Darmstadt

Talk: On integral LSV

jt. Pappas   § 1 **Motivation**

tower of alg. with finite étale

- Mechanism of Shim.-var.

$$SD\ (G, X) \rightsquigarrow (Sh_K(G,X)\ /\ E = E(G,X))_{K \subset G(\mathbb{A}_f)}$$

Furthermore, if p fixed and $K = K^p \cdot K_p$, with K_p quasi-parah.,

should have for $\nu | p$ good model $\mathcal{S}_K(G,X) / \mathcal{O}_{E,(\nu)}$.

- We are concerned w. local analogue: Fix p.

LSD: (G, b, μ), where

- $G / \mathbb{Q}_p$ reductive.
- $b \in G(\breve{\mathbb{Q}}_p)$
- μ = conj-class of minuscule cochar.

Scholze-Weinstein (2020) $\rightsquigarrow$ $(Sh_K(G, b, \mu) / \breve{E})_{K \subset G(\mathbb{Q}_p)}$

tower of rigid-anal. var., with finite étale tran

Predicted by Rapo-Viehmann (2014).

Big contrast between two pictures:

- in global case, explicit descript. as point set. But only in special cases a moduli-theor. description, e.g. if $G =$ then moduli space of princip. pol. ab. var of dim. n + l

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- in local case, a moduli-theor. descr. in all cases but no point set descript. In general, descript. in terms of shtukas. In special cases (EL-type or PEL-type) more concrete descript. in terms of p-divis. groups (Scholze-Weinstein):

$K_0 = G(\Lambda)$.

Prominent case: $G = GL_n$, $\mu = (1^{(d)}, 0^{(n-d)})$ From b get Dieud-mod

$(\breve{\mathbb{Q}}_p^n, b\cdot\sigma) \rightsquigarrow \mathbb{X}$ = p-div. gp of ht = n and dim = d.

$\rightsquigarrow \mathcal{M}$ = corresp. RZ-space = formal scheme over $\mathrm{Spf}\, \breve{\mathbb{Z}}_p$, moduli space of (X, ρ) where
- X as above
- ρ framing.

Then $\mathrm{Sh}_{K_0}(G, b, \mu) = \mathcal{M}^{rig}$. Get remainder of tower by adding level-K-structure.

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Conjecture (Scholze): Extend in tautological way the moduli-pb for $\mathrm{Sh}_K(G, b, \mu)$ over $O_{\breve{E}}$. Then this functor is represen by a formal scheme $\mathcal{M}$ which is normal + flat + loc. f. f. t. / $O_{\breve{E}}$ provided that K is quasi-parahoric. Furthermore

$$\mathcal{M}^{rig} \simeq \mathrm{Sh}_K(G, b, \mu),$$

provided that K is parahoric.

Slight abuse of notation: depends not only on $K \subset G(\mathbb{Q}_p)$, but on smooth gp scheme $\mathcal{G}/\mathbb{Z}_p$ with $K = \mathcal{G}(\mathbb{Z}_p)$.

§ 2 Main result

<u>Theorem</u>: Let (G, b, μ) LSD, let $\mathcal{G}/\mathbb{Z}_p$ quasi-par. gp scheme for G. Assume (G, μ) of abelian type and $p \neq 2$. Then Scholze's functor $\mathcal{M}^{int}_{\mathcal{G},b,\mu}$ is represent. by $\mathcal{M}_{\mathcal{G},b,\mu}$ in sense above. Furthermore:

a) for $x \in \mathcal{M}(\overline{\mathbb{F}}_p)$ ex $y \in \mathbb{M}^{loc}_{\mathcal{G}^\circ,\mu}(\overline{\mathbb{F}}_p)$ s.t.

$$\mathcal{M}^{\wedge}_{/x} \simeq \mathbb{M}^{loc\,\wedge}_{/y}.$$

(<u>local structure</u>)

b) (<u>generic fiber</u>): Let

$$\Pi_{\mathcal{G}} = \mathrm{Ker}\big(H^1_{et}(\mathbb{Z}_p, \mathcal{G}) \to H^1_{et}(\mathbb{Q}_p, G)\big) \quad \text{finite ab.}$$

