Talk: p-adic shtukas and Shimura varieties

Langlands corresp. closely related to determ. of ℓ-adic rep'ns in coho. of Sh-var. These are analyzed through integral models.

But Sh-var. and their integral models are of independent interest.

One pb. for long time: characterize uniquely integral models.

New approach via p-adic Shtukas (Scholze). jt. w. Pappas

(1.) ~~Let $k \supseteq k \supset \mathbb{F}_p$~~

Definition: Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_p}$, with untilt $S^{\#}$ ~~over $W(k)$~~ / A

shtuka over S with leg at $S^{\#}$ = vb. $\mathcal{V}$ on $S \dot{\times} \mathbb{Z}_p$ +

iso

$$\phi_{\mathcal{V}} : \mathrm{Frob}_S^*(\mathcal{V})_{|S \dot{\times} \mathbb{Z}_p \setminus S^{\#}} \longrightarrow \mathcal{V}_{|S \dot{\times} \mathbb{Z}_p \setminus S^{\#}}$$

which is mero. along the Cartier $S^{\#} \subset S \dot{\times} \mathbb{Z}_p$.

Here $S \dot{\times} \mathbb{Z}_p$ is the analytic adic space

$$S \dot{\times} \mathbb{Z}_p = S \dot{\times} \mathrm{Spa}(\mathbb{Z}_p) = \mathrm{Spa}(W(R^+)) \setminus \{\underbrace{[\varpi]}_{\text{Teichm. of pseudo-unif. of } R^+} = 0\}$$

Analogue of Drinfeld's shtuka in fct.-field case for ~~these~~ $X = \mathbb{A}^1_{\mathbb{F}_p}$: there a shtuka over $S \underset{= \mathrm{Spec}(R)}{\in} (Sch/\mathbb{F}_p)$ with leg at $x : S \to X$ is a vb. $\mathcal{V}$ on $X \times_{\mathbb{F}_p} S = \mathrm{Spec}\, R[t]$ + iso

$$\phi_{\mathcal{V}} : \mathrm{Frob}_S^*(\mathcal{V})_{|X \times_{\mathbb{F}_p} S \setminus \Gamma_x} \xrightarrow{\sim} \mathcal{V}_{|X \times_{\mathbb{F}_p} S \setminus \Gamma_x}.$$

So: replace $R[t]$ by $W(R)$ + use analytic geometry

+ remove "$[\varpi] = 0$" to obtain analytic adic space.

Note: $pR = 0$ but $R^\#$ mostly has char 0.

Variants: (i) Let $\mathcal{G}/\mathbb{Z}_p$ connected smooth gp scheme s.t. $G = \mathcal{G} \otimes \mathbb{Q}_p$ reductive.

Then have notion of $\mathcal{G}$-shtuka over $S \in \mathrm{Perf}_k$, with leg at some untilt $S^\#$.

(ii) Let μ = conj.-class of cochar. of G. Then have notion of $\mathcal{G}$-shtuka with leg at $S^\#$ bounded by μ.

(iii) Recall the diamond functor $X \longmapsto X^\diamond$ of Scholze from formal sch. $/\mathbb{Z}_p$, or schemes $/\mathbb{Z}_p$ to v-sheaves on Perf_k:

$$X^\diamond(S) = \{ \text{iso-cl. of } (S^\#, x: S^\# \longrightarrow X) \}.$$

Then can have notion of $\mathcal{G}$-shtuka over X: roughly a functorial rule which to $(S^\#, x) \in X^\diamond(S)$ assoc. $\mathcal{G}$-sht. over S w. leg at $S^\#$.

Examples: a) char 0 Let $(\mathcal{G}, G, \mu)$ as above where μ minuscale $\rightsquigarrow E = E(G, \mu)$. Let X/E loc. noeth. adic space. Then equiv.

- $\mathcal{G}$-shtuka over X w. leg bdd by minuscale μ
- $(\mathbb{P}, H)$, where $\mathbb{P}$ = pro-étale $\mathcal{G}(\mathbb{Z}_p)$-torsor over $X^\diamond$ + equiv. map

$$HT: \mathbb{P} \longrightarrow \mathcal{F}^\diamond_{G, \mu^{-1}}$$

If X = smooth scheme over E, then a pro-étale $\mathcal{G}(\mathbb{Z}_p)$-torsor which is de Rham w. Hodge filtr. bdd by (minuscale) μ, then get such pair $(\mathbb{P}, H)$, and hence $\mathcal{G}$-shtuka bdd by μ.

b) char p Let A = perfect $\mathbb{F}_p$-alg. (no topology), ess. of f.t. over $\bar{\mathbb{F}}_p$

Then equiv. of cat.

$$\left(\text{shtukas over } \mathrm{Spec}\, A\right) \longrightarrow \left(\begin{array}{l}\text{v.b. } \mathcal{V} \text{ on } \mathrm{Spec}(W(A)) + \text{iso} \\ \varphi_{\mathcal{V}} : \mathrm{Frob}^*(\mathcal{V})[\tfrac{1}{p}] \longrightarrow \mathcal{V}[\tfrac{1}{p}]\end{array}\right).$$

c) abelian varieties / X

(2.) LSV + integral models

Let (G, b, μ) local Sh-datum. Scholze defines LSV

$$\mathcal{M}_{G,b,\mu} = \left(\mathcal{M}_{G,b,\mu,K}\right)_K :$$

= rigid-analytic space that represents the functor (diamond) on $\mathrm{Perf}_{\bar{\mathbb{F}}_p}$: $S \mapsto$ $(S^\#, \mathcal{P}, \phi_{\mathcal{P}}, \iota_r)$, where

