Johns Hopkins 19

<u>Talk</u>: p-<u>adic</u> <u>uniformization</u> of <u>unitary</u> <u>Shim curves</u>, I

jt. with S. Kudla, Th. Zink.

2 parts: global + local.

p-adic uniformization of Shimura curves

Complex uniformization: Let X connected smooth projective alg. curve / $\mathbb{C}$ of genus ≥ 2.

Poincaré: ex. discrete co-compact subgp. $\Gamma \subset PGL_2(\mathbb{R})$ s.t.

$$X^{an} \simeq \Omega_{\mathbb{R}} / \Gamma . = X_{\Gamma}^{an}$$

Here

$$\Omega_{\mathbb{R}} = \mathfrak{h}^+ \cup \mathfrak{h}^- = (\mathbb{P}^1_{/\mathbb{R}})^{an} \setminus \mathbb{P}^1(\mathbb{R})$$

Complicated history (→ Gray): Klein, Koebe, Hilbert

Poincaré: Systematic way of constructing such Γ (he used SO_3 - but comes to the same). Let

F = totally real field

w = fixed archim. place

B = quaternion algebra / F with $B \otimes_F F_{w'} = \begin{cases} M_2(\mathbb{R}) & w' = w \\ \mathbb{H} & w' \neq w. \end{cases}$

Get action of $B^\times$ on $\Omega_{\mathbb{R}}$ via

$$B^\times \longrightarrow (B \otimes_F F_w)^\times = GL_2(\mathbb{R}) \longrightarrow PGL_2(\mathbb{R})$$

Let $O_B \subset B$ order, then take $\Gamma =$ image of $O_B^\times$, discrete subgp

If B quaternion division algebra, then Γ co-compact. (automatic if $|F:\mathbb{Q}| > 1$). Shimura curve p-adic uniformization (Cherednik, 1976)

Let B/F as above. Fix p and

v = p-adic place of F such that $B \otimes_F F_v$ division alg.

Assume $\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v}$ unique maximal order.

Then

$$(X_\Gamma \otimes_{\bar{F}} \bar{F}_v)^{an} \simeq (\Omega_{F_v} / \bar{\Gamma}) \otimes_{F_v} \bar{F}_v ,$$

where

$$\bar{\Gamma} \subset PGL_2(F_v) \quad \text{discrete co-compact}$$

Here Ω_{F_v} p-adic analogue of $\Omega_{\mathbb{R}}$ over F_v,

$$\Omega_{F_v} = (\mathbb{P}^1_{/F_v})^{an} \setminus \mathbb{P}^1(F_v). \quad \text{Drinfeld halfplane.}$$

~~Interpretation: LHS is algebraization of complex analytic var.~~ ~~(unique!)~~

after Drinfeld.

~~RHS ~~ rigid-analytic var ( ).

{ Very mysterious, even today – Cherednik gave heuristic argument.
Varshavsky, RZ, Boutot-Zink,

(*) Drinfeld (1976): gave explanation + proof of Ch's theorem for $F = \mathbb{Q}$. Uses adelic version, i.e. finite union of X_Γ's:

Let

$G = B^\times$ as algebr. gp $/\mathbb{Q}$.

Let

$\mathbb{K}$ open compact subgp of $G(\mathbb{A}_f)$

$$X_{\mathbb{K}} = G(\mathbb{Q}) \backslash [\Omega_{\mathbb{R}} \times G(\mathbb{A}_f)/\mathbb{K}].$$

<u>Drinfeld</u>: defines moduli pb for $X_{\mathbb{K}}$ over $\mathbb{Q}$, even over $\mathbb{Z}_{(p)}$ when $\mathbb{K} = \mathbb{K}^p \mathbb{K}_p$ (max.)

