Columbia, March 17

Talk: Modularity of generating series for arithmetic divisors on unitary Shimura varieties

(1) The 4 levels of algebraic geometry

- geo. of varieties defined by polynomials
- geo. of varieties ——— geometric invariants (like dim, genus, etc): leads to curves of genus g, abelian varieties, K3, CY,
- moduli varieties of varieties w. fixed geo. invariants: leads to $\mathcal{M}_g$, $\mathcal{A}_g$, $M \subset \mathcal{A}_g$.
- algebraic cycles / vector bdles on such moduli spaces.

The bottom layer leads to generating series:

e.g. Kontsevich / Witten for $\mathcal{M}_g$.

Here concentrate on $M \subset \mathcal{A}_g$.

Examples / History: • Hirzebruch / Zagier: Let $M/_{\mathbb{C}}$ be Hilbert-Blumenthal surface, i.e. (A, ι, λ) whose

$\dim A = 2$, $\iota : O_F \to \mathrm{End}(A)$ with F = real-quadr. field.

They define special divisors $T(n)^*$ on M^* (compactific.)

and prove $\sum [T(m)] \cdot q^m \in H^2(M, \mathbb{Q})[[q]]$ is modular.

Generalizations by Oda, Kudla / Millson.

Borcherds (1999) (after Gross / Kohnen / Zagier): considers Sh-variety assoc. to orthogonal grps of signature $(n-2,2)$; forms generating series from special divisors w. values in $Ch^1_{\mathbb{Q}}$, and proved their modularity. This is done as follows: construct family of meromorphic diff.-forms whose divisors are expressed as lin. comb. of special divisors. These diff.-forms arise from Borcherds product expansion of Borcherds lift of weakly holo. modular forms. By varying these, impose enough relations to force modularity.

Higher codimension: W. Zhang w. Bruinier / Raum.

<u>Arithmetic analogue</u> (asked by Kudla), when M model over integers

- HZ case : Bruinier / Burgos / Kühn
- Compact Sh.-curves : Kudla / Rapo. / Yang : includes $p=2$!

All problems can be considered for Sh.-varieties to $U(n-1,1)$ instead of $SO(n-2,2)$ (Kudla).

<u>Analogue of Borcherds</u> : E. Hofmann

<u>Analogue of</u> [KRY] : joint work with Bruinier / Howard / Kudla / Yang

- its own sociology
- 5 blind people around elephant.

(2) Moduli space / Special cycles in char 0.

k = imaginary quadratic field $\subset \mathbb{C}$

O_k, $\quad 2 \nmid D \quad$ (Scholze disapproves).

Fix $(s,t) \rightsquigarrow$ algebraic stack $M_{(s,t)}$ over k,

$M_{(s,t)}(S)$ = groupoid of triples (A, ι, λ) where

- A/S abelian scheme of dim $s+t$
- $\iota: O_k \to \mathrm{End}(A)$ s.t.

$$\mathrm{char}(\iota(a) \mid \mathrm{Lie}\, A) = (T-a)^s \cdot (T-\bar{a})^t, \quad \forall a \in O_k$$

- λ princip. polariz. s.t. $\iota(a)^* = \iota(\bar{a})$, $\quad \forall a \in O_k$.

Now fix $n \geq 3$, and hermit. k-vs. of signature $(1,0)$, resp. $(n-1,1)$, with selfdual lattices $(W_0, \mathfrak{a}_0)$, resp. $(W, \mathfrak{a})$.

Moduli space: Open + closed substack

$$M \subset M_{(1,0)} \times_{\mathrm{Spec}\, k} M_{(n-1,1)}$$

of (A_0, A) s.t. $\forall$ geo. point $s \to S$ $\quad \exists$ iso. of hermitian $O_{k,\ell}$-modules,

$$\mathrm{Hom}_{O_k}(T_\ell(A_0), T_\ell(A)) \simeq \mathrm{Hom}_{O_k}(a_0, a) \otimes \mathbb{Z}_\ell \quad , \quad \forall \ell.$$

Here hermitian form on LHS: $(x_1, x_2) = x_2^\vee \circ x_1 \in \mathrm{Hom}_{O_{k,\ell}}(T_\ell(A_0), T_\ell(A_0))$
$\simeq O_{k,\ell}$.

Then $M = M_n$ is smooth of dimension $n-1$ over k.

<u>Remark:</u> This can be generalized to any CM-field F/F_0 (Rapo / Smithling / Zhang). This gives a moduli interpret. for Shimura var. assoc. to

$$V = \mathrm{Hom}_k(W_0, W)$$

$$G = \{ (z, g) \in \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m \times GU(V) \mid \mathrm{Nm}_{F/F_0}(z) = c(g) \in \mathbb{G}_m \}.$$

This is in contrast to Sh.-varieties for $GU(V)$ considered by Kottwitz and Harris / Taylor.

<u>Special divisors:</u> Let $(A_0, A) \in M(S)$, S connected. Then

$\mathrm{Hom}_{O_k}(A_0, A)$ is hermitian O_k-module, positive-definite.

For $m > 0$, let $Z(m)$ = stack of (A_0, A, x) where $(A_0, A) \in M$ and $x \in \mathrm{Hom}_{O_k}(A_0, A)$ with $(x, x) = m$.