- $S^\#$ untilt over $\mathrm{Spa}(\breve{E})$.
- $(\mathcal{P}, \phi_{\mathcal{P}})$ bdd by μ. Here $\mathcal{G}(\mathbb{Z}_p) = K$.
- ι_r : $G_{\mathcal{Y}_{[r,\infty)}(S)} \xrightarrow{\sim} \mathcal{P}|_{\mathcal{Y}_{[r,\infty)}(S)}$ $r \gg 0$ framing.

  s.t. $\phi_{\mathcal{P}}$ corresp. to $b \times \sigma$. open part of $S \dot{\times} \mathbb{Z}_p$.

excludes orth. gp.

Example: If (G, b, μ) of RZ-type, then $\mathcal{M}_{G,b,\mu}$ given by RZ-tower (Scholze).

For integral models, consider again (G, b, μ) as above. For certain K want integral models of $\mathcal{M}_{G,b,\mu,K}$. Namely, let $K = \mathcal{G}(\mathbb{Z}_p)$ of BT-type. Can consider functor as before / exc. $S^\#$ over $\mathrm{Spa}(O_{\breve{E}})$.

Defines v-sheaf $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$.

Conjecture (Scholze): Ex. normal formal scheme $\mathfrak{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ flat + loc. f.f.t / $\mathrm{Spf}\, O_{\breve{E}}$ with $(\mathfrak{M}^{\mathrm{int}})^{\diamond} = \mathcal{M}^{\mathrm{int}}$ (unique if exists)

Examples: • If $(\mathcal{G}, b, \mu)$ of RZ-type, then RZ-space does it. (excludes type (D)).
If (G, μ) of Hodge type, then
• If G unramified, then OK (Zhou, Nie)

• If G tamely ram. + resid. split, then OK (Zhou).
+ ~~exotic case~~ triality, $p = 3$

• Hope: If abelian type + exclude ~~certain cases for~~ $p = 2 \lor 3$.

(3.) Global Sh.-var. + integral models

Let ~~Sh~~ (G, X) Sh-datum (μ) E. Fix $v | p$, let $K = K^p . K$, where $K = \mathcal{G}(\mathbb{Z}_p)$ (smooth)

We always assume $\mathrm{rk}_{\mathbb{Q}} Z = \mathrm{rk}_{\mathbb{R}} Z$. Then get pro-étale $\mathcal{G}(\mathbb{Z}_p)$-local system $\mathbb{P}_K$ over $\mathrm{Sh}_K(G, X)_E$. Liu-Zhu $\Rightarrow$ de Rham, hence by Ex. a) get $\mathcal{G}$-shtuka $\mathcal{P}_{K, E_v}$ over $\mathrm{Sh}_{K, E_v}^{\diamond}$, compat. Hecke corres. prime to p.

Conjecture: Assume $\mathcal{G}$ B-T type. $\exists$ pro-system of normal flat int. models $\mathcal{S}_K$ of Sh_{K, E_v} over O_{E_v} w. étale trans. variable K^p s.t.

(i) $\forall$ dvr R of $(0, p)$ over O_{E_v},

$$\varprojlim_{K^p} \mathrm{Sh}_{K, E}(R[1/p]) = \varprojlim_{K^p} \mathcal{S}_K(R)$$

(ii) $\mathcal{G}$-shtuka $\mathcal{P}_{K, E_v}$ extends to $\mathcal{G}$-sht. $\mathcal{P}_K$

(iii) Let $x \in \mathcal{S}_K(k) \mapsto b_x \in G(\breve{\mathbb{Q}}_p)$. Then ex. iso of "completions"

$$\Theta_x : \left(\mathcal{M}^{\mathrm{int}}_{\mathcal{G}, b_x, \mu / x_0}\right)^{\wedge} \simeq \left(\widehat{\mathcal{S}}_{K / x}\right)^{\diamond}$$

s.t. $\Theta_x^*(\mathcal{P}_K)$ iso to tautological shtuka on $\mathcal{M}^{\mathrm{int}}$

Here b_x as follows: $x^*(\mathcal{P}_{\mathbb{K}})$ is $\mathcal{G}$-shtuka over $\mathrm{Spec}(k)$, hence by Ex. b) get $\mathcal{G}$-torsor $\mathcal{P}$ over $\mathrm{Spec}(W(k))$ + mero Frob. Now trivialize $\mathcal{P}$, then $\mathrm{Frob} = b_x \times \sigma$.

Theorem 1: Any two such integral models are uniquely isomorph.

Theorem 2: Let (G, X) of Hodge type, exclude odd unit. if $p = 2$ and triality if $p = 3$. Then Conjecture holds true.

Remark: If Conj. holds, then get map

$$\Upsilon_{\mathbb{K}} : \mathcal{S}_{\mathbb{K}}(k) \longrightarrow G(\breve{\mathbb{Q}}_p)/G(\breve{\mathbb{Z}}_p)_\sigma .$$

This gives defin. of central leaves / Newton str. / EKOR-str. / KR-str. without abelian varieties.

Even for Siegel moduli space need Scholze's world for charact. !!