⤳ $\mathcal{M}_{\mathbb{K}}$ over $\operatorname{Spec} \mathbb{Z}_{(p)}$. (explain)

<u>Theorem</u>: $$\hat{\mathcal{M}}_{\mathbb{K}} \hat{\otimes}_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p \simeq G_-(\mathbb{Q}) \backslash [\hat{\Omega}_{\mathbb{Q}_p} \hat{\otimes}_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p \times G(\mathbb{A}_f^p)/\mathbb{K}^p]$$

Here $\hat{\Omega}_F$ = integral model of Ω_F

<u>Principle</u>: Show that formal scheme $\hat{\Omega}_F \hat{\otimes}_{O_F} \breve{O}_F$ represents a moduli problem of p-divisible groups. Very difficult! → Zink?

(*) (except. iso)

~~Work of KRZ + ~~~~(Contrib of Scholze~~ ~~close to proving full theorem.~~

~~Let K/F CM-field, d = [F:Q].~~

~~V = 2-dim. K-vs + hermit form, s.t. { sgn(V_w) = (1,1); V_{w'} definite, w' ≠ w.~~

~~Hdx ...~~

~~More precisely, fix r: Hom(K, Q̄) → {0,1,2}~~

~~s.t. r_φ + r_φ̄ = 2 and r_φ = { 1 φ|w; 0 or 2 φ|w', w'≠w.~~

~~Then want sgn(V_φ) = (r_φ, 2 − r_φ). ⤳ E = E_r ⊃_w F.~~

~~Also assume inv_v(V) = −1.~~

~~Let G = GU(V) (multiplier in Q^×) ⤳ Shimura variety.~~

Assume v non-split

<u>Choice of $\mathbb{K}$</u>:

Assume $p \neq 2$, if K_v/F_v ramified.

Notation: Let K/F CM-field, $d = [F:\mathbb{Q}]$.

Fix $r: \mathrm{Hom}(K, \overline{\mathbb{Q}}) \longrightarrow \{0,1,2\}$ s.t.

$$r_\varphi + r_{\bar\varphi} = 2, \ \forall \varphi \quad \text{and} \quad r_\varphi = \begin{cases} 1 & \varphi | w \\ 0 \text{ or } 2 & \varphi | w', \ w' \neq w. \end{cases}$$

Fix V = 2-dim. K-vs + anti-herm. form s.t.

$$\mathrm{sgn}(V_\varphi) = (r_\varphi, 2 - r_\varphi), \ \forall \varphi.$$

$\rightsquigarrow E = E_r$ and $F \underset{w}{\hookrightarrow} E$.

Recall $v | p$. Assume: v non-split and V_v non-split.

Also assume $p \neq 2$ when K_v / F_v ramified.

Let $G = GU(V)$ (multipl. in $\mathbb{Q}^\times$) $\rightsquigarrow$ Shimura var.

$Sh(G, \{h_r\})_{\mathbb{K}}$, with model over E.

The Shim-var represents foll. moduli pb. on (Sch/E):

$S \mapsto (A, \iota, \bar\lambda, \bar\eta)$, where

- A abelian scheme of dim. $2d$, up to isog.
- $\iota: O_K \longrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$ s.t. Lie A satisfies (KC_r).
- $\bar\lambda$ = $\mathbb{Q}$-class of polariz., compat. w. ι
- $\bar\eta: \hat V(A) \simeq V \otimes \mathbb{A}_f$ mod $\mathbb{K}$ K-lin. similitude.
  (for $\pi_1(x, s)$.)

For good choice of $\mathbb{K}$ get integral model.

Write

$$V \otimes \mathbb{Q}_p = \bigoplus_{\mathfrak{p}} V_{\mathfrak{p}} \qquad \text{orth., over } F.$$

Note: $\mathfrak{p}_0$ one of them.

Choose lattices

$$\Lambda_{\mathfrak{p}} \subset V_{\mathfrak{p}}$$

s.t.

- $\Lambda_{\mathfrak{p}}$ selfdual if $\mathfrak{p}$ split, or ramified, or unramif. + $\mathrm{inv}_{\mathfrak{p}} V = 1$
- $\Lambda_{\mathfrak{p}}$ almost selfdual if $\mathfrak{p}$ unram. + $\mathrm{inv}_{\mathfrak{p}} V = -1$.

