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Talk: Tame $\mathcal{G}$-bdles on curves

jt. G. Pappas.

Not.: k perfect field, X curve over k = smooth, proj., geo. conn.

$X \rightsquigarrow K = k(X)$, $x \rightsquigarrow \mathcal{O}_x = \hat{\mathcal{O}}_{X,x}$, K_x

§1 $\overset{\text{relat.}}{\text{BT}}$ $\overset{\text{its}}{\text{gp}}$ schemes

Def.: A BT gp scheme over X $\underset{\text{Df}}{=}$ smooth gp scheme $\mathcal{G}/X$

s.t.

- $G = \mathcal{G} \times_X \operatorname{Spec} K$ conn. red. $/K$
- $\forall x$: $\mathcal{G}_x = \mathcal{G} \times_X \operatorname{Spec} \mathcal{O}_x$ is a quasi-parahoric $/\mathcal{O}_x$

A BT-gp scheme parahoric $\underset{\text{Df}}{=}$ $\mathcal{G}_x$ parah., $\forall x$.

Explanation: BT assoc. to gp scheme G_x / local field K_x the (extended) BT-building $\mathcal{B}(G_x, K_x)$, to any point $v \in \mathcal{B}$ they assoc. conn. smooth gp scheme $\mathcal{G}_v$ s.t. $\mathcal{G}_v(\mathcal{O}_x) \underset{\cancel{\text{finite index}}}{\subseteq} \operatorname{Stab}_{G_x(K_x)}(v)$. parahoric to v.

finite index

Quasi-parahoric = conn. comp. is parahoric.

Note: If BT has G ss-sc, then autom. parahoric

<u>Examples of BT :</u>

- Any reductive gp scheme $/X$ is BT (parahoric), e.g. $\mathcal{G} = GL(V)$: hence no finite classific.
- Let $\pi : X' \longrightarrow X$ tame Galois cov., i.e. $p \nmid |\Gamma|$. Let $\mathcal{G}'$ BT over X' + semi-linear Γ-action $\rightsquigarrow$

$$\mathcal{G} = \mathrm{Res}_{X'/X}(\mathcal{G}')^{\Gamma} \qquad \text{BT over } X.$$

- Conversely, let $\mathcal{G}$ parahoric BT over X. Get parahoric BT $\mathcal{G}'$ over X', charact. by
  - let $B \subset X$ branch. Then

$$\mathcal{G}' | X' \setminus \pi^{-1}(B) = \mathcal{G} \times_X (X' \setminus \pi^{-1}(B))$$

  - for $x' \mapsto x$, $\mathcal{G}'_{x'}$ corresp. to

$$v_{\mathcal{G}_x} \in \mathcal{B}(G, K_x) \subset \mathcal{B}(G', K'_{x'}).$$

In ss-sc case, both constr. are inverse to each other.

" <u>Γ-related</u> "

- Let $\mathcal{G}$ parah. BT, let $x \in X$ and $P_x \subset \bar{\mathcal{G}}_x$ psgp $\rightsquigarrow$ $\mathcal{G}' \to \mathcal{G}$ iso outside x and $\mathrm{im}(\bar{\mathcal{G}}'_x \to \bar{\mathcal{G}}_x) = P_x$.

<u>Thm 1</u>: Let $\mathcal{G}/X$ BT s.t. G abs. simple + sc. Assume $p \neq 2$ if G class., $p \neq 2, 3$ if G not E_8, $p \neq 2, 3, 5$.
Then, after finite ext. of base field k, ex tame $X' \to X$ and reductive gp scheme $\mathcal{G}'$ s.t. $\mathcal{G}+\mathcal{G}'$ are Γ-related.

Proof based on (M. Larsen, P. Gille, Serre)

<u>Prop.</u>: Let G ss-sc over local field F, let $\mathcal{G}/\mathcal{O}_F$ parahoric. Then ex. F'/F s.t. $G' = G \otimes_F F'$ split and Chevalley form $\mathcal{G}'/\mathcal{O}_{F'}$ s.t.

$$\mathcal{G}(\mathcal{O}_F) = G(F) \cap \mathcal{G}'(\mathcal{O}_{F'}).$$

If G tamely ramif. + exclude small primes as above, then can take F'/F tamely ramif.

subdivide apt.

§ 2 $\mathcal{G}$-bundles

<u>Def.</u>: A <u>$\mathcal{G}$-bdle</u> $/X \underset{\text{df}}{=} \mathcal{G}$-torsor over X.

<u>Examples</u>: • Let $\mathcal{G}$ reductive s.t. G <u>constant</u>, i.e. $G = G_0 \otimes_k K$. Assume that $\mathcal{G}$ <u>strong inner form</u> of G_0, i.e. ex G_0-torsor $\mathcal{P}_0$ s.t.

$$\mathcal{G} = \underline{\mathrm{Aut}}(\mathcal{P}_0)$$

Then equiv.

$$\{ \mathcal{G}\text{-torsors}/X \} \longleftrightarrow \{ G_0\text{-torsors} \}.$$

e.g.

$GL(V)$-bdles $=$ GL_n-bdles $=$ v.b. of rank n.

- Can interpret Primvariety in terms of $\widehat{\mathrm{Res}}^{\text{ker Norm}}_{X'/X}(\mathbb{G}_m)$!
- Classical lit.: parabolic bdles – but without weights.