Natural map $Z(m) \longrightarrow M$ finite + unramified

<u>Toroidal compactification:</u> The stack M is non-compact. But ex.

$$M \hookrightarrow M^*$$

canonical open embedding where M^* smooth + projective, and

$$\partial M^* = M^* \setminus M \qquad \text{smooth divisor.}$$

Connected comp. of ∂M^* are enumerated by cusps $\mathcal{C}$, i.e. connected components of bdary in minimal compactific. of M). To each $\mathcal{C}$

$\rightsquigarrow$ $\mathfrak{a}_{\mathcal{C}}$ = hermit. $\mathcal{O}_k$-module of signature $(n-2, 0)$.

<u>Now we are there:</u> for $m > 0$

$$Z(m)^* = \text{Z-closure of } Z(m) \text{ in } M^*$$

$$B(m) = \frac{m}{n-2} \cdot \sum_{\mathcal{C}} \#\{x \in \mathfrak{a}_{\mathcal{C}} \mid (x,x) = m\} \cdot \partial M^*_{\mathcal{C}}.$$

$$Z(m)^{tot} = Z(m)^* + B(m).$$

(Hofmann):

<u>Theorem:</u> The generating series

$$\sum_{m \geq 0} [Z(m)^{tot}] q^m \in Ch^1_{\mathbb{Q}}(M^*)[[q]]$$

is a modular form

Here $[Z(0)^{tot}] = [\omega^{-1}]$, where ω = canon. ext. of $\left(\begin{array}{l}\text{line bdle of}\\ \text{mod. forms of}\\ \text{wt 1 (stack!)}\end{array}\right.$

③ Integral models

The definition of M makes sense over Spec $\mathcal{O}_k$ (demand existence only for $\ell \neq \mathrm{char}\, k(s)$). Good model over Spec $\mathcal{O}_k[D^{-1}]$, but not flat over Spec $\mathcal{O}_k$. Two ways of remedying this:

- <u>Pappas</u>: impose condition on Lie A which is automatic over $\mathcal{O}_k[D^{-1}]$

$$\Lambda^2\big(\iota(a) - a \mid \mathrm{Lie}\, A\big) = 0 \quad , \quad \forall a \in \mathcal{O}_k .$$

- <u>Krämer</u>: add auxiliary datum,

  $\mathcal{F}_A \subset \mathrm{Lie}\, A$ $\mathcal{O}_k$-stable direct summand $\mathcal{O}_k$-submodule of rank $n-1$

  s.t. $\mathcal{O}_k$ acts via structure map on $\mathcal{F}_A$, via conjugate on $\mathrm{Lie}\, A/\mathcal{F}_A$.

Get cartesian diagram of $\mathcal{O}_k$-schemes,

$$\begin{tikzcd}
\mathcal{M}_{Kra} \arrow[r, hook] \arrow[d] & \mathcal{M}^*_{Kra} \arrow[d, "\pi"] \\
\mathcal{M}_{Pap} \arrow[r, hook] & \mathcal{M}^*_{Pap}
\end{tikzcd}$$

<u>Properties</u>:

- $\mathcal{M}^*_{Kra}$ regular, flat, $\partial \mathcal{M}^*_{Kra}$ smooth relative divisor

- $\mathcal{M}^*_{\mathrm{Pap}}$ normal, flat, with normal fibers. Every irreducible divisor w. empty generic fiber ~~has nonempty~~ meets the boundary.
- π is an isomorph. outside the singular locus. This singular locus is discrete, supported in fibers over $p \mid D$. Preimage of singular point is an irreducible divisor in $\mathcal{M}_{\mathrm{Kra}}$.

As before define for $m > 0$

$$\mathcal{Z}(m)^*_{\mathrm{Kra}} = \mathbb{Z}\text{-closure of } \mathcal{Z}(m) \text{ in } \mathcal{M}^*_{\mathrm{Kra}}$$

Then $\mathcal{Z}(m)^*_{\mathrm{Kra}}$ is a divisor. As before get $[\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)] \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{M}^*_{\mathrm{Kra}})$

<u>Theorem:</u> Let $\chi_k : (\mathbb{Z}/D)^\times \longrightarrow \{\pm 1\}$ be Dirichlet char. corresp. to k.

The formal generating series

$$\sum_{m \geq 0} [\mathcal{Z}(m)^*_{\mathrm{Kra}}] \cdot q^m \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{M}^*_{\mathrm{Kra}})[[q]]$$

is a holo modular form of wt. n, level $\Gamma_0(D)$ and character χ_k.

(i.e., get such after applying linear form $Q : \mathrm{Ch}^1_{\mathbb{Q}} \longrightarrow \mathbb{C}$).

About the proof, in partic., why do we need $\mathcal{M}^*_{Pap}$:

Consider meromorphic differential form of Borcherds as meromorphic diff. form on $\mathcal{M}^*_{Kra}$, or $\mathcal{M}^*_{Pap}$. Want that lin. comb. of special divisors or bdary — i.e., only relative divisors, no vertical components. This can be tested on $\mathcal{M}^*_{Pap}$!

$\mathcal{M}^*_{Pap}$:
- disadvantage: not regular, $\mathcal{Z}^*_{Pap}(m)$ not divisor at sing. points
- advantage: all vertical divisors meet boundary.

Final remark: Have similar result for $\widehat{Ch}^1_{\mathbb{Q}}(\mathcal{M}^*_{Kra})$ (even though not finite-dimensional).