Darmstadt
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Talk: On the quasi-canonical AFL

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SK — WZ

TY — MR

jt. with CL + WZ.

§1 FL + AFL

Not.: $p \neq 2$, F/F_0 unram., $n \geq 1$.

$$G' = \mathrm{Res}_{F/F_0}(GL_n \times GL_{n+1}).$$

For $W = F/F_0$-hermit., $\dim W = n+1$, $u \in W$ non-isot.

$\rightsquigarrow W^\flat$, $G_W = U(W^\flat) \times U(W)$.

Fix basis of $W^\flat$ and add u: $W^\flat \simeq F^n$, $W \simeq F^{n+1}$

$$G_W \otimes_{F_0} F = GL_n \times GL_{n+1}.$$

$[G'_{rs}], [G_{W,rs}]$

Upto iso have W_0 and W_1. If $\|u_0\| = \|u_1\|$, have

matching $[G'_{rs}] = [G_{W_0,rs}] \sqcup [G_{W_1,rs}]$ orbit corr.

Dual to this transfer matching

$\varphi \in C_c^\infty(G')$ transfers to $(f_0, f_1) \in C_c^\infty(G_{W_0}) \times C_c^\infty(G_{W_1})$

$$\omega(\gamma)\cdot S_\varphi \rightsquigarrow \mathrm{Orb}_\gamma(\varphi) = \begin{cases} \mathrm{Orb}_g(f_0) & \text{if } \gamma \leftrightarrow g \in G_{W_0} \\ \mathrm{Orb}_g(f_1) & \quad\quad\quad\quad \in G_{W_1}. \end{cases}$$

Thm (FL): Assume $\|u_0\| = \|u_1\| = 1$. Then

$$\mathbb{1}_{K'} \text{ transfers to } (\mathbb{1}_{K_\bullet}, 0).$$

Here: $u_0 \in \Lambda_0 \subset W_0$ and

$$K_\bullet = \mathrm{Stab}_{G_{W_0}}(\Lambda_0^\flat, \Lambda_0), \quad K' = \mathrm{Stab}_{G'}(\Lambda_0^\flat, \Lambda_0).$$

Yun, WZ, B-P, ZZ.

Thm (AFL): Assume $\|u_0\| = \|u_1\| = 1$. For $\gamma \leftrightarrow g \in G_{W_1}$,

$$-\langle Z(u_1), g\, Z(u_1)\rangle_{Z(u_1)\times \mathcal{N}_{n+1}} = \frac{1}{2}\,\partial \mathrm{Orb}_\gamma(\mathbb{1}_{K'}).$$

WZ, AM-WZ, ZZ.

Explanation of LHS: Let $\mathcal{N}_n$ = RZ-space of $\{(X, \iota, \lambda, \varrho)\}$ where

- X formal O_{F_0}-module of ht $2n$, dim n
- $\iota : O_F \to \mathrm{End}(X)$ of sign. $(1, n-1)$
- λ princip. O_{F_0}-polariz., comp. with ι
- $\varrho : X \dashrightarrow \mathbb{X}_n$ framing, where $\mathbb{X}_n$ basic.

Then: $\mathcal{N}_n$ formal scheme, loc. f.f.t $/ O_{\breve{F}}$, f. smooth of dim. $n-1$ (relative).

Apply this to $n+1$ instead of n. Then

$$\mathrm{Aut}(\mathbb{X}_{n+1}) = U(W_1). \implies U(W_1) \text{ acts on } \mathcal{N}_{n+1}$$

For $0 \neq u_1 \in W_1$ $\leadsto$ KR divisor $\mathcal{Z}(u_1) \subset \mathcal{N}_{n+1}$.

If $\|u_1\| = 1$, then $\mathcal{Z}(u_1)$ f. smooth of relat. dim $n-1$. Hence

$\mathcal{Z}(u_1) \times \mathcal{N}_{n+1}$ regular and action of G_{W_1} and

$$\langle \mathcal{Z}(u_1), g\,\mathcal{Z}(u_1)\rangle_{\mathcal{Z}(u_1)\times\mathcal{N}_{n+1}} = \chi\left(\mathcal{Z}(u_1)\times\mathcal{N}_{n+1},\ \mathcal{Z}(u_1) \cap^{\mathbb{L}} g\,\mathcal{Z}(u_1)\right) \cdot \log q$$

defined + finite for $g \in G_{W_1, rs}$.