Then

$$\mathcal{M}^{rig}_{\mathcal{G},b,\mu} \simeq \coprod_{\bar\beta \in \Pi_{\mathcal{G}}} Sh_{K_\beta}(G, b, \mu).$$

<u>Remarks</u>:
- Also have same result for $p = 2$, under certain hypoth.
- $\mathbb{M}^{loc}_{\mathcal{G}^\circ,\mu}$ = local model in sense of Scholze or Pappas-Zhu
  proj. scheme flat,
  = normal w. reduced fiber, CM for $p \neq 2$:

  Anschütz, Gleason, Haines, Lourenço, Richarz,
- For $\bar\beta \in \Pi_{\mathcal{G}} \rightsquigarrow$ quasi-par. $\mathcal{G}_\beta/\mathbb{Z}_p$ s.t. $\mathcal{G}_\beta(\breve{\mathbb{Z}}_p)$ conj. $\mathcal{G}(\breve{\mathbb{Z}}_p)$.
- Last formula reminisc. of Kottwitz form. for PEL-Sh-var, there index set is $\mathrm{Ker}\big(H^1(\mathbb{Q}, G) \to \prod_v H^1(\mathbb{Q}_v, G)\big)$.

  trivial, if $\mathcal{G}$ parahoric.

Explanations:

abelian type

- (G,μ) of Hodge type $\underset{\mathrm{Df.}}{=}$ $\exists\ i : (G,\mu) \hookrightarrow (GL_n, \mu_d)$
- (G,μ) of abelian type $\underset{\mathrm{Df.}}{=}$ $\exists\ (G_1,\mu_1)$ of Hodge type + iso $(G_{1\,ad}, \mu_{1\,ad}) \simeq (G_{ad}, \mu_{ad})$.

Remark: Call G acceptable $\underset{\mathrm{Df.}}{=}$ write $G_{ad} = V = \prod \mathrm{Res}_{F_i/\mathbb{Q}_p} \overline{H}_i$ ($\prod G_i$), where $\overline{H}_i$ absol. siple. Then $\forall$ split over tame ext. of F_i: autom. if $p \geq 5$. Let (G,μ) of abelian type and $p \neq 2$ s.t. $\mu_{ad,i}$ non-trivial, $\forall i$. Then G acceptable! Use Serre: G_i not a triality group!

Scholze's functor

Scholze's functor: On category $\mathrm{Perfd}_{\mathbb{F}} = \{$ complete Tate ring, which is a perfect $\mathbb{F}$-algebra $\}^{op}$!

$\mathcal{G}$ = smooth gp scheme $/\mathbb{Z}_p$

(Tate ring = Huber ring s.t. $\exists$ pseudo-uniformizer =

$= K\langle T_1, \ldots, T_n\rangle / I$, where K non-arch. field).

$\mathcal{M}(S) = \{$ iso-cl. of $(S^\#, \mathcal{P}, \phi_{\mathcal{P}}, r)$ where

- $S^\#$ = untilt of S over $\mathrm{Spa}\, O_{\breve{E}}$
- $(\mathcal{P}, \phi_{\mathcal{P}})$ = $\mathcal{G}$-shtuka over S w. leg at $S^\#$ bded by μ, i.e. $\mathcal{G}$-torsor on $S \dot{\times} \mathrm{Spa}(O_{\breve{E}}) \underset{\text{affine}}{=} \mathrm{Spa}\, W(S) \setminus \{[\varpi] = 0\}$