Let

$$K_p = \mathrm{Stab}_{G(\mathbb{Q}_p)}\left(\bigoplus \Lambda_{\mathfrak{p}}\right).$$

Let

$$\mathbb{K}^p \subset G(\mathbb{A}_f^p) \text{ arbitr. and } \mathbb{K} = \mathbb{K}^p . K_p .$$

Let ν place of E inducing v of F. Define functor $\mathcal{A}_{\mathbb{K}}$ on $(Sch/O_{E,(\mathfrak{p}_\nu)})$: assoc. to S quadruples $(A, \iota, \bar{\lambda}, \bar{\eta}^p)$, where

- A abelian scheme of dim $= 2d = 2[F:\mathbb{Q}]$ over S, up to isog. prime to p
- $\iota : O_K \longrightarrow \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ s.t. Lie A sat. Kottwitz cond. (KC_r). $\prod_{\varphi} (T - \varphi(a))^{r_\varphi}$
- $\bar{\lambda} = \mathbb{Q}$-class of polariz., compat. with ι

for $\mathcal{I}^{p} \times \bar{\eta}$.

- $\bar{\eta}^{p} : V^{p}(A) \simeq V \otimes \mathbb{A}_f^{p}$, a $\mathbb{K}^{p}$-class of anti-Hermit. similitudes.

<u>Conditions imposed</u>:

- ex $\lambda \in \bar{\lambda}$ of degree prime to p s.t. $|\mathrm{Ker}\, \lambda_p| = \langle$

next time → • Lie A satisfies (EC_r) $\forall s \in S$

- $\mathrm{inv}_p^{+}(A_s, \iota_s, \lambda_s) = \mathrm{inv}_p(V)$, $\forall\, \mathfrak{p} \mid p$. $(*)$ →

<u>Main theorem</u>: (i) $\mathcal{A}_{\mathbb{K}}$ is representable by a projective flat $\mathcal{O}_{E,(\mathfrak{p}_v)}$-scheme with semi-stable red. It $\mathbb{C}$-fiber identified with $Sh(G, \{h_r\})_{\mathbb{K}}$.

(ii) Let $\hat{\mathcal{A}}_{\mathbb{K}}$ = completion along special fiber. Then ex.

$$\hat{\mathcal{A}}_{\mathbb{K}} \hat{\otimes}_{\mathcal{O}_{E_v}} \mathcal{O}_{\breve{E}_v} \simeq \bar{G}(\mathbb{Q}) \backslash \left[(\hat{\Omega}_{F_v} \hat{\otimes}_{\mathcal{O}_{F_v}} \mathcal{O}_{\breve{E}_v}) \times G(\mathbb{A}_f)/\mathbb{K} \right],$$

compat. with action of $G(\mathbb{A}_f^{p})$ + descent datum.

<u>Remarks</u>: a) if $\mathfrak{p}_v$ only prime above p, then can drop inv.

b) In [RZ] consider case where v split – even in higher dimension, for higher rank unitary gps.

c) <u>Really</u>, do not replace Drinfeld but reduce to Drinfeld. Main work is local.

d) Action of $\bar{G}(\mathbb{Q})$ on $G(\mathbb{Q}_p)/\mathbb{K}_p$ <u>not</u> from action of $\bar{G}(\mathbb{Q})$ on $G(\mathbb{Q}_p)$!

(*) <u>Explanation of</u> $\mathrm{inv}_{\mathfrak{p}}^{+}(A,\iota,\lambda)$, <u>where</u> $A_{/k=\bar{k}}$.