__Def__ (Balaji–Seshadri): Let $\pi : X' \longrightarrow X$ tame and $\mathcal{G}'$ and $\mathcal{G}$ parahoric BT which are Γ-related.

A $(\mathcal{G}', \Gamma)$-__bdle on X'__ $\underset{\text{Df.}}{=}$ $\mathcal{G}'$-bdle on X' + semi-linear Γ-action (as torsor).

__Equivalent definition__: Let $\mathcal{X} = [X'/\Gamma]$ stack, with gp scheme $\mathcal{G}_{\mathcal{X}}$ over it. Then

$$(\mathcal{G}', \Gamma)\text{-bdle on } X' = \mathcal{G}_{\mathcal{X}}\text{-bdle on } \mathcal{X}.$$

$H^1(X', \Gamma; \mathcal{G}') = H^1(\mathcal{X}, \mathcal{G}_{\mathcal{X}})$

__Example__: Have fully faithful

$$\{\mathcal{G}\text{-bdles on } X\} \longrightarrow \{(\mathcal{G}', \Gamma)\text{-bdles on } X'\}.$$

Z: not equiv. if $X' \to X$ ramified. E.g.

$\mathrm{Res}_{X'/X}(\mathcal{P}')^{\Gamma}$ __not__ $\mathcal{G}$-bdle

(Damiolini)

Consider $q : \mathcal{X} \to X$. Leray-SSqu. gives

<u>Prop.</u>: Assume $k = \bar{k}$. Exact sequence of pointed sets

$$0 \to H^1(X, \mathcal{G}) \to H^1(\mathcal{X}, \mathcal{G}_{\mathcal{X}}) \xrightarrow{\tau} T \; ,$$

where $T = \prod_{x \in B} T_x$ with

$$T_x = H^1(\Gamma_{x'}, \bar{\mathcal{G}}_x^{\,red}(k)).$$

Here $\bar{\mathcal{G}}_x$ = special fiber of $\mathcal{G}_x$, $\bar{\mathcal{G}}_x^{\,red}$ = max. reductive quot.

<u>Thm 2</u>: Assume G ss-sc.

a) The map τ is surjective,

$$\tau : H^1(\mathcal{X}, \mathcal{G}_{\mathcal{X}}) \longrightarrow T$$

<u>locally type map</u>

b) Fiber

$$\tau^{-1}(t) = H^1(X, \mathcal{G}_t) \, ;$$

where $\mathcal{G}_t$ is ~~p~~ BT with generic fiber G, with $v_{\mathcal{G}_t, x} \in \mathcal{B}(G, K_x) \subset \mathcal{B}(G', K'_{x'})$ in same $G(K'_{x'})$-orbit as $v_{\mathcal{G}_x}$, $\forall x \in B$.

§3 Families

Consider $\mathrm{Bun}_{\mathcal{G}}$ and $\mathrm{Bun}_{(\mathcal{G}', \Gamma)}$: smooth algebraic stacks (Heinloth).

Thm 3: Same sit. as in Thm 2, $X' \to X$, $\mathcal{G}$, $\mathcal{G}'$.

a) The local type map
$$\tau : \mathrm{Bun}_{(\mathcal{G}', \Gamma)}(k) \longrightarrow T$$
is locally constant.

b) After choice of section of τ, canon. disjoint
$$\mathrm{Bun}_{(\mathcal{G}', \Gamma)} = \coprod_{t \in T} \mathrm{Bun}_{\mathcal{G}_t} .$$
~~Thus~~ Each summand connected (Heinloth).

Remark: Proof of a) based on Scholze argument used in our proof of Anschütz purity for tame gps.

Spec $W(\mathcal{O}_C)$ $\qquad C/\mathbb{F}_p$ non-arch. field.

Explanation of my abstract

Weil (1938), "Génér. des fcts. abéliennes":

says that 19^{th} cent. studies only line bdles on X (Abel, Jacobi, Riemann, Picard, Albanese). Comes down to: so far, only $\pi_1(X)^{ab}$, e.g.

$$\mathrm{Hom}(\pi_1(X), \mathbb{C}^\times).$$

Suggests: replace $\mathbb{C}^\times$ by any complex Lie gp.

But: what are corresp. geom. objects? "matrix divisor".

Groth. (1956): Sur le mémoire d'A. Weil "...."

Considers complex var. $X = X'/\Gamma$, G cx. Lie gp.

G-bdle on X $\rightsquigarrow$ Γ-equiv. G-bdle on X'

- $H^1(X', \Gamma; G)$
- local description of $H^1(\Gamma_{x'}, G)$ if $\dim X = 1$.
- if Γ acts freely, can describe (G, Γ)-bdles as G-bdles. Asks: how to describe (G,Γ)-bdles in terms of objects on X when there is ramific

Answer: in terms of BT gp schemes $\mathcal{G}$ on X and $\mathcal{G}$-bdles.

Main open question: Recall Verlinde formula, gives

$\dim H^0(\mathrm{Bun}_G, \mathcal{L})$ (Faltings + others)

$\mathcal{L}$ positive line bdle (natural upto power)

Aim: generalize this explicitly to $\mathrm{Bun}_{\mathcal{G}}$.

(Mukhopadhay, Hong - Kumar)