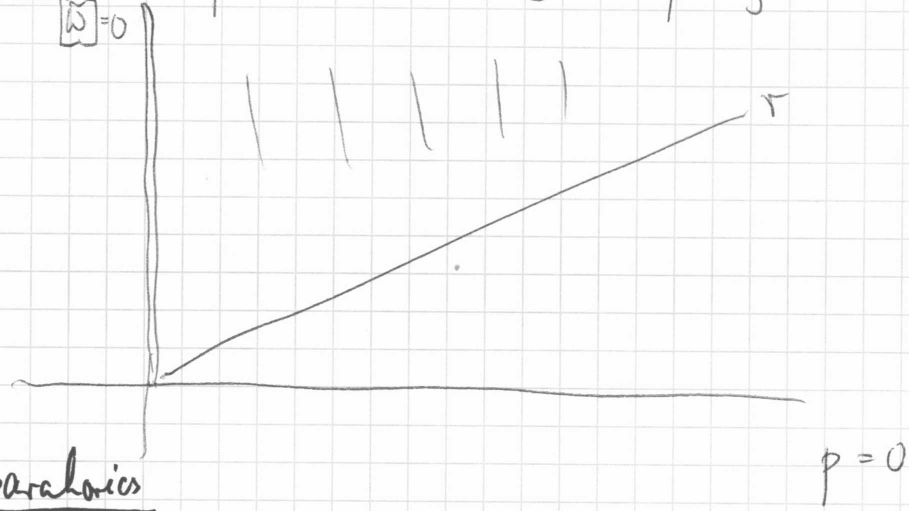
+ $\phi_{\mathcal{P}} : \mathrm{Frob}^{a}(\mathcal{P})\,|\, S \dot{\times} \mathrm{Spa}\, O_{\breve{E}} \setminus S^\# \longrightarrow \mathcal{P}|$ ______

mero along $S^\#$, bded by μ.

- i_r = a framing,

$$i_r : \mathcal{G}_{y_{[r,\infty)}}(S) \xrightarrow{\sim} \mathcal{P}|_{y_{[r,\infty)}}(S) \qquad r \gg 0 \text{ germ.}$$

compat. with $b \times \mathrm{Frob}_S$, resp. ϕ_P.



- **Quasi-parahorics**

Quasi-parahorics: $G/\mathbb{Q}_p$ reductive

A quasi-parahoric subgp of $G(\breve{\mathbb{Q}}_p)$ = a subgp $\breve{K}$ squeezed as

$$G(\breve{\mathbb{Q}}_p)^0 \cap \mathrm{Stab}_{\mathcal{F}} \subset \breve{K} \subset G(\breve{\mathbb{Q}}_p)^1 \cap \mathrm{Stab}_{\mathcal{F}}, \quad \mathcal{F} \subset \mathcal{B}(G_{ad}, \breve{\mathbb{Q}}_p)$$

= stabilizes facet + cont. parahoric w. finite index.

$$\kappa : G(\breve{\mathbb{Q}}_p) \longrightarrow \pi_1(G)_I = \Omega_G.$$

BT: $\exists!$ smooth gp scheme $\breve{\mathcal{G}}/\breve{\mathbb{Z}}_p$ w. generic fiber $\breve{G}$ s.t. $\breve{\mathcal{G}}(\breve{\mathbb{Z}}_p) = \breve{K}$.

A $\mathbb{Q}_p$-quasi-parah. of $G \overset{=}{\underset{\mathrm{Def.}}{}}$ σ-stable quasi-par.

$$\rightsquigarrow K = G(\mathbb{Q}_p) \cap \breve{K} = \mathcal{G}(\mathbb{Z}_p) \quad \text{where } \mathcal{G}/\mathbb{Z}_p \text{ descended.}$$

Note: Map $\mathcal{G} \mapsto \mathcal{G}(\mathbb{Z}_p)$ is inj. on parahorics but not on quasi-parah.

<u>Remark</u>: In analogous fct-field case,

$$\mathcal{G} \text{ quasi-parah.} \iff LG/\mathcal{G} \text{ ind-proper (Richarz).}$$

Let $\mathcal{G}$ quasi-parahoric. Recall $\Omega_G = \pi_1(G)_I$. Then

$$\pi_0(\mathcal{G}) \subset \Omega_G . \quad \text{Let } C_{\mathcal{G}} = \Omega_G / \pi_0(\mathcal{G}).$$