- if $\operatorname{char} k \neq p$, then

$$\mathrm{inv}_{\mathfrak{p}}^{+}(A,\iota,\lambda) = \mathrm{inv}_{\mathfrak{p}} V_{\mathfrak{p}}(A)$$

- if $\operatorname{char} k = p$, then let $M_{\mathfrak{p}} =$ rational $\mathfrak{p}$-part of Dieud.-module of A, with action of $K_{\mathfrak{p}}$, via ι. Set

$$P_{\mathfrak{p}} = \bigwedge^{2}_{K_{\mathfrak{p}} \otimes W(k)} M . \qquad \text{free } K_{\mathfrak{p}} \otimes W\text{-module, rk } 1.$$

Polarization λ induces <u>hermitian</u> form ψ on $P_{\mathfrak{p}}$. Then ex generator z of $P_{\mathfrak{p}}$ with

$$V z = p z .$$

Then

$$\psi(z,z) \in F_{\mathfrak{p}}^{\times} .$$

Then

$$\mathrm{inv}_{\mathfrak{p}}(A,\iota,\lambda) = \psi(z,z) \in F_{\mathfrak{p}}^{\times}/\mathrm{Nm}\, K_{\mathfrak{p}}^{\times} \quad \text{well-def.}$$

<u>Change by</u> $\mathrm{sgn}(\tau)$. Then $= \begin{cases} (-1)^{d-1} & \text{if } \mathfrak{p} = \mathfrak{p}_0 \\ (-1)^{d} & \text{if } \mathfrak{p} \neq \mathfrak{p}_0 . \end{cases}$

<u>Prop.</u>: $s \longmapsto \mathrm{inv}_{\mathfrak{p}}(A_s, \iota_s, \lambda_s)$ is loc. constant //

<u>Still don't understand</u>:

- How to compare coverings on LHS to RHS.
- Related to this: what happens if $\mathbb{K}_p$ made smaller at places $v' \neq v$?

For $\mathfrak{f} | p$, ~~$\mathfrak{f} \neq \mathfrak{f}_0$~~, fix

$$K_{\mathfrak{f}}^* \subset K_{\mathfrak{f}} \quad \text{finite index, "=" for } \mathfrak{f} = \mathfrak{f}_0$$

and let

$$\mathbb{K}_p^* = G(\mathbb{Q}_p) \cap \prod K_{\mathfrak{f}}^*, \quad \mathbb{K} = \mathbb{K}^p \cdot \mathbb{K}_p$$

~~$\mathcal{A}_{\mathbb{K}^*}$~~ $\breve{E}_v^* =$ Galois ext. of $\breve{E}_v$ — <u>abelian</u> ?!

$$+ \ \breve{\mathcal{A}}_{\mathbb{K}^*} \text{ over } \mathcal{O}_{\breve{E}_v^*} \text{ s.t. } \breve{\mathcal{A}}_{\mathbb{K}^*}[\tfrac{1}{p}] \simeq Sh_{\mathbb{K}^*} \otimes_E \breve{E}_v^*$$

+ Analogue:

$$\breve{\mathcal{A}}_{\mathbb{K}^*} \otimes \mathcal{O}_{\breve{E}_v^*} \simeq \tilde{G}(\mathbb{Q}) \backslash \left[\left(\hat{\Omega}_{F_v} \hat{\otimes}_{\mathcal{O}_{F_v}} \mathcal{O}_{\breve{E}_v^*} \right) \times G(\mathbb{A}_f^p) / \mathbb{K}^* \right].$$

Not yet proved.

The p-adic uniformization of Shimura curves, II.

<u>This part is local:</u> $F/\mathbb{Q}_p$ finite, $d = [F:\mathbb{Q}_p]$. K/F quadratic. $p \neq 2$ if K/F ram.

For $S \in (Sch/O_{\breve{F}})$, consider triples (X, ι, λ), where

- X = p-div. gp over S, with strict O_F-action, of ht $= [F:\mathbb{Q}_p]\cdot 4$ and dim $= 2$, i.e. $ht_F(X) = 4$
- $\iota : O_K \to End_{O_F}(X)$ s.t.
$$\det(\iota(a) \mid Lie\, X) = (T-a)(T-\bar{a}), \ \forall a \in O_K.$$
- $\lambda : X \longrightarrow X^\vee$ relative polarization s.t. Rosati induces $a \mapsto \bar{a}$.

