Oxford, Oct. 19

Talk: On arithmetic transfer conjectures

$p > 2$. F/F_0 quadr. ext. of p-adic fields.

$n \geq 1$. • Let $W = F/F_0$ - herm. vs of dim n.

Let $v \in W$ non-isotropic "special vector" $\Rightarrow$

$$W^\flat \;,\; U^\flat \subset U. \qquad \underline{\text{Notation}}: W_0 \text{ and } W_1.$$

$$G_W = U^\flat \times U.$$

• Let $G' = \mathrm{Res}_{F/F_0}(GL_{n-1/F} \times GL_{n/F})$.

Actions: $U^\flat \times U^\flat$ on G_W

$\mathrm{Res}_{F/F_0}(GL_{n-1}) \times (GL_{n-1} \times GL_n)$ on G'

$\Rightarrow$ let $\gamma \in G'(F_0)$, $g \in G_W(F_0)$: $\underline{\text{Notions}}$

• γ, resp. g regular semi-simple.

• γ matches g

jt. w. Kudla, Smithling, Zhang.

<u>Motivation</u>: • reciprocity law

• structure of RZ-spaces

• Sunshine rays from Wei

① The AFL conjecture

Assume F/F_0 unramified. Choose special vector $v \in W_1$, $\|v\| = 1$.

<u>Conjecture</u>: Assume $\gamma \in G'(F_0)_{rs}$ matches $g \in G_{W_1}(F_0)_{rs}$. Then

$$\omega(\gamma) \cdot \partial \mathrm{Orb}_\gamma \left(\mathbb{1}_{GL_{n-1}(O_F) \times GL_n(O_F)} \right) = -2 \cdot \langle \Delta, g\Delta \rangle.$$

<u>LHS</u>: $\displaystyle\int_{H_1'(F_0) \times H_2'(F_0)} f'(h_1^{-1} \gamma h_2) \cdot \eta(h_2) \cdot |\det h_1|^s \, dh_1 \, dh_2$

$\eta(h_2) \in \{\pm 1\}$.

$\omega(\gamma)$ transfer factor $\in \{\pm 1\}$. + Haar measure.

<u>RHS</u>: Let

$\mathcal{N}_n$ = RZ-space par. $(X, \iota, \lambda, \varrho)$, where

X = strict formal O_{F_0}-module

$\iota : O_F \hookrightarrow \mathrm{End}(X)$ of sign $(1, n-1)$

λ = polariz. on X, compat. with ι, <u>principal</u>.

$\varrho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\bar{k}} \bar{S}$ framing

Then $\mathcal{N}_n$ formal scheme $/ \mathrm{Spf}\, O_{\breve{F}}$, form. smooth of rel. dim. $n-1$.

If $n = 1 \Rightarrow \mathcal{N}_n = \mathrm{Spf}\, O_{\breve{F}}$, with univers. object $(\mathcal{E}, \iota_\mathcal{E}, \lambda_\mathcal{E}, \varrho_\mathcal{E})$

$\mathrm{Aut}(\mathbb{X}, \iota_\mathbb{X}, \lambda_\mathbb{X})$ acts on $\mathcal{N}_n$

Closed embedding

$$\delta : \mathcal{N}_{n-1} \hookrightarrow \mathcal{N}_n \ : \ Y \mapsto Y \times \overline{\mathcal{E}}$$

Let

$$\Delta \subset \mathcal{N}_{n-1} \times \mathcal{N}_n \qquad \text{graph of } \delta.$$

Now $\quad G_{W_1}(F_0) = U(W_1^\flat) \times U(W_1)$ acts.

$$\langle \Delta, g\Delta \rangle = \chi(\mathcal{O}_\Delta \otimes^{\mathbb{L}} \mathcal{O}_{g\Delta}) \cdot \log q_0$$

<u>Results</u>:
- $n = 2, 3$ (Zhang): local proof.
- n arbitr. but g minuscule (Rapo-Terstiege-Zhang, He-Li-Zhu): local
- $F_0 = \mathbb{Q}_p$, $n \geq p$. (Zhang): global.

② An AT conjecture.

F/F_0 unramif. Let $K_n^{[1]} = \{ g \in GL_n(\mathcal{O}_F) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ block sizes $n-1, 1 \}$.

Special vector $v \in W_0$, $\|v\| = \pi$.

<u>Conjecture</u>: $\exists\, c$ s.t.

a) $\exists f' \in C_c^\infty(G'(F_0))$ s.t. $f' \sim \mathbb{1}_{GL_{n-1}(\mathcal{O}_F) \times K_n^{[1]}}$ with

$$\omega(\gamma)\, \partial\mathrm{Orb}_\gamma(f') = c \cdot \langle \Delta^{[1]}, g\Delta^{[1]} \rangle$$

whenever γ matched to $g \in G_{W_0}(F_0)_{rs}$.

b) $\forall f'$ ———————————— $\exists f'_{corr}$

———————————— $+ \mathrm{Orb}_\gamma(f'_{corr})$.