Then

$$\Pi_{\mathcal{G}} = \mathrm{Ker}\left(\pi_0(\mathcal{G})_\phi \longrightarrow \Omega_\phi\right), \qquad (*)$$

hence get exact sequence

$$0 \to \Omega^\phi / \pi_0(\mathcal{G})^\phi \to C_{\mathcal{G}}^\phi \to \Pi_{\mathcal{G}} \to 0$$

<u>Claim</u>: Start with $\bar{\beta} \in \Pi_{\mathcal{G}}$, lift to $\beta \in \pi_0(\mathcal{G})$ in $(*)$. Then

$\exists\ \gamma \in \Omega$ with

$$\beta = (1-\phi)\gamma.$$

Now $\Omega \subset \tilde{W} = N(\breve{\mathbb{Q}}_p) / T(\breve{\mathbb{Q}}_p)_1^0$.

<u>Claim</u>: $\exists\ \dot{\gamma} \in N(\breve{\mathbb{Q}}_p)$ ~~with~~ which lifts γ such that

$$\phi(\dot{\gamma})^{-1} \cdot \dot{\gamma} \in \breve{K}.$$

$\leadsto$

$$\breve{K}_\gamma = \dot{\gamma} \breve{K} \dot{\gamma}^{-1}.$$

Then

- $\breve{K}_\gamma$ indep't of lift $\dot{\gamma}$
- $\breve{K}_\gamma$ is σ-invariant

- $G(\mathbb{Q}_p)$-conj. class of $\breve{K}_{\mathcal{G}}$ only depends on $\bar\beta \in \Pi_{\mathcal{G}}$.

<u>Integral strengthening of decomp. in b)</u>

<u>Theorem</u>: $\mathcal{M}_{\mathcal{G},b,\mu} \cong \coprod_{\bar\beta\in\Pi_{\mathcal{G}}} \mathcal{M}_{\mathcal{G}^\circ_\beta, b,\mu} / \pi_0(\mathcal{G}_\beta)^\phi$.

$\exists$ action of $\pi_0(\mathcal{G}_\beta)^\phi$ on $\mathcal{M}_{\mathcal{G}^\circ_\beta, b,\mu}$

via twisting of $\mathcal{G}$-shtuka. And

<u>Remark</u> a) In generic fiber get

$$\mathcal{M}^{rig}_{\mathcal{G},b,\mu} = \coprod_{\bar\beta} \mathcal{M}^{rig}_{\mathcal{G}^\circ_\beta, b,\mu} / \pi_0(\mathcal{G}_\beta)^\phi .$$

$$= \coprod_{\bar\beta} \mathcal{M}^{rig}_{\mathcal{G}_\beta,b,\mu}\ Sh_{K_\beta}(G,b,\mu)$$

<u>Rmk</u>: b) gives a reduction procedure to parahoric moduli schemes.

§3 Proofs

<u>Strategy of proof</u>:

- Can define formal completion of functor $\mathcal{M}_{\mathcal{G},b,\mu}$ at $x \in \mathcal{M}_{\mathcal{G},b,\mu}(k)$ (extends Gleason, uses Anschütz purity – but have a much simpler proof (global paper)
- Show that $\hat{\mathcal{M}}_{/x}$ is representable by formal spectrum of normal local ring, $\forall x$. This is done first in Hodge type case, using Kisin-Pappas, resp. Kisin-Zhou. + RZ-case <u>uses displays</u>
- This implies representability in Hodge type case,

when $\mathcal{G}(\breve{\mathbb{Z}}_p) = GL(\Lambda \otimes \breve{\mathbb{Z}}_p) \cap G(\mathbb{Q}_p)$.

- Compatib. with ~~Deal with~~ ad-iso's $(G, b, \mu) \longrightarrow (G', b', \mu')$

- ~~Deal~~ Compat. for $M_{\mathcal{G}^\circ}$ $\longrightarrow M_{\mathcal{G}}$

Modelled on de Jong : $\mathcal{Y} \underset{cl.}{\subset} \mathcal{X}$ $\simeq$ $\begin{cases} Y \subset X \\ T \subset \mathcal{X} \\ V_t \subset \mathcal{X} \end{cases}$