Impose:

- $|Ker\, \lambda| = \begin{cases} 1 & K/F \text{ ramified} \\ q_F^2 & K/F \text{ unramified} \end{cases}$
- $inv(X, \iota, \lambda) = -1$. (automatic when K/F unramified).

<u>Then:</u> if $S = Spec\, \bar{k}_F$, only one triple $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ up to isogeny. Furthermore, is isoclinic of slope $1/2$, and
$$Aut^0(\mathbb{X}) \simeq U(F).$$

[KR], [K], [KRZ]

<u>Theorem:</u> Consider the functor on $\mathrm{Nilp}_{O_{\breve{F}}}$,

$$\mathcal{M}: S \mapsto \{\text{iso-cl. of } (X, \iota, \lambda, \varrho)\},$$

where (X, ι, λ) as above and where

$$\varrho: X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\mathrm{Spec}\, \bar{k}_F} \bar{S}$$

is quasi-is. of ht $=0$, respecting ι and λ.

There ex.! isomorphism

$$\mathcal{M} \simeq \hat{\Omega}_F \hat{\otimes}_{O_F} O_{\breve{F}}$$

which is equivariant w.r.t. ~~some~~ fixed isomorphism

$$SU(F) \simeq SL_2(F).$$

<u>Remarks:</u> (i) Invariant $\in \{\pm 1\}$, defined fiber-wise (then locally constant). If (X, ι, λ) over $k = \bar{k}$, then corresp. to <u>relative</u> Dieudonné module $(M, \iota, \langle\, ,\rangle)$, where

- M free $W_{O_F}(k)$-module of rk 4, with F, V with
$$VF = FV = \pi_F, \quad \text{with } \pi_F M \subset VM \overset{2}{\subset} M$$
- $\iota: O_K \to \mathrm{End}_{O_F}(M)$ s.t.
$$\mathrm{char}(\iota(a) \mid M/VM) = (T-a)\cdot(T-\bar{a}), \quad a \in O_K$$

$$\langle \, , \rangle : M \times M \longrightarrow W_{O_F}(k) \quad \text{s.t.}$$

$$\langle Vx, Vy \rangle = \pi_F \cdot \sigma^{-1}(\langle x, y \rangle)$$

and $\langle \iota(a)x, y \rangle = \langle x, \iota(\bar{a})y \rangle$.

Using relative Dieudonné model, the same procedure as last year

k),

this is a $K \otimes W_{O_F}(k)$-module free of rk 1, with hermitian form ψ_P. And a relative Dieud.-module which is isoclinic: hence ex $z \in P$ with generator

$$\Lambda^2 V(z) = \pi_F \cdot z.$$

Then $\psi_P(z,z) \in F^\times$. Then

$$\operatorname{inv}(M, \iota, \lambda) = \psi_P(z,z) \in F^\times / \operatorname{Nm} K^\times = \{\pm 1\}.$$

(ii) Theorem is proved by relating to Drinfeld's moduli pb. of s.f. O_B-modules.

(iii) $p=2$ ⇝ Kirch's thesis.

<u>Our moduli pb is different</u>: Fix $\varphi_0 : F \longrightarrow \overline{\mathbb{Q}}_p$.

$r : \mathrm{Hom}(K, \overline{\mathbb{Q}}_p) \longrightarrow \{0,1,2\}$ s.t.

$r_\varphi + r_{\overline{\varphi}} = 2$ and
$$r_\varphi = \begin{cases} 1 & \text{if } \varphi = \varphi_0 \text{ or } \overline{\varphi}_0 \text{ extensions} \\ 0 \text{ or } 2 & \text{if } \varphi \notin \{\varphi_0, \overline{\varphi}_0\}. \end{cases}$$

Let $E = E_r \subset \overline{\mathbb{Q}}_p$. For $S \in (Sch/O_E)$ consider (X, ι, λ), where

- X p-div. gp of ht $= 4d$ and dim $= 2d$, where $d = [F:\mathbb{Q}_p]$,
- $\iota : O_K \longrightarrow \mathrm{End}(X)$ s.t. (KC_r) holds.
- $\lambda : X \longrightarrow X^\vee$ polariz. s.t. Rosati induces $a \mapsto \overline{a}$.