<u>RHS</u>: Let $\mathcal{N}_n^{[1]}$ = like $\mathcal{N}_n$, except that $\deg \lambda = q_0^2$.

Have cl. embedding

$$\delta^{[1]} : \mathcal{N}_{n-1} \hookrightarrow \mathcal{N}_n^{[1]} : Y \longmapsto Y \times \bar{\mathcal{E}}^{[1]},$$

where $\mathcal{E}^{[1]} = (\mathcal{E}, \iota_{\mathcal{E}}, \pi \cdot \lambda_{\mathcal{E}})$.

$\Delta^{[1]}$ = graph of $\delta^{[1]}$.

Note: $\mathcal{N}_{n-1} \times \mathcal{N}_n^{[1]}$ regular (but not form. smooth).

$\mathcal{N}_2^{[1]}$ well-known RZ-space,

$$\mathcal{N}_2^{[1]} \simeq \hat{\Omega}^2_{F_0} \times_{\mathrm{Spf}\, O_{F_0}} \mathrm{Spf}\, O_{\breve{F}} .$$

Here W_0 split, $W_0^\flat$ non-split, hence

$$G_{W_0}(F_0) = U(W_0^\flat) \times U(W_0) \text{ acts on } \mathcal{N}_{n-1} \times \mathcal{N}_n^{[1]}.$$

<u>Results</u>: • $n = 2, 3$: local ([RSZ])

• n arbitrary, $p \geq n$, $F_0 = \mathbb{Q}_p$, intersection non-degenerate

③ <u>Another AT conjecture</u>

F/F_0 unram., special vector $v \in W_1$, $\|v\| = \pi$.

<u>Conjecture</u>: Same as before, except:

• $\mathbb{1}_{GL_{n-1}(O_F) \times K_n^{[1]}}$ replaced by $\mathbb{1}_{K_{n-1}^{[1]} \times GL_n(O_F)}$

• $\Delta^{[1]}$ replaced by $\cancel{\tilde{\mathcal{N}}_{n-1}^{[1]}}$, ~~where~~ $\tilde{\mathcal{N}}_{n-1}^{[1]}$

~~$\tilde{\Delta}^{[1]}$ = "graph" of $\tilde{\mathcal{N}}_{n-1}^{[1]} \to \mathcal{N}_n$ in $\tilde{\mathcal{N}}_{n-1}^{[1]} \times$~~.

Here fix isogeny $\mathbb{Y} \times \overline{\mathbb{E}}^{[1]} \xrightarrow{\alpha} \mathbb{X}$ of degree q_0^2, and
$\tilde{\mathcal{N}}_{n-1}^{[1]}$ = locus in $\mathcal{N}_{n-1}^{[1]} \times \mathcal{N}_n$ where α extends.

Note that $W_1^\flat$ ~~non~~ split, W_1 non-split $\Rightarrow$ G_{W_1} acts.

<u>Known</u>: Very little! Know a little about the geometry of the maps

$$\begin{tikzcd} & \tilde{\mathcal{N}}_{n-1}^{[1]} \arrow[dl] \arrow[dr] & \\ \mathcal{N}_{n-1}^{[1]} & & \mathcal{Z}(v) \subset \mathcal{N}_n. \end{tikzcd}$$

More variants:

- ~~replace graph of~~ $\tilde{\mathcal{N}}^{[1]}_{n-1}$ by graph of $\mathcal{Z}(v) \to \mathcal{N}_n$

($\mathcal{Z}(v)$ regular by Terstiege)

- ~~replace graph of~~ $\tilde{\mathcal{N}}^{[1]}_{n-1}$ by graph in $\hat{\mathcal{N}}^{[1]}_{n-1} \times \mathcal{N}_n$

(don't know whether $\hat{\mathcal{N}}^{[1]}_{n-1}$ regular $\to$ cf. geom. above.

(4). Let $0 \leq r \leq n-1$, F/F_0 unram. Let

$$K_n^{[r]} = \qquad \text{blocks } (n-r, r).$$

Conjecture: Same as before, except:

- $f' \sim \mathbb{1}_{K^{[r]}_{n-1} \times GL_n(O_F)}$

- $\left\langle \tilde{\mathcal{N}}^{[r]}_{n-1}, \; g \cdot \hat{\mathcal{N}}^{[r]}_{n-1} \right\rangle$.

Here $\tilde{\mathcal{N}}^{[r]}_{n-1}$ = locus in $\mathcal{N}^{[r]}_{n-1} \times \mathcal{N}_n$ where isogeny $\mathbb{Y} \times \bar{\mathbb{E}}^{\varepsilon} \to \mathbb{X}$ (of degree $q_0^{r+\varepsilon}$) lifts to $Y \times \bar{\mathcal{E}}^{\varepsilon} \to X$.