A_{λ_i} for λ_i s.t.

$$\mathcal{H}_{K_0} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathbb{Z}_p\left[A_{\lambda_i} \mid \dots \right].$$

<u>For AFL</u> however use $\mathcal{H}(U_{W_0}) = \mathbb{Z}\left[T_{\lambda_1}, \dots, T_{\lambda_m}\right]$,
n $\qquad m = \left[\frac{n}{2}\right]$

where $\lambda_i = \left(1^{(i)}, 0^{(n-2i)}, (-1)^{(i)}\right)$.

Std. chain

Let $K_i = \mathrm{Stab}$ (vertex lattice of type $2i$),

let $A_i = \mathbb{1}_{K_0 K_i} \cdot \mathbb{1}_{K_i K_0}$. (note $\pi_1(G)_\Gamma = \mathbb{Z}$

<u>Lemma:</u> $A_1, \dots, A_m$ is a polynomial basis of $\mathcal{H}$ and in fact the elts $A_1, \dots, A_m$ are linear comb. of $T_1, \dots, T_m$ by unipotent upper triang. matrix.

<u>Concluding remarks:</u>

- We defined a correspondence for every atomic elt. But we don't know about relations, e.g. commutativity (similar pb. in [FP]). $\to$ [YL]

- For except. gps, A_{λ_i} not always linear combination (i.e. $X' = X$, $Y' = Y + X^2$). Also $r \geqslant 2$ possible.

- I explained pb. for Shim-var, resp. local Shim-var. But there are also other contexts (affine flag var., local models).