Impose:

- $|\mathrm{Ker}\,\lambda|$ as before: $|k_E|^2$ ~~[illegible]~~ / ~ 1
- $\mathrm{inv}^r(X, \iota, \lambda) = -1$. (~~again~~ autom. if K/F unram.).

<u>Again</u>: if $S = \mathrm{Spec}\, \overline{k}_E$, only one $(X, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ up to isogeny, and isoclinic of slope $1/2$ and
$$\mathrm{Aut}^0(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) \simeq U(F).$$

- Lie X satisfies <u>Eisenstein</u> (EC_r). – see below.

Extend φ_0 to $\breve{\varphi}_0 : O_{\breve{F}} \longrightarrow O_{\breve{E}}$

Theorem: Consider the functor $\mathcal{M}_r : \mathrm{Nilp}_{O_{\breve{E}}} \longrightarrow (\mathrm{Sets})$, analogue.

Then ex.! isom. (contracting functor)

$$\mathcal{M}_r \xrightarrow{\sim} \mathcal{M} \hat{\otimes}_{O_{\breve{F}}, \breve{\varphi}_0} O_{\breve{E}},$$

equiv. w.r.t. $U(F)$. In partic., $\mathcal{M}_r$ flat + semi-st. reduction.

Remarks: (i) Instead of working with p-divisible gps, work with displays. Over $k = \bar{k}$, this means

Dieud.-module $\longmapsto$ relative Dieud.-module.

(ii) Special case of general pb. about local Shimura varieties: Let $(G, \{\mu\}, b)$ be local Shim.-datum s.t. b basic. Modify μ into $\mu' = \mu \cdot \varepsilon$, by a central character, and b'. Then $\mathbb{M}(G, \{\mu'\}, b')$ should be twist of $\mathbb{M}(G, \{\mu\}, b)$.

(iii) In previous paper, R+Z considered the original Drinfeld moduli pb. (no polariz., but for $D^{\times}_{1/n}$). Could construct contracting functor only in special fiber. Scholze extended this, using

- Scholze-Weinstein descr. of p-div. gps
- integral p-adic Hodge theory
- RZ-local models.

(iv) In the present case, Scholze has variant of this proof (uses unpublished early vesion of our paper). But our proof is more direct and less eclectic.

<u>Explanation of</u> (EC_r): Let $K^t \subset K$ and let $\Psi = \mathrm{Hom}(K^t, \bar{\mathbb{Q}}_p)$.

Let, for $\psi \in \Psi$,

$$A_\psi = \{ \varphi : K \to \bar{\mathbb{Q}}_p \mid \varphi_{|K^t} = \psi, \ r_\varphi = 2 \}$$

$$B_\psi = \{ \quad \text{———} \quad r_\varphi = 0 \}.$$

Fix uniformizer Π of K, with Eisenstein poly.

$$E(T) = \prod_\varphi (T - \varphi(\Pi)) \in O_{K^t}[T].$$

For $\psi \in \Psi$, let

$$E_\psi(T) = \prod_{\varphi_{|K^t} = \psi} (T - \varphi(\Pi))$$

$$E_{A_\psi}(T), \quad E_{B_\psi}(T).$$

Also, let

$$S_\psi(T) = \begin{cases} 1 & \psi \neq \psi_0, \bar{\psi}_0 \\ (T - \varphi_0(\Pi)) \cdot (T - \bar{\varphi}_0(\Pi)), & \psi = \psi_0, \ K/F \text{ ram.} \\ (T - \varphi_0(\Pi)), & \psi = \psi_0, \ K/F \text{ unr.} \\ (T - \bar{\varphi}_0(\Pi)) & \psi = \bar{\psi}_0, \ K/F \text{ unr.} \end{cases}$$

Then

$$E_\psi = S_\psi \cdot E_{A_\psi} \cdot E_{B_\psi}.$$

Then all factors $\in O_{E'}[T]$.

