What is ... an RZ-Space?

~~1970's~~ RZ-space = formal scheme / $\mathrm{Spf}(O_{\breve{E}})$.

<u>Formal scheme</u> $\mathfrak{X}$ = certain ind-schemes $X_0 \subset X_1 \subset X_2 \subset \dots$ all infinit thickg of reduced scheme

$(X_0, \mathcal{O})$, X_0 = reduced scheme + sheaf of local rings w. I-adic top. s.t. $\mathcal{O}/\mathcal{I} = \mathcal{O}_{X_0}$.

<u>Example</u>: Let Y scheme, $X_0 \subset Y$ closed $\rightsquigarrow$ $Y_{/X_0} = \mathfrak{X}$.

Let $\mathrm{Nilp}_{O_{\breve{E}}}$ = $O_{\breve{E}}$-scheme s.t. $p\mathcal{O}$ loc. nilpot. ideal

<u>Yoneda</u>: (Formal schemes / $\mathrm{Spf}\, O_{\breve{E}}$) $\hookrightarrow$ Funct$(\mathrm{Nilp}_{O_{\breve{E}}}, \mathrm{Sets})$.

<u>Theorem (Drinfeld 1976)</u>: Let $D_{\frac{1}{n}}$ w. inv $= 1/n$. Consider foll. functor on $\mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$: $S \mapsto \{$iso-cl. of (X, ι, ϱ) where

- X/S p-div. gp of ht n^2, dim $= n$
- $\iota: O_D \longrightarrow \mathrm{End}(X)$
- $\varrho: X \times_S \bar{S} \dashrightarrow \mathbb{X} \times_{\mathrm{Spec}\, \bar{\mathbb{F}}_p} \bar{S}$ / quis ht $= 0$

s.t. action of ι on Lie X is special. $\}$

Here $\mathbb{X}$ fixed (unique up to O_D-lin. quis)

Then this functor representable by $\hat{\Omega}^n_{\mathbb{Q}_p}$.

Here $\hat{\Omega}^n_{\mathbb{Q}_p}$ = formal scheme Deligne-Drinfeld-Mumford. Properties:

- $\left(\hat{\Omega}^n_{\mathbb{Q}_p}\right)^{rig} = \mathbb{P}^{n-1}_{\mathbb{Q}_p} \setminus \bigcup_{H/\mathbb{Q}_p} H$
- $\hat{\Omega}^n$ p-adic formal scheme. action of $PGL_n(\mathbb{Q}_p)$ extends
- $\hat{\Omega}^n$ regular formal scheme
- special fiber reduced + dual graph = $\mathcal{B}(PGL_n, \mathbb{Q}_p)$; each irred. comp. success. bl-up of $\mathbb{P}^{n-1}_{\mathbb{F}_p}$

~~1976~~ Also variant: replace $\mathbb{Z}_p$ by O_E.

Tried to vary this: diff. invariant, diff. special cond. One aim: find formal scheme $/\breve{\mathbb{Z}}_p$ s.t. generic fiber = $Gr_{\mathbb{Q}_p} \setminus \bigcup$ Schubert var.

Could not make this work! Finally turned around: instead of solving a pb, want to formulate a pb.

Thm (Rapo+Zink): Consider foll. functor on $Nilp_{\breve{\mathbb{Z}}_p}$: $S \mapsto \{$ iso-cl. of (X, ϱ) where

- X/S p-divis. gp.

- $\varrho : X \times_S \bar{S} \dashrightarrow \mathbb{X} \times_{\mathrm{Spec}\, \bar{\mathbb{F}}_p} \bar{S}$ quis

Here $\mathbb{X} / \bar{\mathbb{F}}_p$ fixed p-div. gp.

This functor is repres. by formal scheme, loc. form. of finite type/$\breve{\mathbb{Z}}_p$

local rings $\mathbb{Z}_p[[T_1, \dots, T_r]]\langle\langle S_1, \dots, S_t\rangle\rangle / I$.

Variants à la Drinfeld w. action of O_p + polariz.

Fix n and $n = p+q$, $p > 0, q > 0$

Example: Let $K/\mathbb{Q}_p$ quadr. ext. Functor $S \mapsto$ on $\mathrm{Nilp}_{O_{\breve{K}}}$

$(X, \iota, \lambda, \varrho)$ where

- X = p-div. gp ht $= 2n$, dim $= n$
- $\iota : O_K \longrightarrow \mathrm{End}(X)$ s.t.

$$\mathrm{char}(\iota(a) \mid \mathrm{Lie}\, X) = (T - a)^p \cdot (T - \bar{a})^q, \quad \forall a \in O_K$$

- $\lambda : X \longrightarrow X^\vee$ anti-sym. s.t. $\mathrm{Ros}_{\lambda} | O_K = a \mapsto \bar{a}$.

  $\mathrm{Ker}\, \lambda \subset X[p]$ fixed rank.

Then if $K/\mathbb{Q}_p$ unramif., this is repr. by formal scheme flat / $O_{\breve{K}}$

+ ~~formally regular~~ formally smooth if $p = 1$ or $q = 1$

Görtz

Global motivation

Drinfeld: Let $\mathbb{B}$ = indef. quat.-alg. $/\mathbb{Q}$ s.t. $\mathbb{B}\otimes\mathbb{Q}_p \simeq D_{\frac{1}{2}}$. Consider ab. var. of dim 2 + action of $O_{\mathbb{B}}$ special + ~~Tkm ot~~ level-str. prime to p. The functor on $(Sch/\mathbb{Z}_{(p)})$ repr. by projective scheme M and

$$\hat{M}_{/M\otimes_{\mathbb{Z}_{(p)}}\mathbb{F}_p} \times_{\mathrm{Spf}\,\mathbb{Z}_p} \mathrm{Spf}\,\breve{\mathbb{Z}}_p \simeq \bar{\Gamma}\backslash\hat{\Omega}^2_{\mathbb{Q}_p}, \quad \bar{\Gamma}\subset PGL_2(\mathbb{Q}_p)$$

p-adic unformiz. by RZ-space.

RZ: General unif.-theorem of Sh-var. along basic locus

But in general basic locus proper subset of special fiber.

Drinfeld's situation holds almost never, e.g.

- If RZ-space p-adic, then it's Drinfeld space   GHR
- global   If basic locus = whole special fiber, then it's Drinfeld   Kottwitz
- local   If $(RZ)_{red}$ "same dim",   GHR + Scholze.

Further connections:

- Local models
- ADLV

- ~~Kottwitz conjecture~~ LSV + Scholze
- Kottwitz conjecture on coho of rigid-anal.
- Special cycles on RZ spaces.
- Classes of RZ-spaces w. "simple descript." of reduced scheme

(fully HN-decompos.)