§2 Quasi-canonical FL / AFL

Now let $\|u_0\| = \|u_1\| = \|\pi\|$. Let almost selfdual

$$K_{n+1} = \mathrm{Stab}_{U(W_0)}(\Lambda_0), \quad K_n^{[1]} = \mathrm{Stab}_{U(W_0^\flat)}(\Lambda_0^\flat),$$

$$\tilde{K}_n^{[1]} = \mathrm{Stab}(\Lambda_0^\flat) \cap \mathrm{Stab}(\Lambda_0) \triangleleft K_n^{[1]}$$

Thm (qc FL): Let

$$\varphi = \left(q^{2(n+1)} - 1\right) \cdot \mathbb{1}_{\tilde{K}'^{[1]} \times K'_{n+1}} + \frac{1}{c_1'}\left((-1)^{n+1} + 1\right) \cdot \mathbb{1}_{K'_n \times K'^{+}_{n+1}} \in C_c^\infty(G').$$

Then φ transfers to $\left(\mathbb{1}_{\tilde{K}_n^{[1]} \times K_{n+1}}, 0\right)$.

Here $K_n'^{[1]}, K'_{n+1}, \tilde{K}_n'^{[1]}$ analogous to previous gps.

In addition have $K'^{+}_{n+1} = \mathrm{Stab}(\Lambda_0^\flat \oplus \langle u_0 \rangle)$.

$c_1' \hat{=}$ Haar measure.

Prop.: Let $\|u_1\| = |\pi|$. Then $Z(u_1)$ regular formal scheme, f. smooth outside closed subsch. of dim. 0.

keep!

Thm (gc AFL): If $\gamma \leftrightarrow g \in G_{W_1}$, then

$$-\langle Z(u_1), g Z(u_1) \rangle_{Z(u_1) \times \mathcal{N}_{n+1}} = \frac{1}{2} \partial \mathrm{Orb}_\gamma(\varphi) + \mathrm{Orb}_\gamma(\varphi_{corr})$$

$$\varphi_{corr} = \begin{cases} \frac{n+1}{c_1'} \mathbb{1}_{K_n' \times K_{n+1}'^{+}} \cdot \log q & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Proofs by reduction to FL/AFL.

§3 AT.

Consider $\mathcal{N}_n^{[1]} = \{(X, \iota, \lambda, \rho)\}$ where $\mathrm{Ker}\,\lambda \subset X[\pi]$ of order q^2.

$$\tilde{\mathcal{N}}_n^{[1]} = \{ (Y, X) \in \mathcal{N}_n^{[1]} \times \mathcal{N}_{n+1} \mid \mathbb{Y} \times \mathbb{E} \to \mathbb{X} \text{ extends to } Y \times \bar{\mathcal{E}} \to X \text{ isogeny} \}$$

Diagram (KR 2012)

$$\mathcal{N}_n^{[1]} \xleftarrow{\pi_1} \tilde{\mathcal{N}}_n^{[1]} \xrightarrow{\pi_2} \mathcal{N}_{n+1}.$$

Note: Top is integral model of member of RZ-tower corresp. to $\tilde{K}_n^{[1]} \triangleleft K_n^{[1]}$ explains $\tilde{K}_n^{[1]}$.

Prop.: $\tilde{\mathcal{N}}_n^{[1]}$ regular formal scheme, action of G_{W_1}

Thm (AT): $-\langle \hat{\mathcal{N}}_n^{[1]}, g\,\tilde{\mathcal{N}}_n^{[1]} \rangle_{\tilde{\mathcal{N}}_n^{[1]} \times \mathcal{N}_{n+1}}$ = above RHS.

If $\gamma \leftrightarrow g \in G_{W_1}$

Proof: π_2 factors through $Z(u_1)$, iso over smooth locus.

Now apply projection formula