Let $E' = K^t \cdot E \subset \bar{\mathbb{Q}}_p$. Assume $S \in (Sch/O_{E'})$, then

$$\mathrm{Lie}\, X = \bigoplus_{\psi \in \Psi} \mathrm{Lie}_\psi(X).$$

(EC_r): $\quad S_\psi \cdot \mathbb{E}_{A_\psi}(r(\Pi) \mid \mathrm{Lie}_\psi X) = 0, \quad \forall \psi$

$\Lambda^{4-[K^t_\psi:F^t]}(\mathbb{E}_{A_\psi}(r(\Pi) \mid \mathrm{Lie}_\psi X) = 0$, d.h.

$\mathrm{rg} \leq \begin{cases} 1 & K/F \text{ ram.} \\ 2 & K/F \text{ unr.} \end{cases}$

If $S \in (\mathrm{Sch}/O_E)$ define (EC_r) via base change $O_E \to O_{E'}$.

__Lemma:__ (i) (EC_r) is indep't of choice of Π.

(ii) When S is E-scheme, then (KC_r) implies (EC_r).

(iii) When $S \in (\mathrm{Sch}/\mathrm{Spf}\, O_E)$, then

- if K/F unramif., then $(EC_r) \Rightarrow (KC_r)$
- if K/F ramif., ~~then~~ assume (EC_r) holds, then also

(KC_r) iff $\quad \mathrm{Tr}\left(r(\Pi) \mid \mathrm{Im}(\mathbb{E}_{A_{\psi_0}}(\Pi) \mid \mathrm{Lie}_{\psi_0} X\right) = 0.$

__Remark:__ The Kottwitz cond. (KC_r) has become standard. But if S not in char p, then very difficult to understand: this is why in (iii) had to restrict to special fiber.

Here is the contracting functor when $S = \mathrm{Spec}\, k$, $k = \bar{k}$:

Proceeds in 2 steps

Step 1: Let (M, ι) Dieudonné module of X. Then construct a new Dieudonné module (M', ι') s.t. $\iota'_{|O_F}$ strict, and $\dim = 2$ via $O_F \xrightarrow{\varphi_0} O_E \longrightarrow k$, as follows:

Decompose

$$M = \bigoplus M_\psi \, , \quad \text{with}$$

$$F_\psi : M_\psi \to M_{\psi\sigma} \, , \quad \text{resp. } V_\psi : M_{\psi\sigma} \longrightarrow M_\psi \, .$$

Set

$$M' = M \, , \quad \text{with}$$

$$F'_\psi = \Pi^{a_\psi} \cdot F_\psi \, , \quad V'_\psi = \Pi^{-a_\psi} \cdot V_\psi \, . \quad + \underline{\text{polarization}}$$

Remark: This even works for displays, via

$$F'_\psi = \tilde{E}_{A_\psi}(\Pi) \cdot F_\psi \, ,$$

where

$$\tilde{E}_{A_\psi}(T) = \prod_{\varphi \in A_\psi} \left(T - [\varphi(\Pi)] \right).$$

Step 2: Let (M', ι') Dieud-module s.t. $\iota'_{|O_F}$ strict. Then construct a relative Dieudonné module (M'', ι'').

Decompose

$$M' = \bigoplus_{i \in \mathbb{Z}/f} M'_i$$

Then M_0' is $O_F \otimes_{O_{F^t}} W(k) = W_{O_F}(k)$-module. Structures ⟹

$$\pi_F M_0' \subset V^t M_0' \subset M_0'.$$

Now set

$$M'' = (M_0', \pi_F V^{-t}, V^t).$$

#

<u>Main point</u>: polariz. induces <u>relative</u> polarization!

<u>Remark</u>: Again, this works for displays (⇝ Ahsendorf functor, analogue of Drinfeld functor for Cartier modules).